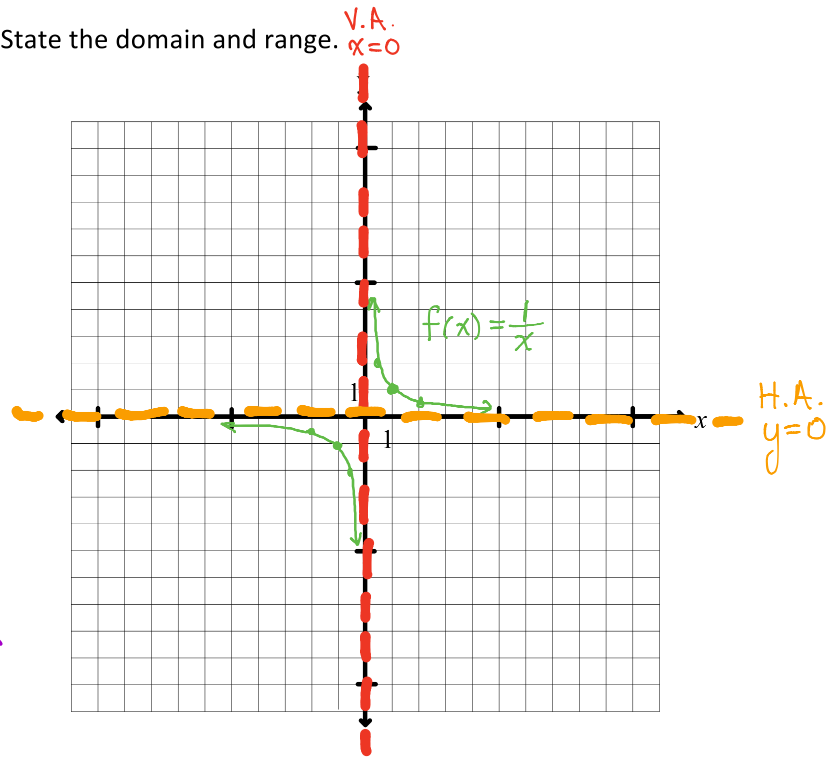
Transformations of Reciprocal Functions

A. Characteristics of the Reciprocal Function *f(x) = 1/x*

1. Graph the function $f(x)=\frac{1}{x}$ on the grid below. State the domain and range.

f(x) = 1/x

x	y
-2	$-\frac{1}{2}$
-1	-1
-1/2	-2
0	undefined
1/2	2
1	1
2	$\frac{1}{2}$



D = $\{x \in \mathbb{R} \mid x \neq 0\}$

R = $\{y \in \mathbb{R} \mid y \neq 0\}$

let y=0:
$0 = \frac{1}{x}$
$0 \cdot x = 1$
$0 = 1$

The graph of the reciprocal function is *a hyperbola* found in quadrant I and quadrant III of the Cartesian plane. Basic properties of the reciprocal function ***f(x) = 1/x*** are as follows:

- the value 1/0 is **undefined**, so the expression in the *denominator cannot equal zero*
- any value of *x* that would yield 1/0 is a restriction on the function, and is visualized on the graph as a **vertical asymptote** (note: *in this course,* an ***asymptote*** is a line representing a value that the function will approach but *never reach*)
- if the numerator is a constant, no input value of *x* will result in an output value of zero, and is visualized on the graph as a **horizontal asymptote**
- the **domain** is any real number *except* the singularity: D = {x| x ∈ R, x ≠ 0} → v.a.: x = 0
- the **range** is any real number *except* zero: R = {y| y ∈ R, y ≠ 0} → h.a.: y = 0

2. When graphing reciprocal functions of the form $y=\frac{a}{k(x-d)}+c$, label the following on the graph:

$y = a\left[\frac{1}{k(x-d)}\right] + c$

- the equation of the function
- the equation of the vertical asymptote, *x = d*
- the equation of the horizontal asymptote, *y = c*
- clearly mark each key point with a dot
- label any intercepts

B. Transforming Reciprocal Functions

Reciprocal functions of the form $y = \frac{1}{x}$ can also be transformed according to: $y = \frac{a}{k(x-d)} + c$.

1. Sketch the following transformed functions on the grids below. List the transformations in order on the base function $y = \frac{1}{x}$. State the domain and range.

a) $y + 5 = -\frac{1}{x+2}$

$y = -\frac{1}{x+2} - 5$

Description:

V.R. across the x-axis

H.T. 2 units left V.A. $x = -2$

V.T. 5 units down H.A. $y = -5$

(x, y) → ($x - 2$, $-y - 5$)

$y = \frac{1}{x}$ →

$(-2, -\frac{1}{2}) \rightarrow (-4, -4\frac{1}{2})$

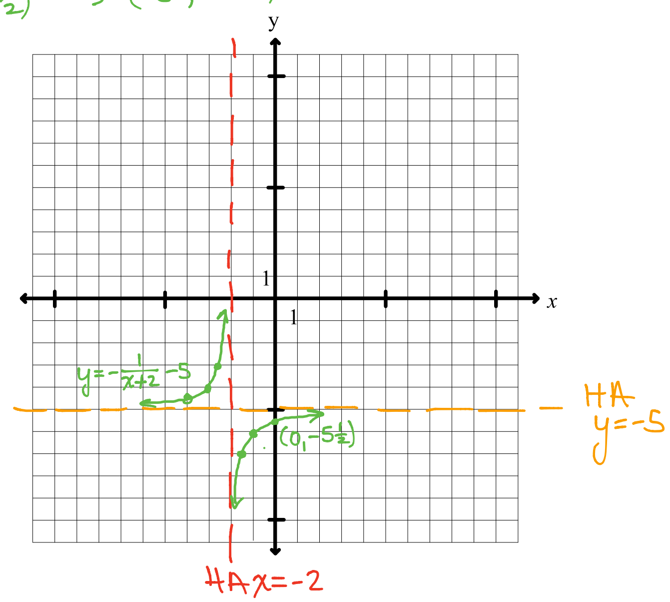
$(-1, -1) \rightarrow (-3, -4)$

$(-\frac{1}{2}, -2) \rightarrow (-2\frac{1}{2}, -3)$

$(\frac{1}{2}, 2) \rightarrow (-1\frac{1}{2}, -7)$

$(1, 1) \rightarrow (-1, -6)$

$(2, \frac{1}{2}) \rightarrow (0, -5\frac{1}{2})$



D = $\{x \in \mathbb{R} \mid x \neq -2\}$

R = $\{y \in \mathbb{R} \mid y \neq -5\}$

b) $y = \frac{2}{-x+3} + 4$

$y = 2\left[\frac{1}{-(x-3)}\right] + 4$

Description:

HR across the y-axis

VE by a factor of 2

HT 3 units right VA $x = 3$

VT 4 units up HA $y = 4$

(x, y) → ($-x + 3$, $2y + 4$)

$(-2, -\frac{1}{2}) \rightarrow (5, 3)$

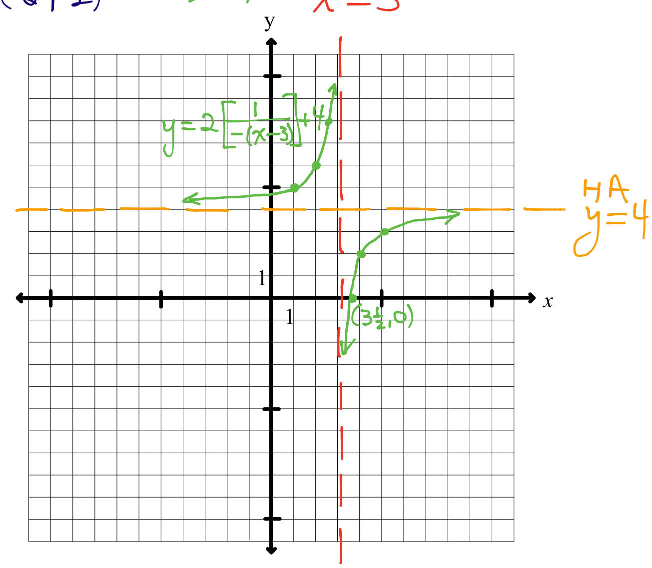
$(-1, -1) \rightarrow (4, 2)$

$(-\frac{1}{2}, -2) \rightarrow (3\frac{1}{2}, 0)$

$(\frac{1}{2}, 2) \rightarrow (2\frac{1}{2}, 8)$

$(1, 1) \rightarrow (2, 6)$

$(2, \frac{1}{2}) \rightarrow (1, 5)$ VA $x = 3$



D = $\{x \in \mathbb{R} \mid x \neq 3\}$

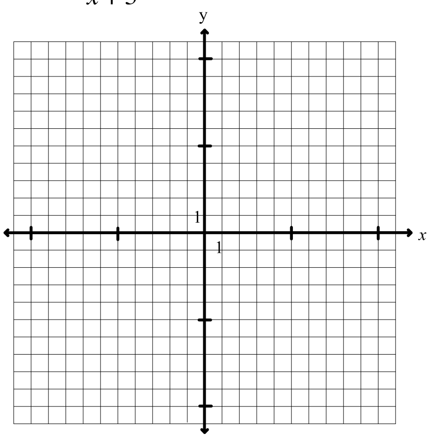
R = $\{y \in \mathbb{R} \mid y \neq 4\}$

HW: "WORKSHEET: Transformations of Reciprocal Functions" + Unit 1 T.H.Q #2 #1abc.

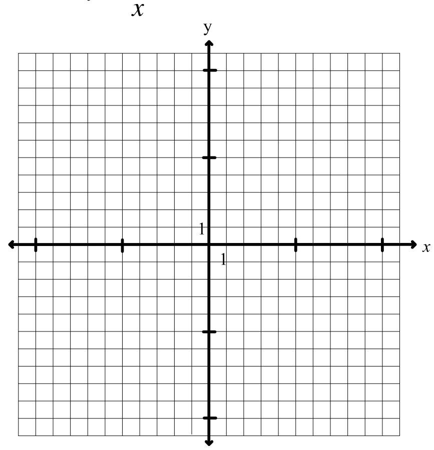
WORKSHEET: Transformations of Reciprocal Functions

Sketch a **graph** for each function below and state the **domain**, **range**, **vertical asymptote**, and **horizontal asymptote**. ***Show your work on a separate sheet of paper.***

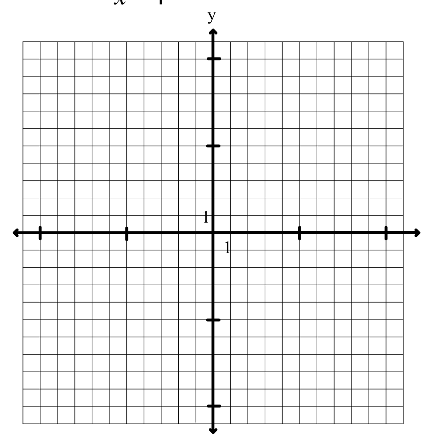
a. $y = \dfrac{1}{x+3}$



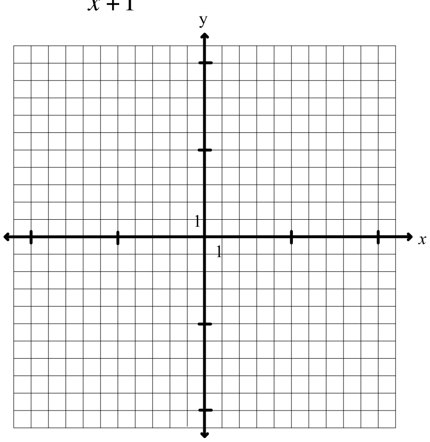
b. $y = \dfrac{1}{x} - 3$



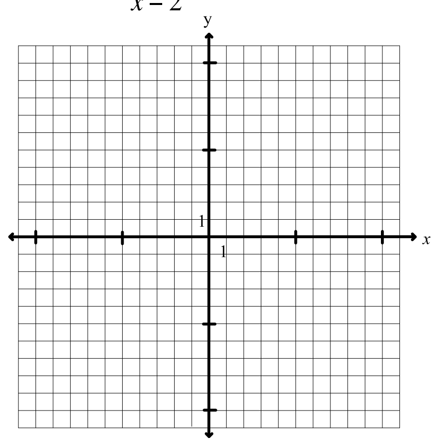
c. $y = \dfrac{2}{x-4}$



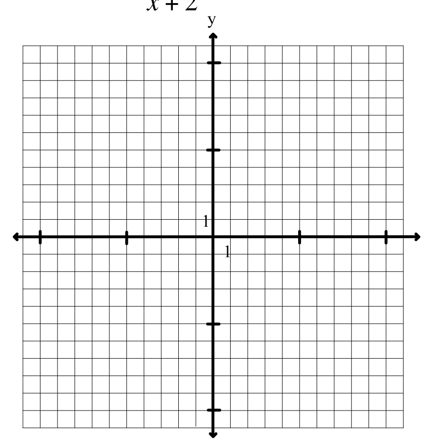
d. $y = \dfrac{3}{x+1} + 1$



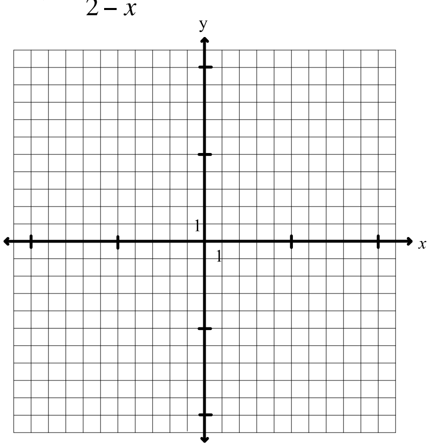
e. $y = \dfrac{-4}{x-2}$



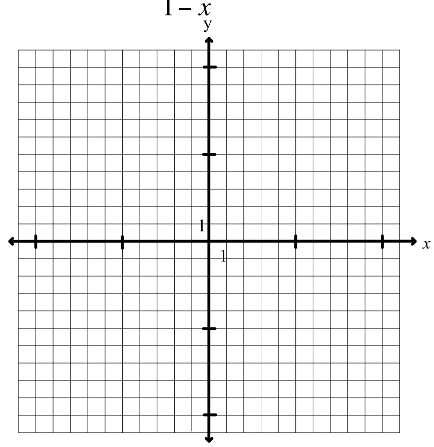
f. $y = 3 + \dfrac{1}{x+2}$



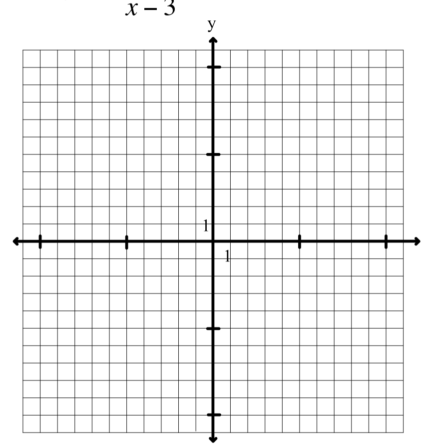
g. $y = \dfrac{2}{2-x}$



h. $y = 1 - \dfrac{1}{1-x}$



i. $y = -\dfrac{2}{x-3} + 2$



Answers:

1. ii)	Domain $x \in R$	Range $y \in R$	Equation of the Asymptotes
a.	$x \neq -3$	$y \neq 0$	$x = -3, y = 0$
b.	$x \neq 0$	$y \neq -3$	$x = 0, y = -3$
c.	$x \neq 4$	$y \neq 0$	$x = 4, y = 0$
d.	$x \neq -1$	$y \neq 1$	$x = -1, y = 1$
e.	$x \neq 2$	$y \neq 0$	$x = 2, y = 0$
f.	$x \neq -2$	$y \neq 3$	$x = -2, y = 3$
g.	$x \neq 2$	$y \neq 0$	$x = 2, y = 0$
h.	$x \neq 1$	$y \neq 1$	$x = 1, y = 1$
i.	$x \neq 3$	$y \neq 2$	$x = 3, y = 2$

Functions and Their Inverses

A. Introduction

We have seen reflections across the x-axis (i.e. across the line y = 0) and the y-axis (i.e. across the line x = 0), but the **inverse of a function** can be visualized as a reflection across the line *y = x*. To achieve this reflection, the coordinates for each point in the function (x , y) are inverted to create corresponding points (y , x).

The inverse of a function, f(x), is denoted by $f^{-1}(x)$.

If we look at a table of values for a function and its inverse function, we see a pattern:

f(x) = x + 3	
x	***f(x)***
2	5
3	6
4	7
5	8

$f^{-1}(x) = x - 3$	
x	***$f^{-1}(x)$***
5	2
6	3
7	4
8	5

Notice that the input values for the first function become the output values for the second function. The inverse function "undoes" what the original function did! The **domain** for the original function becomes the range of the inverse, and the **range** of the original function becomes the domain of the inverse.

B. Finding the Inverse

There are 3 ways to find the inverse of a function: *1) interchanging* the coordinates of each point*; 2) graphing* the function and its inverse; and *3) finding the inverse algebraically*.

I. Interchanging Coordinates

1. Given that *g* = {(-2, -8), (0, -2), (3, 4), (4, 7)}

a) g^{-1} = {(-8,-2), (-2,0), (4,3), (7,4)}

b) State the domain and range of *g* and g^{-1}.

D_g = {-2, 0, 3, 4}

R_g = {-8, -2, 4, 7}

$D_{g^{-1}}$ = {-8, -2, 4, 7}

$R_{g^{-1}}$ = {-2, 0, 3, 4}

c) Are *g* and g^{-1} both functions? Explain.

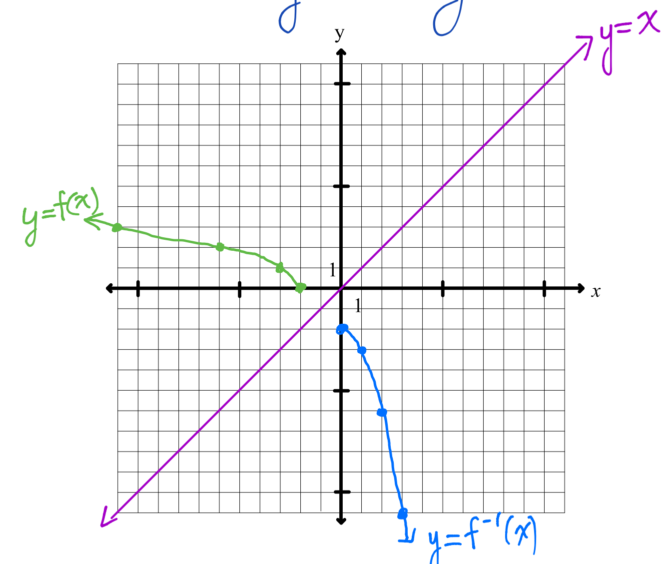
yes, no x-value has more than 1 y-value for either g or g^{-1}

II. Graphing the Inverse of a Function

- Draw the line y = x
- Determine key points on *f(x)*
- Interchange *(x, y)* for *(y, x)*
- Plot the corresponding points as the inverse

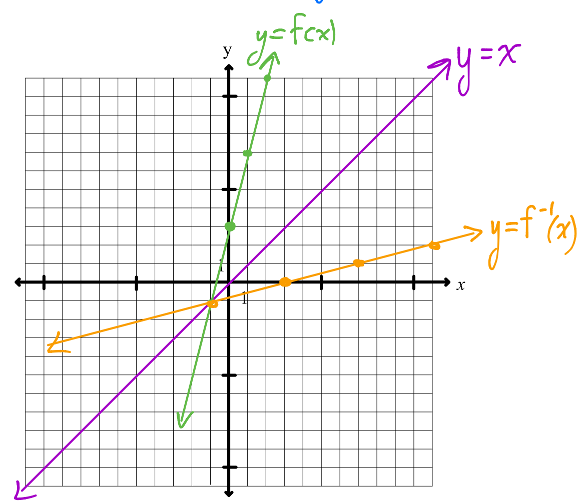
1. Graph $f(x) = \sqrt{-(x+2)}$, and sketch its inverse.

H.R. across y-axis, H.T 2 units left } on $y = \sqrt{x}$



III. Finding the Inverse of a Function <u>Algebraically</u>

- Replace *f(x)* with *y*
- Interchange *x* and *y* in the equation
- Rearrange to isolate the 'new' *y*
- State whether or not the inverse is a function



1. Find the inverse of *f(x) = 4x + 3*. Graph both relations.

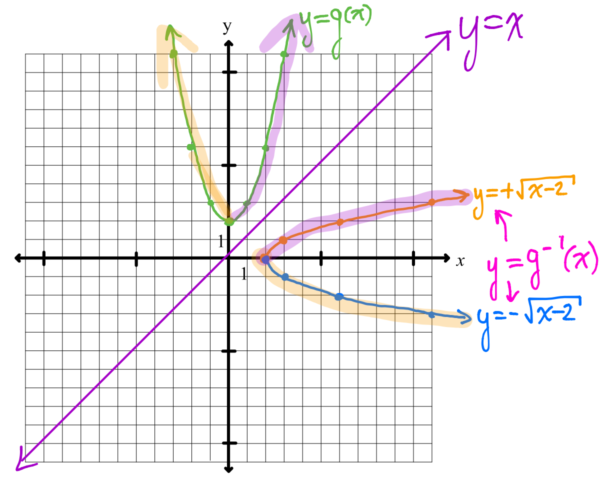
Let $y = 4x+3$

For inverse: $x = 4y+3$

$\frac{-4y}{-4} = \frac{x+3}{-4}$ $\frac{}{-4}$

$y = \frac{1}{4}x - \frac{3}{4}$ $\left(\text{or } y = \frac{x-3}{4}\right)$

$\therefore f^{-1}(x) = \frac{1}{4}x - \frac{3}{4}$, this is a function.



2. a) Find the inverse of $g(x) = x^2 + 2$. Graph both relations.

Let $y = x^2+2$

For inverse: $x = y^2+2$

$x-2 = y^2$

$\pm\sqrt{x-2} = y$

$\therefore g^{-1}(x) = \pm\sqrt{x-2}$

b) Is the inverse a function? No

c) State the domain and range of g^{-1}.

$D_{g^{-1}} = \{x \in \mathbb{R} \mid x \geq 2\}$

$R_{g^{-1}} = \{y \in \mathbb{R}\}$

d) Restrict the domain of *g(x)* so that its inverse is a function.

$D_g = \{x \in \mathbb{R} \mid x \geq 0\}$

or $D_g = \{x \in \mathbb{R} \mid x \leq 0\}$

3. Find the inverse of the following functions algebraically. For a) and b) use the range of the function to restrict the domain of the inverse, if necessary.

a) $f(x) = \sqrt{x+4}$

Let $y = \sqrt{x+4}$

For inverse: $x = \sqrt{y+4}$

$(x)^2 = (\sqrt{y+4})^2$

$x^2 = y+4$

$x^2 - 4 = y$

$\therefore f^{-1}(x) = x^2 - 4,\ x \geq 0$

$R_f = \{y \in \mathbb{R} \mid y \geq 0\}$

$D_{f^{-1}} = \{x \in \mathbb{R} \mid x \geq 0\}$

b) $g(x) = \dfrac{1}{x-3}$

Let $y = \frac{1}{x-3}$; $R_g = \{y \in \mathbb{R} \mid y \neq 0\}$

For inverse: $\frac{x}{1} = \frac{1}{y-3}$

$x(y-3) = 1$

$y - 3 = \frac{1}{x}$

$y = \frac{1}{x} + 3$

$g^{-1}(x) = \frac{1}{x} + 3$

$D_{g^{-1}} = \{x \in \mathbb{R} \mid x \neq 0\}$

c) $h(x) = \dfrac{2x+3}{x-1}$

Let $y = \frac{2x+3}{x-1}$

For inverse:

$\frac{x}{1} = \frac{2y+3}{y-1}$

$x(y-1) = 1(2y+3)$

$xy - x = 2y + 3$

$xy - 2y = 3 + x$

$y(x-2) = 3 + x$

$y = \frac{3+x}{x-2}$

$\therefore h^{-1}(x) = \frac{3+x}{x-2} \leftarrow x \neq 2$

HW: A) p. 215-219 #2, 3bdg, 10(iv)abc & (v)abc, 11, 15ac, 16(iii)abc & (viii)abc, 30; B) Find the inverse of: $y = \dfrac{x+5}{x-2},\ x \neq 2$