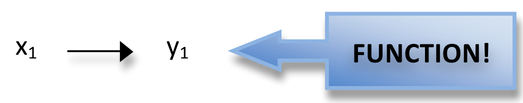
Functions and Function Notation

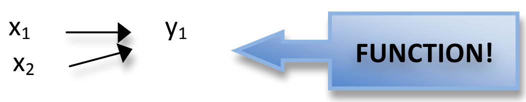
A. Functions and Relations

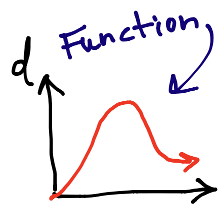
A **relation** is defined as an identified *pattern* or *relationship* between two variables. Relations can be represented in various ways: as a set of **ordered pairs**, a **table of values**, a **graph**, or an **equation**.

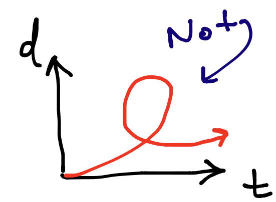
A **function** is a type of relation between two variables, an *input* variable and an *output* variable, in which each value of the *input* variable corresponds to no more than **ONE** value of the *output* variable.

∴ A relation is **NOT** a function if *one x value has 2 or more different y-values associated* with it.









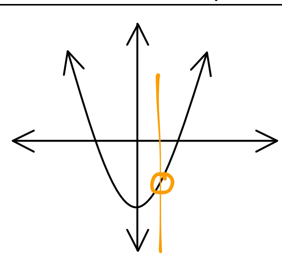
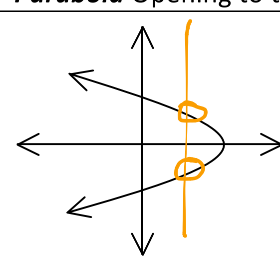
B. The Vertical Line Test (or VLT!)

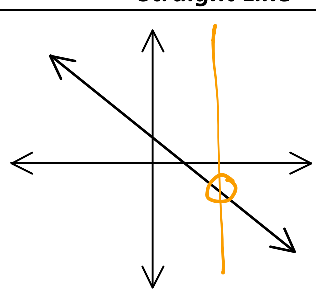
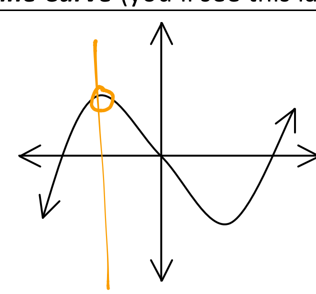
An easy way of determining whether or not a relation is a function is to use its graph and the Vertical Line Test.

The **Vertical Line Test** states that a relation is a function if you can draw a vertical line that passes through **ZERO** points or **ONE** point on the graph of the relation.

∴ A relation ***fails*** the vertical line test if the line intersects the graph at **more than one** point!

Ex. 1: Let's look at some examples below…

Parabola Opening Up	***Parabola*** Opening to the Left
	
Does this pass the Vertical Line Test? YES / NO Therefore, is this relation a function? YES / NO	Does this pass the Vertical Line Test? YES / NO Therefore, is this relation a function? YES / NO

Straight Line	***Sine Curve*** (you'll see this later)
	
Is this relation a function? YES / NO	Is this relation a function? YES / NO

But a vertical line is NOT!

C. What is Function Notation?

In an equation such as $y = 2x + 3$, __y__ depends on __x__ and is said to be a *function* of *x*. Since we are dealing with functions, and not all relations represent functions, we're going to use a special type of notation when writing equations that represent functions. It's called **FUNCTION NOTATION**!

Equation	Function Notation
$y = 3x + 1$	$f(x) = 3x + 1$
$d = 3t^2 - 2t + 1$	$d(t) = 3t^2 - 2t + 1$
$A = \pi r^2$	$A(r) = \pi r^2$
$v = t^2 - 21$	$v(t) = t^2 - 21$

"*f(x)*" is read "*f of x*" or "*f at x*". It represents the height of the function at a given independent (x) value.

Using **function notation** is similar to using equations involving ***x*** and ***y*** values. To find a ***y***-value given an ***x***-value simply requires ***substitution***. Thus, we can write ordered pairs (*x, f(x)*) which are the same as (*x, y*) since ***y = f(x).***

D. Performing Operations Using Function Notation

i) **Given an *x*-value**, we can use substitution to **find the height** of the function at that particular *x*-value (i.e. we can find the corresponding *y*-coordinate on the Cartesian plane).

Ex. 2: If $f(x) = -3x + 2$, find:

a) $f(-1) = -3(-1) + 2$

$= 3 + 2$

$= 5$

$\boxed{\therefore f(-1) = 5}$

b) $f(0) = -3(0) + 2$

$= 0 + 2$

$= 2$

$\boxed{\therefore f(0) = 2}$

Ex. 3: If $g(x) = -2x^2 - 5x + 2$, find:

a) $g(-1) = -2(-1)^2 - 5(-1) + 2$

$= -2[1] + 5 + 2$

$= -2 + 5 + 2$

$= 5$

$\boxed{\therefore g(-1) = 5}$

b) $2[g(2)] = 2[-2(2)^2 - 5(2) + 2]$

$= 2[-2(4) - 10 + 2]$

$= 2[-16]$

$= -32$

$\boxed{\therefore 2[g(2)] = -32}$

ii) **Given the height** of the function, we can use substitution to **solve the equation for x** at that particular height (i.e. we can find the corresponding *x*-coordinate on the Cartesian plane).

Ex. 4: If $f(x) = 5x - 1$ and $g(x) = x^2 + 3x$, find the value(s) of *x* when

a) $f(x) = 24$

$y = f(x)$

$\therefore y = 24$

$x = ?$

$24 = 5x - 1$

$\frac{25}{5} = \frac{5x}{5}$

$\boxed{\therefore x = 5}$

b) $g(x) = 18$

$y = g(x)$

$\therefore y = 18$

$x = ?$

$18 = x^2 + 3x$

$0 = x^2 + 3x - 18$

$0 = (x+6)(x-3)$

$x + 6 = 0$ or $x - 3 = 0$

$\boxed{\therefore x = -6 \text{ or } x = 3}$

c) $g(x) = 0$

$0 = x^2 + 3x$

$0 = x(x+3)$

$x = 0$ or $x + 3 = 0$

$\boxed{\therefore x = -3, 0}$

d) $f(x) = g(x)$

$5x - 1 = x^2 + 3x$

$0 = x^2 + 3x - 5x + 1$

$0 = x^2 - 2x + 1$

$0 = (x-1)(x-1)$

$x - 1 = 0$ or $x - 1 = 0$

$\boxed{\therefore x = 1, 1}$

iii) **Given an expression that replaces the x-value** of the function, we can use substitution to **write and simplify** a new expression representing a new function!

Ex. 5: Given that $f(x) = 2x^2 - x + 1$ and $g(x) = \sqrt{4x+1}$:

a) write and simplify $f(x+3)$

$f(x+3) = 2(x+3)^2 - (x+3) + 1$

$= 2(x+3)(x+3) - x - 3 + 1$

$= 2(x^2 + 6x + 9) - x - 2$

$= 2x^2 + 12x + 18 - x - 2$

$= 2x^2 + 11x + 16$

$\boxed{\therefore f(x+3) = 2x^2 + 11x + 16}$

b) write and simplify $g(a^2 - a)$

$g(a^2 - a) = \sqrt{4(a^2 - a) + 1}$

$= \sqrt{4a^2 - 4a + 1}$

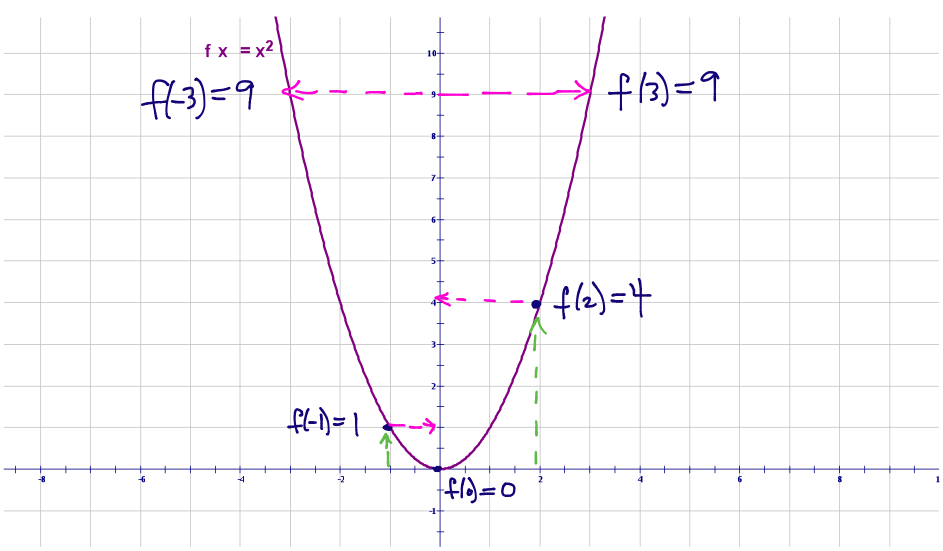
$= \sqrt{(2a-1)^2}$

$= 2a - 1$

$\boxed{\therefore g(a^2 - a) = 2a - 1}$

iv) **Given the graph** of the function, we can **interpret** the height at varying *x*-values, and vice versa.

Ex. 6: Find the required values using the graph of $f(x) = x^2$ below.



a) $\underbrace{f(2)}_{y=?} = $ 4 ($x = 2$)

b) $\underbrace{f(-1)}_{y=?} = $ 1 ($x = -1$)

c) $\underbrace{f(0)}_{y=?} = $ 0 ($x = 0$)

d) $\underbrace{f(x)}_{y=9} = 9,\ x = $ 3, −3 (?)

e) $\underbrace{f(x)}_{y=1} = 1,\ x = $ 1, −1 (?)

HW: p. 178-181 #1, 7-10(odds), 12, 26, 32

Domain and Range

A. Describing Relations and Functions Using Number Sets

In order to understand how relations behave, we need to understand the sets of numbers that they can inhabit:

N: denotes the set of ***natural*** numbers, which include all positive counting numbers but NOT zero;
N = {1, 2, 3, . . .}

W: denotes all ***whole*** numbers, including positive counting numbers and zero;
W = {0, 1, 2, 3, . . .}

I: denotes the set of ***integers***, which includes all positive and negative whole numbers, as well as zero;
I = {. . . , -3, -2, -1, 0, 1, 2, 3, . . .}

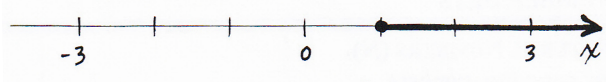
R: denotes the set of all ***real*** numbers, including all integers; fractions/rational numbers [Q] (terminating decimals and repeating decimals); and irrational numbers [$\overline{Q}$](non-terminating, non-repeating decimals)

B. Domain

The domain of a relation is the set of all ___input___ (_x_) elements of a relation. The domain describes the values that are acceptable for the **independent** variable.

Domains can be communicated in ***words***, using a ***number line***, or as a ***set*** (giving a *list* of numbers or using *inequality* statements). To create a set, you need to know proper set notation.

Ex. 1: Given the number line …

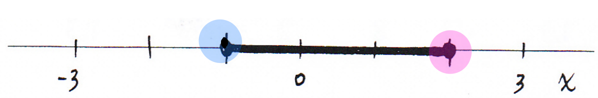


Words: *"x is any real number greater than or equal to 1"*

Set Notation: $D = \{x \mid x \in R, x \geq 1\}$

The domain
is
the set of
all x values
such that
x is an element of the real numbers
where/and
x is greater than or equal to 1

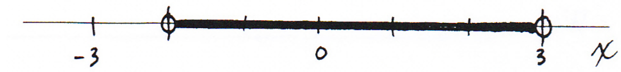
Ex. 2: Given the number line …



Words: *"x is any real number greater than or equal to –1 and less than or equal to 2"*

Set Notation: $D = \{x \mid x \in R, -1 \leq x \leq 2\}$

Ex. 3: Given the number line …



Words: *"x is any real number greater than −2 and less than 3"*

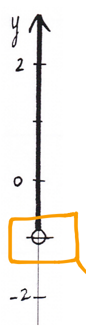
Set Notation: $D = \{x \mid x \in \mathbb{R}, -2 < x < 3\}$

C. Range

The range of a relation is the set of all output (y) elements of a relation. The range describes the values that are acceptable for the **dependent** variable.

Ranges can also be communicated in ***words***, using a ***number line***, or as a ***set***.

Ex. 4: Given the number line …

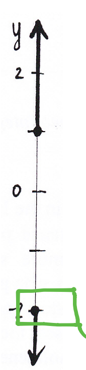


Words: *"y is any real number greater than −1"*

Set Notation: $R = \{y \mid y \in \mathbb{R}, y \boxed{>} -1\}$

open circle means "not equal to"

Ex. 5: Given the number line …



Words: *"y is any real number greater than or equal to 1 or less than or equal to −2"*

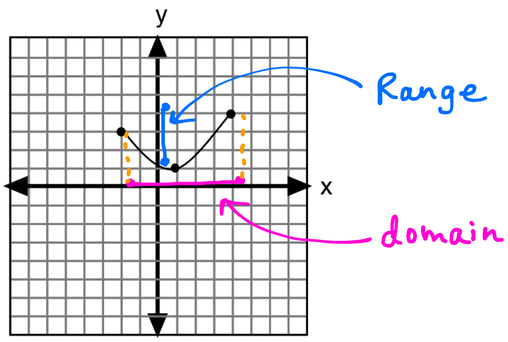
Set Notation: $R = \{y \mid y \in \mathbb{R}, y \geq 1, y \boxed{\leq} -2\}$

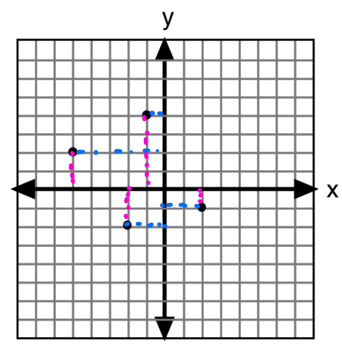
closed circle means "equal to"

D. Bringing Domain and Range Together

Now that we've worked with domain and range separately, let's bring them together!

Ex. 6: Use *set notation* to describe the domain and range of the following relations:





x	*y*
– 4	7
– 3	2
– 8	5
– 4	– 1

Function? Yes (Passes VLT)

D = $\{x \mid x \in \mathbb{R}, -2 \le x \le 4\}$

R = $\{y \mid y \in \mathbb{R}, 1 \le y \le 4\}$

Function? Yes (Passes VLT)

D = $\{-5, -2, -1, 2\}$

R = $\{2, -2, 4, -1\}$

Function? No (x-values are repeated)

D = $\{-4, -3, -8\}$

R = $\{7, 2, 5, -1\}$

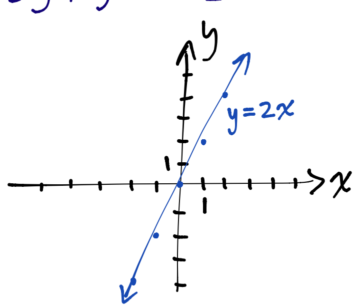
Ex. 7: Write the domain and range for the following relations:

a) $0 = 2x - y$ *Function?* Yes

$y = 2x \leftarrow$ Linear relation

D = $\{x \mid x \in \mathbb{R}\}$

R = $\{y \mid y \in \mathbb{R}\}$

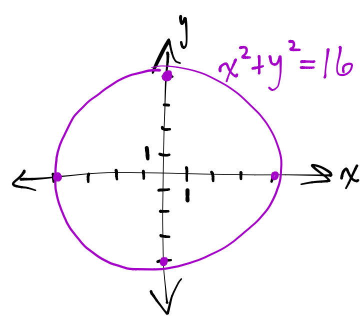


b) $16 = x^2 + y^2$ *Function?* No

$4^2 = x^2 + y^2 \leftarrow$ Circle with radius = 4

D = $\{x \mid x \in \mathbb{R}, -4 \le x \le 4\}$

R = $\{y \mid y \in \mathbb{R}, -4 \le y \le 4\}$

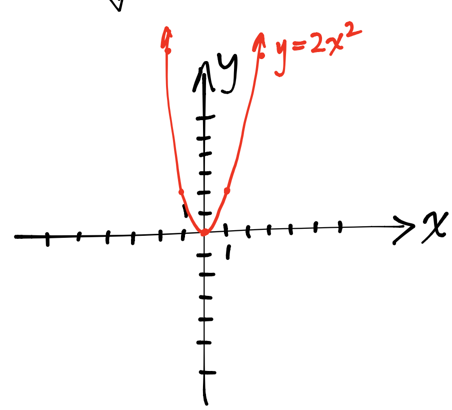


c) $0 = 2x^2 - y$ *Function?* Yes

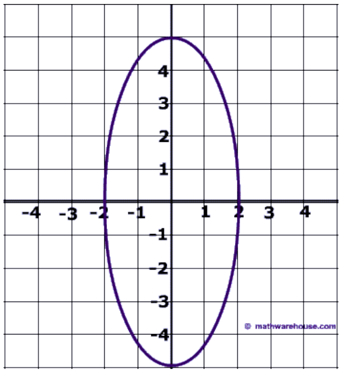
$y = 2x^2 \rightarrow$ Quadratic relation

D = $\{x \mid x \in \mathbb{R}\}$

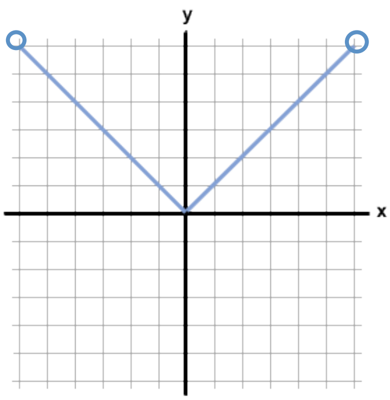
R = $\{y \mid y \in \mathbb{R}, y \ge 0\}$



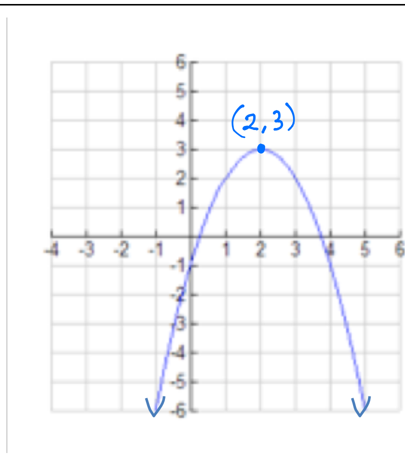
Ex. 8: Describe the domain and range of the following relations in the manner indicated:



Domain – words:	*x is any real number greater than or equal to –2 and less than or equal to 2*
Domain – set notation:	$D = \{x / x \in \mathbb{R}, -2 \le x \le 2\}$
Range – set notation:	$R = \{y \mid y \in \mathbb{R}, -5 \le y \le 5\}$
Function (with reason)?	No → fails the vertical line test.



Domain – words:	*x is any real number greater than* -6 *and less than* 6
Domain – set notation:	$D = \{x \mid x \in \mathbb{R}, -6 < x < 6\}$
Range – set notation:	$R = \{y \mid y \in \mathbb{R}, 0 \le y < 6\}$
Function (with reason)?	YES → passes the V.L.T.



Domain – words:	x is any real number
Domain – set notation:	$D = \{x \mid x \in \mathbb{R}\}$
Range – set notation:	$R = \{y \mid y \in \mathbb{R}, y \le 3\}$
Function (with reason)?	YES → passes the V.L.T.

HW: p. 178-181 #2-5, 17, 18, 21, 25

Workshop: Graphing Quadratic Functions Given Any Form

A. Characteristics of the Quadratic Function *f(x)* = x^2

The graph of the quadratic function is a *parabola*. Basic properties of the quadratic function ***f(x)* = x^2** are as follows:

- **standard form** is written as $y = ax^2 + bx + c$, where (0, c) is the *y-intercept.*
- **factored form** is written as $y = a(x – r)(x – s)$, where (r, 0) and (s, 0) are the *x-intercepts, zeros* or *roots*.
- **vertex form** is written as $y = a(x – h)^2 + k$, where (h, k) is the *vertex*.
- the **direction of opening** can be determined by inspecting the '**a**' value:
  - when a > 0, the parabola opens up and the vertex is a minimum.
  - when a < 0, the parabola opens down and the vertex is a maximum.

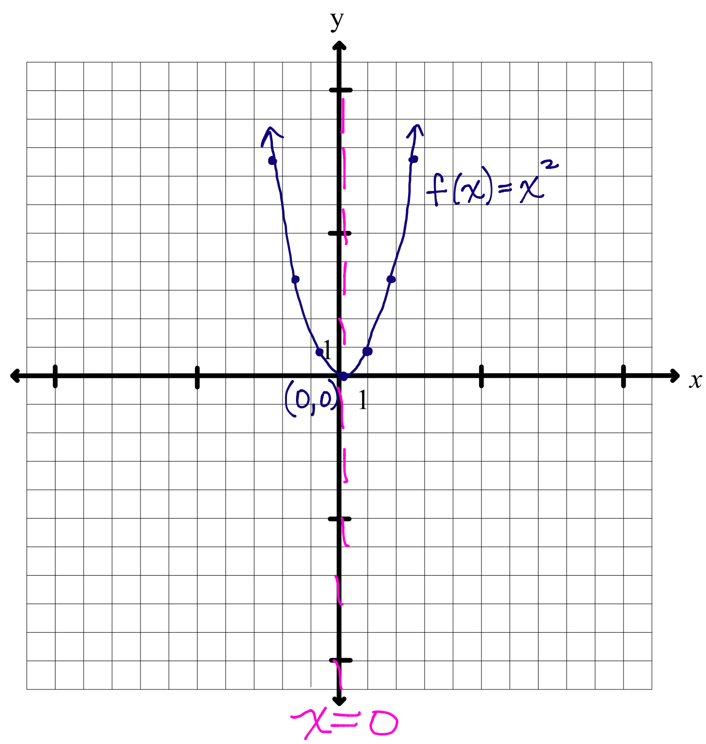
- the **axis of symmetry** is a vertical line drawn through the vertex across which the parabola is symmetrical; it has the equation $x = h$.
- the **domain** is D = {x| x ∈ R} → real life may have restrictions eg no negative time!
- the **range** is R = {y| y ∈ R, y ≥ k **or** y ≤ k}

Ex. 1: Graph the function *f(x)* = x^2 on the grid below. State the domain and range. When graphing quadratic functions, include the following on the graph:

- the equation of the function
- the vertex and any intercepts
- the axis of symmetry (sketched and labeled)
- key points (clearly marked with a dot)

***f(x)* = x^2**

x	y	
-3	9	$f(-3)=(-3)^2$
-2	4	$f(-2)=(-2)^2$
-1	1	$f(-1)=(-1)^2$
0	0	$f(0)=(0)^2$
1	1	$f(1)=(1)^2$
2	4	$f(2)=(2)^2$
3	9	$f(3)=(3)^2$



① Plot vertex (h,k)
② Move "x" units ⟷
③ move "ax^2" units ↕

D = $\{x \mid x \in \mathbb{R}\}$

R = $\{y \mid y \in \mathbb{R}, y \geq 0\}$

B. Vertex Form of a Quadratic Function

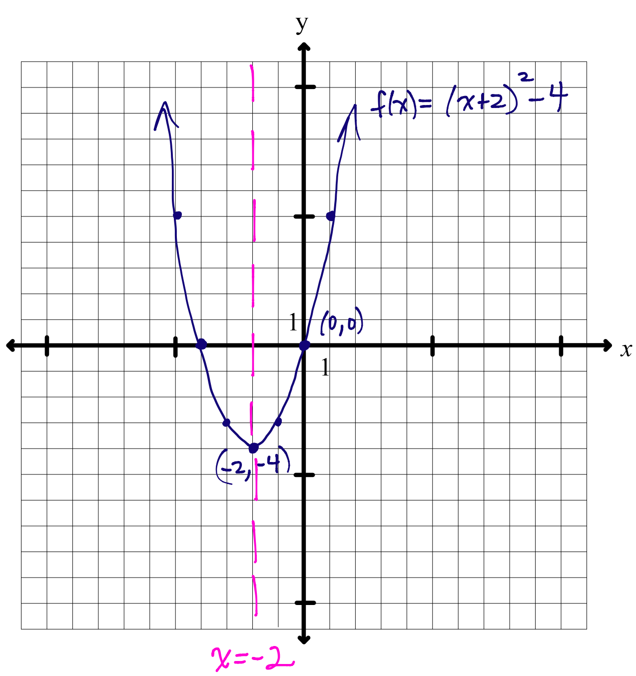
	Characteristics	**What To Do . . .**
Vertex Form	$f(x) = a(x - h)^2 + k$ ***or*** $y = a(x - h)^2 + k$ • In this form, the only <u>point</u> evident is the <u>vertex</u>, (h, k) • the a value determines the direction of opening and the step pattern	• Plot and label the vertex, (h, k) • From the vertex, move horizontally (⇔) 1 unit, and vertically (⇳) $1^2(a)$ units (on either side); then, move horizontally 2 units, and vertically $2^2(a)$ units; [repeat with 3 units and $3^2(a)$ units, if necessary, to get accurate end behaviour] • Sketch and label the axis of symmetry, $x = h$

Ex. 2: Sketch the following quadratic functions on the grids below. State the vertex (h, k), equation of the axis of symmetry (x = h), domain, and range.

a) $f(x) = (x + 2)^2 - 4$

Vertex: $(-2, -4)$

Axis of symmetry: $x = -2$



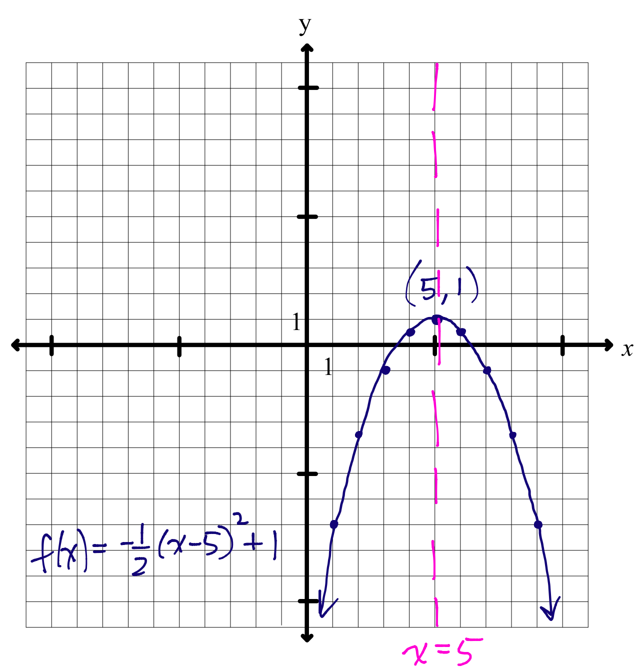
D = $\{x \mid x \in \mathbb{R}\}$

R = $\{y \mid y \in \mathbb{R}, y \geq -4\}$

b) $f(x) = -\frac{1}{2}(x - 5)^2 + 1$

Vertex: $(5, 1)$

Axis of symmetry: $x = 5$



D = $\{x \mid x \in \mathbb{R}\}$

R = $\{y \mid y \in \mathbb{R}, y \leq 1\}$

C. Factored Form of a Quadratic Function

	Characteristics	**What To Do . . .**

Factored Form	$f(x) = a(x - r)(x - s)$ ***or*** $y = a(x - r)(x - s)$
	
	• In this form, the only <u>points</u> evident are the <u>x-intercepts</u>, $(r, 0)$ and $(s, 0)$
	• the x-value of the vertex ("*h*") can be calculated by finding the value of $\frac{r+s}{2}$
	• the y-value of the vertex ("*k*") can be calculated by finding the value of *f(h)*
	• the *a* value determines the direction of opening and the step pattern
		• Plot and label the x-intercepts, $(r, 0)$ and $(s, 0)$
		• Find and plot the vertex, (h, k), using:
		  ○ $h = \frac{r+s}{2}$
		  ○ $k = f(h)$
		• From the vertex, move horizontally (⇔) 1 unit, and vertically (⇕) $1^2(a)$ units (on either side); then, move horizontally 2 units, and vertically $2^2(a)$ units; [repeat with 3 units and $3^2(a)$ units, if necessary, to get accurate end behaviour]
		• Sketch and label the axis of symmetry, $x = h$

Ex. 3: Sketch the following quadratic functions on the grids below. State the *x*-intercepts, vertex (*h*, *k*), equation of the axis of symmetry (*x* = h), and range.

a) $f(x) = 2(x - 3)(x - 7)$

① $h = \frac{3+7}{2}$

$h = 5$

② $f(5) = 2(5-3)(5-7)$

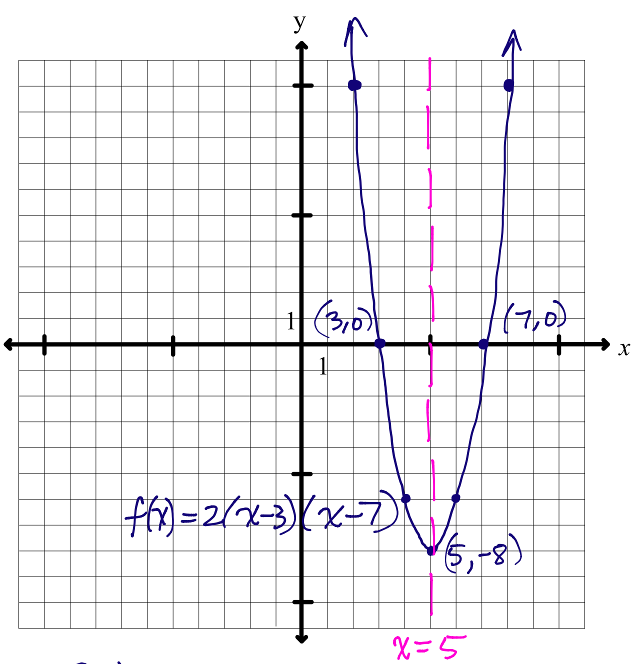
$= 2(2)(-2)$

$= -8$

x-int(s): $x = 3, 7$

Vertex: $(5, -8)$

Axis of symmetry: $x = 5$



R = $\{y \mid y \in \mathbb{R}, y \geq -8\}$

b) $f(x) = -\frac{1}{3}x(x - 4)$

① $h = \frac{0+4}{2}$

$h = 2$

② $f(2) = -\frac{1}{3}(2)(2-4)$

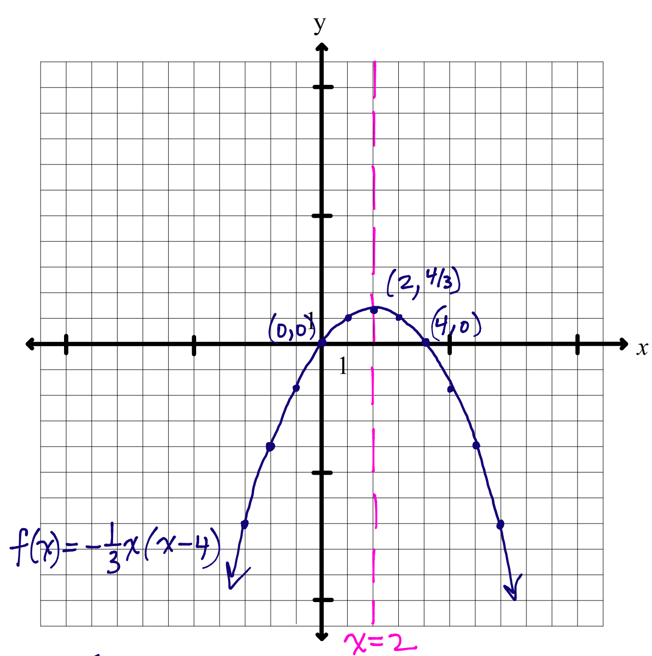
$= -\frac{1}{3}(2)(-2)$

$= \frac{4}{3}$

x-int(s): $x = 0, 4$

Vertex: $(2, 4/3)$

Axis of symmetry: $x = 2$



R = $\{y \mid y \in \mathbb{R}, y \leq 4/3\}$

D. Standard Form of a Quadratic Function

	Characteristics	**What To Do . . .**
Standard Form	$f(x) = ax^2 + bx + c$ ***or*** $y = ax^2 + bx + c$ • In this form, the only point evident is the y-intercept, (0, *c*) • the vertex (*h, k*) can be determined by completing the square • the *a* value determines the direction of opening and the step pattern	• Plot and label the y-intercept, (0, *c*) • Find, plot, and label the vertex, (*h, k*), by first completing the square • From the vertex, move horizontally (⇔) 1 unit, and vertically (⇕) $1^2(a)$ units (on either side); then, move horizontally 2 units, and vertically $2^2(a)$ units; [repeat with 3 units and $3^2(a)$ units, if necessary, to get accurate end behaviour] • Sketch and label the axis of symmetry, *x = h*

Ex. 4: Complete the square on the following quadratic functions to find the vertex; then, graph them on the grids below. Label the *y*-intercept, vertex (*h*, *k*), and equation of the axis of symmetry (*x* = h) on the graph. State the range.

a) $f(x) = x^2 - 4x + 3$

$$\left[\frac{(-4)}{2}\right]^2$$

$$f(x) = (x^2 - 4x + 4 - 4) + 3$$

$$f(x) = (x^2 - 4x + 4) - 4 + 3$$

$$f(x) = (x-2)^2 - 1$$

$\therefore$ vertex is $(2,-1)$

b) $f(x) = -3x^2 + 12x - 5$

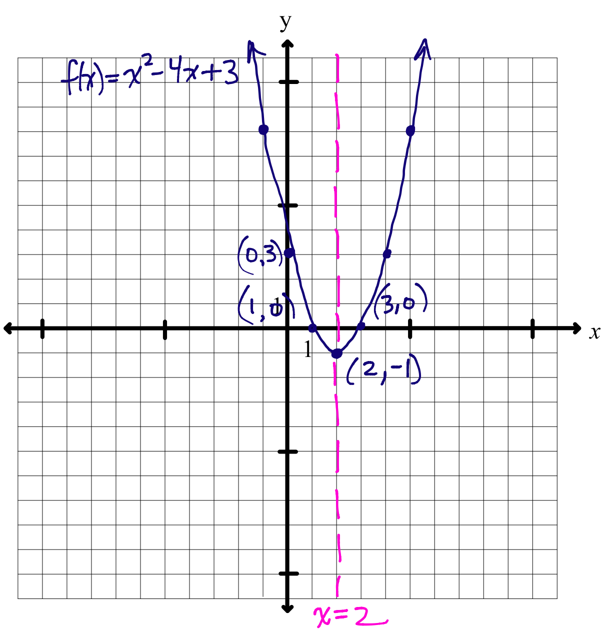
$$f(x) = -3(x^2 - 4x) - 5$$

$$f(x) = -3(x^2 - 4x + 4 - 4) - 5$$

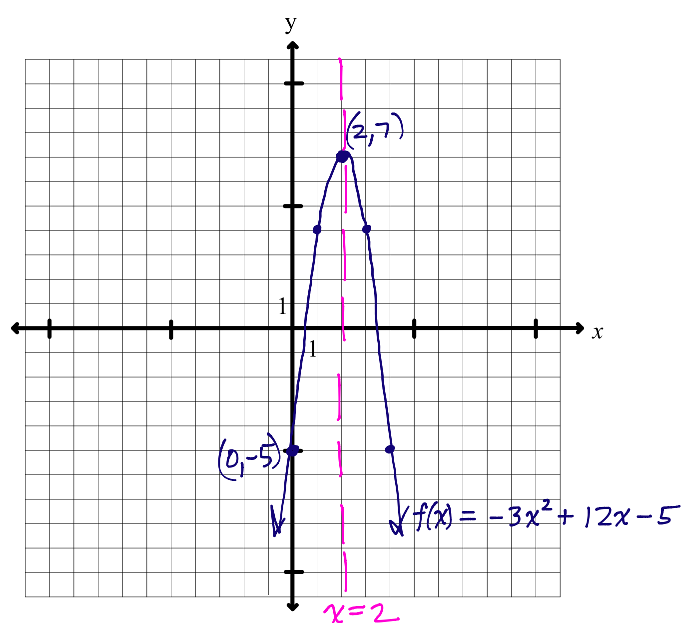
$$f(x) = -3(x^2 - 4x + 4) + 12 - 5$$

$$f(x) = -3(x-2)^2 + 7$$

$\therefore$ vertex is $(2,7)$



R = $\{y \mid y \in \mathbb{R},\ y \geq -1\}$



R = $\{y \mid y \in \mathbb{R},\ y \leq 7\}$

HW: WORKSHEET: "Graphing Quadratic Functions Given Any Form"