

Transformations of Functions

A **Transformation** is an operation that moves (or maps) a figure from an original position to a new position.

Transformations that we will consider are reflections, stretches, and translations on any base function $y = f(x)$:

$$y = a f[k(x - d)] + c$$

Diagram illustrating the components of the transformation equation $y = a f[k(x - d)] + c$:

- a : Vertical stretch \neq reflection
- f : any function of x (e.g., $y = x^2$)
- k : horizontal stretch \neq reflection
- $x - d$: horizontal translation
- d : horizontal translation
- c : vertical translation

Base Function	$y = x^2$	$y = \sqrt{x}$	$y = \frac{1}{x}$	$y = \sin x$	$y = (b)^x$
Transformed Function	$y = a(k(x - d))^2 + c$	$y = a\sqrt{k(x - d)} + c$	$y = a\frac{1}{k(x - d)} + c$	$y = a\sin k(x - d) + c$	$y = a(b)^{k(x - d)} + c$

Perform combinations of transformations in the following order:

1. Reflections on the Function $y = f(x)$:

Reflection	Form	Effect
Vertical	$y = -f(x)$ $(-a)$	Compared to $y = f(x)$, the graph of $y = -f(x)$ is a vertical reflection across the x-axis. The point (x, y) on $y = f(x)$ becomes the point $(x, -y)$ on $y = -f(x)$. (VR)
Horizontal	$y = f(-x)$ $(-k)$	Compared to $y = f(x)$, the graph of $y = f(-x)$ is a horizontal reflection across the y-axis. The point (x, y) on $y = f(x)$ becomes the point $(-x, y)$ on $y = f(-x)$. (HR)

2. Stretches on the Function $y = f(x)$:

Stretch	Form	Effect
Vertical	$y = a f(x)$	If $a > 1$, the graph is vertically expanded by a factor of a . (VE) If $0 < a < 1$, the graph is vertically compressed by a factor of a . (VC) The point (x, y) on $y = f(x)$ becomes the point (x, ay) on $y = af(x)$.
Horizontal	$y = f(kx)$	If $k > 1$, the graph is horizontally compressed by a factor of $\frac{1}{k}$. (HC) If $0 < k < 1$, the graph is horizontally expanded by a factor of $\frac{1}{k}$. (HE) The point (x, y) on $y = f(x)$ becomes the point $(\frac{1}{k}x, y)$ on $y = f(kx)$.

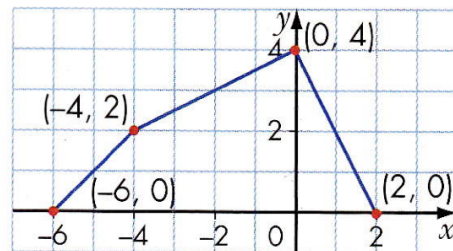
3. Translations on the Function $y = f(x)$:

Translation	Form	Effect
Vertical	$y = f(x) + c$	Compared to the graph of $y = f(x)$, the graph of $y = f(x) + c$ is a vertical translation of c units. (VT) When $c > 0$ the graph is vertically translated up c units . When $c < 0$ the graph is vertically translated down c units . The point (x, y) on $y = f(x)$ becomes the point $(x, y + c)$ on $y = f(x) + c$.
Horizontal*	$y = f(x - d)$	Compared to the graph of $y = f(x)$, the graph of $y = f(x - d)$ is a horizontal translation of d units. (HT) When $d > 0$ the graph is horizontally translated to the RIGHT d units . When $d < 0$ the graph is horizontally translated to the LEFT d units . The point (x, y) on $y = f(x)$ becomes the point $(x + d, y)$ on $y = f(x - d)$.

*If necessary, **factor the coefficient of the x-term** to identify the horizontal translation more easily.

A. Translations on $y = f(x)$

1. The graph of $y = f(x)$ is shown to the right. Sketch the following transformed functions on the grids below and state their domain and range:



a) $y = f(x) + 6$

Description of transformations on $y = f(x)$:

vertical translation

6 units up

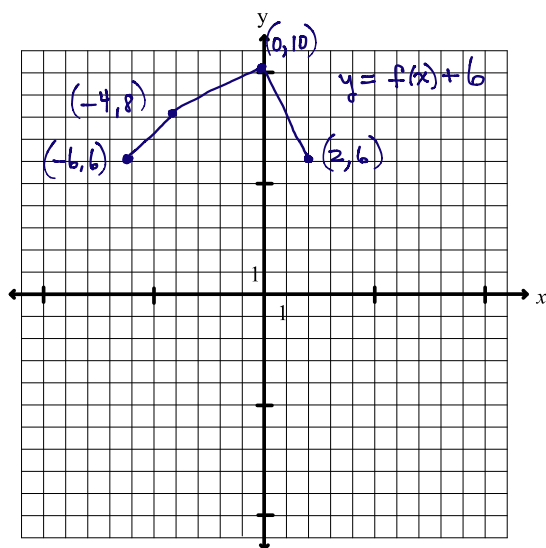
$(x, y) \rightarrow (x, y + 6)$

$(-6, 0) \rightarrow (-6, 6)$

$(-4, 2) \rightarrow (-4, 8)$

$(0, 4) \rightarrow (0, 10)$

$(2, 0) \rightarrow (2, 6)$



$D = \{x | x \in \mathbb{R}, -6 \leq x \leq 2\}$

$R = \{y | y \in \mathbb{R}, 6 \leq y \leq 10\}$

b) $y = f(x + 2)$

Description of transformations on $y = f(x)$:

horizontal translation

2 units left

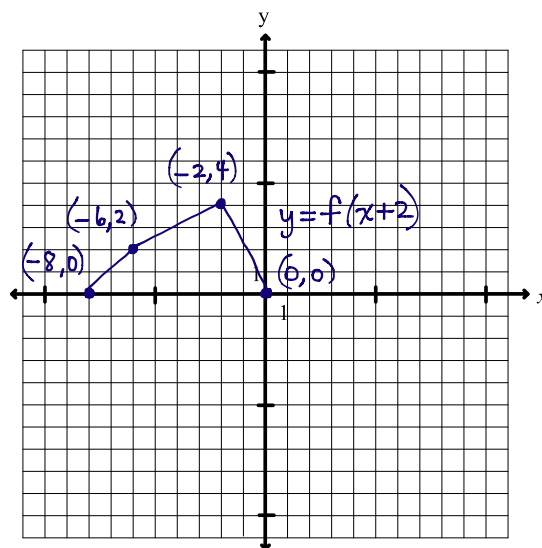
$(x, y) \rightarrow (x - 2, y)$

$(-6, 0) \rightarrow (-8, 0)$

$(-4, 2) \rightarrow (-6, 2)$

$(0, 4) \rightarrow (-2, 4)$

$(2, 0) \rightarrow (0, 0)$

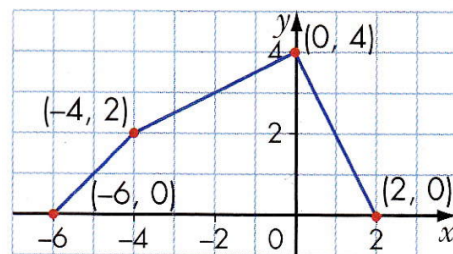


$D = \{x | x \in \mathbb{R}, -8 \leq x \leq 0\}$

$R = \{y | y \in \mathbb{R}, 0 \leq y \leq 4\}$

B. Stretches and Reflections on $y = f(x)$

1. The graph of $y = f(x)$ is shown to the right. Sketch the following transformed functions on the grids below and state their domain and range:



a) $y = -3f(x)$

Description of transformations on $y = f(x)$:

- vertical reflection across the x -axis
- vertical expansion by a factor of 3

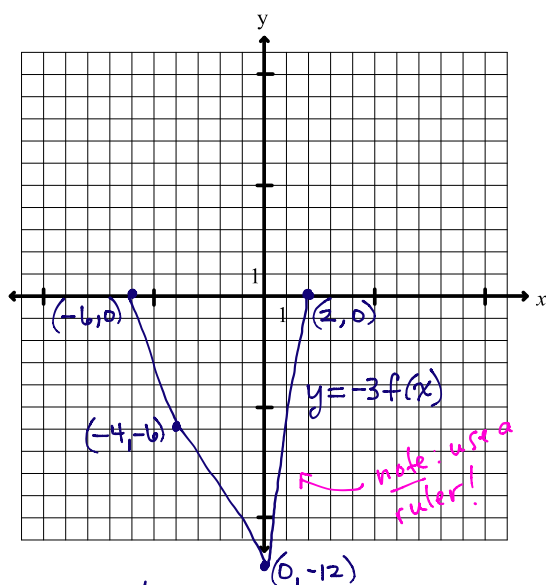
$(x, y) \rightarrow (x, -3y)$

$(-6, 0) \rightarrow (-6, 0)$

$(-4, 2) \rightarrow (-4, -6)$

$(0, 4) \rightarrow (0, -12)$

$(2, 0) \rightarrow (2, 0)$



$D = \{x \mid x \in \mathbb{R}, -6 \leq x \leq 2\}$

$R = \{y \mid y \in \mathbb{R}, -12 \leq y \leq 0\}$

b) $y = f(-2x)$

Description of transformations on $y = f(x)$:

- horizontal reflection across the y -axis
- horizontal compression by a factor of $\frac{1}{2}$

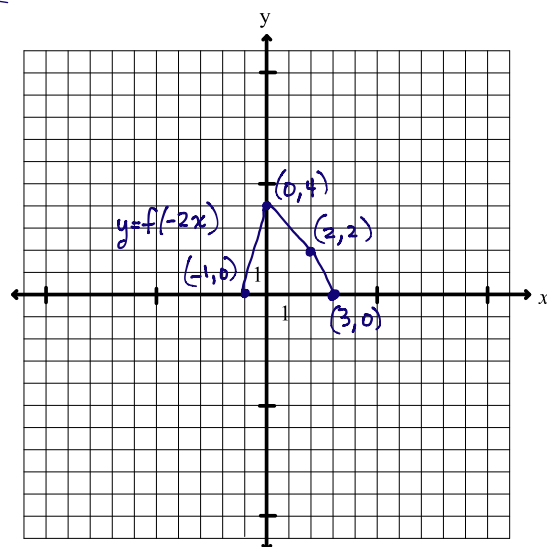
$(x, y) \rightarrow (-\frac{1}{2}x, y)$

$(-6, 0) \rightarrow (3, 0)$

$(-4, 2) \rightarrow (2, 2)$

$(0, 4) \rightarrow (0, 4)$

$(2, 0) \rightarrow (-1, 0)$



$D = \{x \mid x \in \mathbb{R}, -1 \leq x \leq 3\}$

$R = \{y \mid y \in \mathbb{R}, 0 \leq y \leq 4\}$

C. Combinations of Transformations

When the function $y = f(x)$ has been transformed to $y = af[k(x-d)] + c$, each point (x, y) on the base function becomes point $\left(\frac{1}{k}x - d, ay + c\right)$ on the transformed function.

- Describe how the graphs of the following functions can be obtained from the graph of the function $y = f(x)$ and write the point-by-point transformation.

a. $y = f(3x) + 8$

Description of transformations on $y = f(x)$:

HC by a factor of $\frac{1}{3}$

VT 8 units up

$(x, y) \rightarrow \left(\frac{1}{3}x, y + 8\right)$

b. $y = -f\left[\frac{1}{2}(x-4)\right]$

Description of transformations on $y = f(x)$:

VR across the x-axis

HE by a factor of 2

HT 4 units right

$(x, y) \rightarrow (2x + 4, -y)$

c. $y = 4f(-x-2) - 5$

$y = 4f[-(x+2)] - 5$

Description of transformations on $y = f(x)$:

VE by a factor of 4

HR across the y-axis

HT 2 units left

VT 5 units down

$(x, y) \rightarrow (-x - 2, 4y - 5)$

d. $y = 2f(6-3x) - 12$

$y = 2f[-3x+6] - 12$

$y = 2f[-3(x-2)] - 12$

Description of transformations on $y = f(x)$:

VE by a factor of 2

HR across the y-axis

HC by a factor of $\frac{1}{3}$

VT 12 units down

$(x, y) \rightarrow \left(-\frac{1}{3}x + 2, 2y - 12\right)$

- The function $y = f(x)$ has been transformed to $y = af[k(x-d)] + c$. Determine the values of a , k , d , and c for each of the following transformations:

- a vertical compression by a factor of $\frac{1}{2}$ and a horizontal translation 7 units left:

$y = \frac{1}{2}f(x+7)$

- a reflection across the x-axis, a vertical expansion by a factor of 3, a horizontal compression by a factor of $\frac{1}{4}$, and a vertical translation 10 units up:

$y = -3f[4x] + 10$

Transformations of Quadratic Functions

A. Transforming Quadratic Functions

Quadratic functions of the form $y = x^2$ can also be transformed according to: $y = a[k(x-d)]^2 + c$.

Ex. 1: Sketch the following transformed functions on the grids below. List the transformations in order on the base function $y = x^2$. State the vertex (d, c), equation of the axis of symmetry ($x = d$), domain, and range.

a) $y = (x+3)^2 - 2$

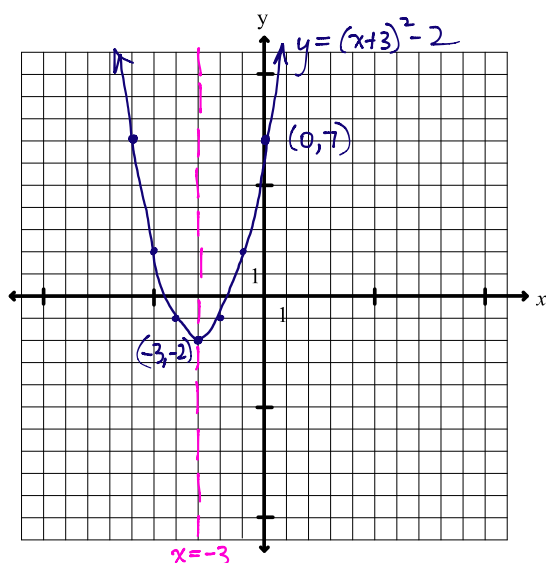
Description:

- HT 3 units left
- VT 2 units down

$(x, y) \rightarrow (x-3, y-2)$

Vertex: $(-3, -2)$

Axis of symmetry: $x = -3$



D = $\{x/x \in \mathbb{R}\}$

R = $\{y/y \in \mathbb{R}, y \geq -2\}$

b) $y = -2(x-4)^2 + 5$

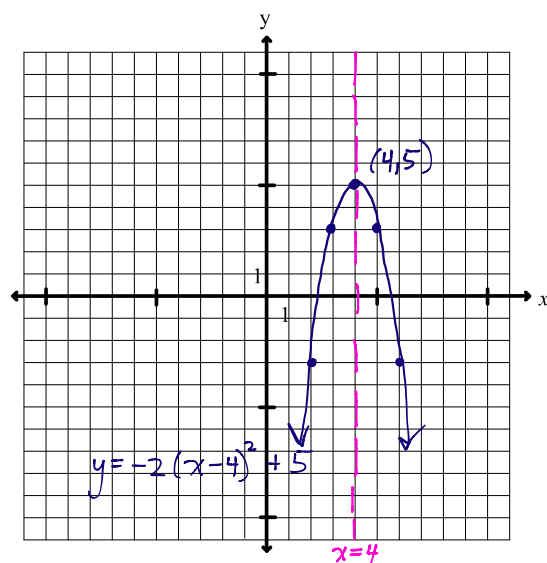
Description:

- VR across the x-axis
- VE by a factor of 2
- HT 4 units right
- VT 5 units up

$(x, y) \rightarrow (x+4, -2y+5)$

Vertex: $(4, 5)$

Axis of symmetry: $x = 4$



D = $\{x/x \in \mathbb{R}\}$

R = $\{y/y \in \mathbb{R}, y \leq 5\}$

Ex. 2: Sketch the following transformed functions on the grids below. List the transformations in order on the base function $y = x^2$. State the vertex (d, c), equation of the axis of symmetry ($x = d$), domain, and range.

a) $y = \left(\frac{1}{3}x + 1\right)^2$
 factor 'k'
 $y = \left[\frac{1}{3}(x+3)\right]^2$

b) $y = -(2x - 10)^2 + 6$

factor 'k'
 $y = -[2(x-5)]^2 + 6$
 $y = -[2^2(x-5)^2] + 6$
 $y = -4(x-5)^2 + 6$
 * You can "promote" the k-value by squaring it!

Description:

- HE by a factor of 3
 - HT 3 units left

$(x, y) \rightarrow \left(\frac{3x-3}{3}, y\right)$

Vertex: $(-3, 0)$

Axis of symmetry: $x = -3$

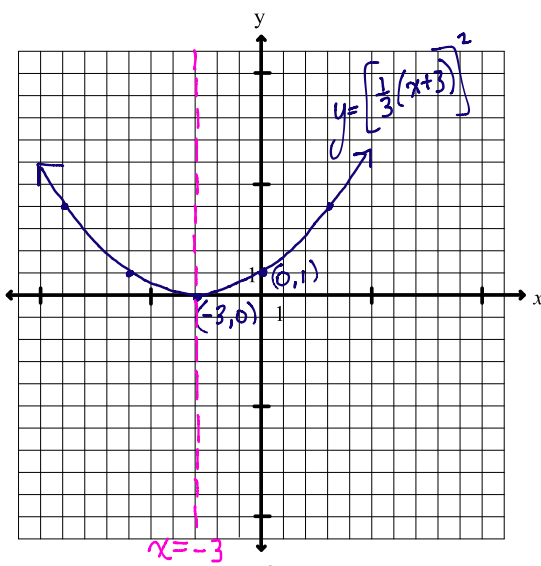
Description:

- VR across the x-axis
 - VE by a factor of 4
 - HT 5 units right
 - VT 6 units up

$(x, y) \rightarrow (x+5, -4y+6)$

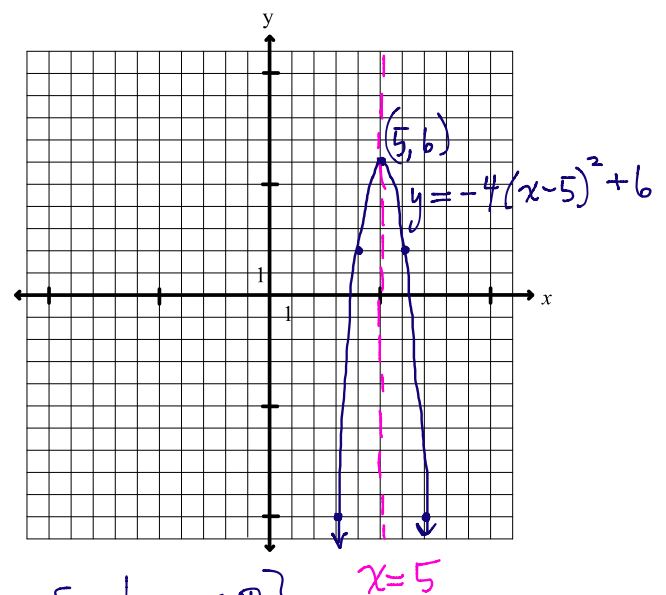
Vertex: $(5, 6)$

Axis of symmetry: $x = 5$



D = $\{x | x \in \mathbb{R}\}$

R = $\{y | y \in \mathbb{R}, y \geq 0\}$



D = $\{x | x \in \mathbb{R}\}$

R = $\{y | y \in \mathbb{R}, y \leq 6\}$

Ex. 3: i) Graph $f(x) = -x^2 + 6x$ on the grid provided.

Factor to find x-intercepts:

$$f(x) = -x(x-6)$$

∴ the zeros are: 0 and 6

Use zeros to find vertex:

$$i) h = \frac{0+6}{2}$$

$$h = 3$$

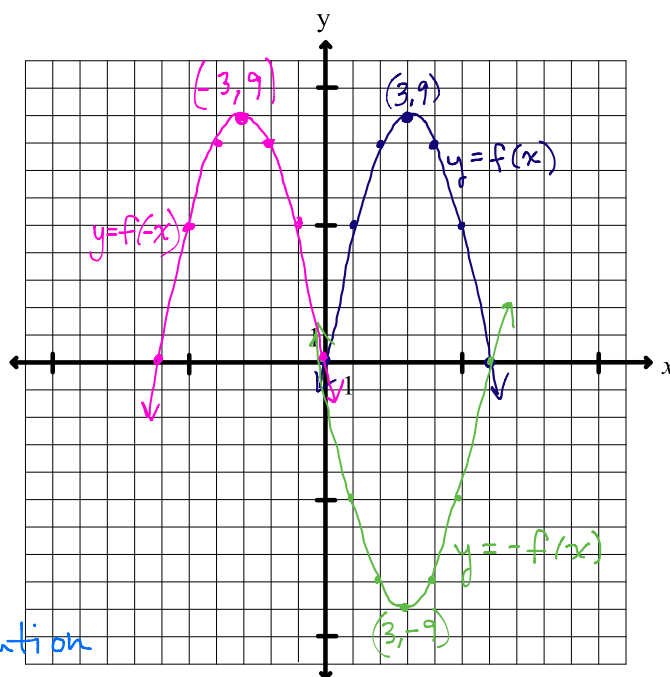
$$ii) \text{ find } f(3):$$

$$f(3) = -(3)^2 + 6(3)$$

$$= -9 + 18$$

$$= 9$$

∴ the vertex is (3,9)
and the a-value is -1.



ii) Write equations for a) $y = -\boxed{f(x)}$ and b) $y = \boxed{f(-x)}$.
Graph these functions on the grid provided.

a) $y = -f(x)$

$$y = -[-x^2 + 6x]$$

$$y = x^2 - 6x$$

b) $y = f(-x)$

$$y = -(-x)^2 + 6(-x)$$

$$y = -x^2 - 6x$$

Ex. 4: Given $f(x) = x^2$, a) sketch a graph of $y = -\overbrace{f(2x-10)}^{\text{factor "k"}} + 6$ and b) write an equation for this transformed parabola.

$$y = -f[2(x-5)] + 6$$

Transformations on $y = f(x)$ are:

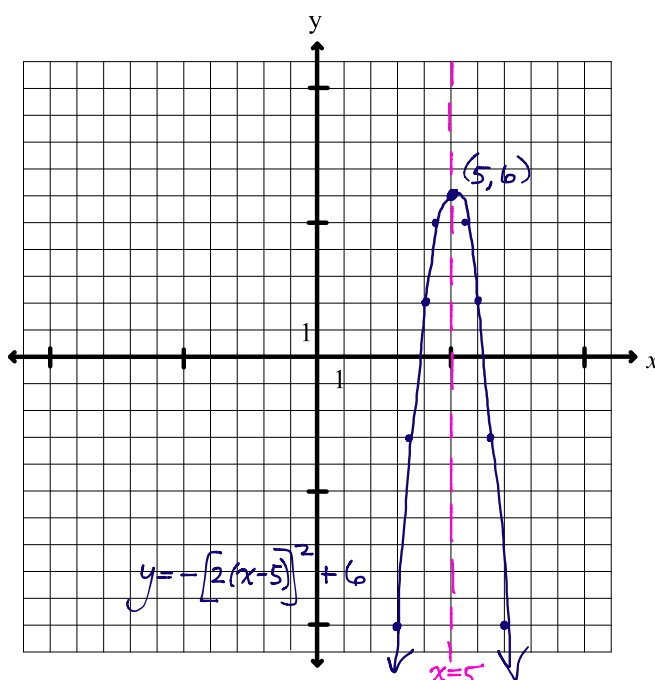
- a) -VR across the x-axis
- HC by a factor of $\frac{1}{2}$
- HT 5 units right
- VT 6 units up

∴ vertex is at (5,6) &
the parabola opens
down

$$b) f(x) = x^2 \rightarrow y = -f[2(x-5)] + 6$$

$$a = -1, k = 2, d = 5, c = 6 \text{ and } y = a[k(x-d)]^2 + c$$

$$\therefore \text{the equation is } y = -[2(x-5)]^2 + 6$$



Transformations of Square Root Functions

A. Characteristics of the Square Root Function $f(x) = \sqrt{x}$

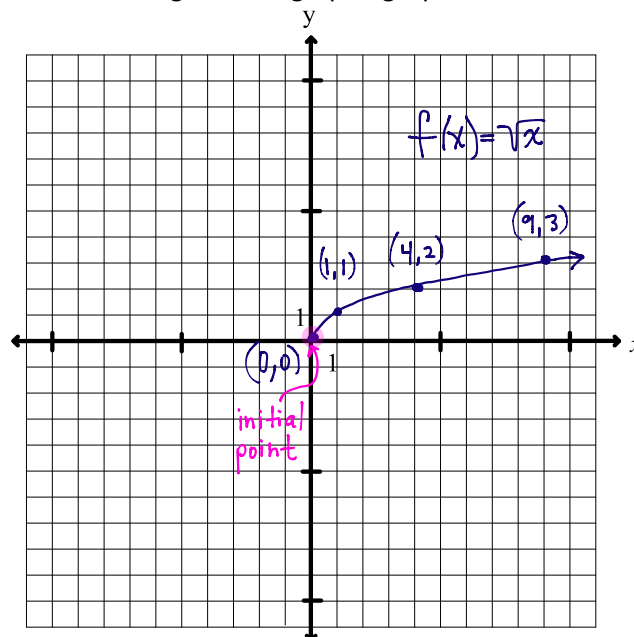
1. Graph the function $f(x) = \sqrt{x}$ on the grid below. State the domain and range. When graphing square root functions, label the following on the graph:

- the equation of the function
- the “vertex” of the half-parabola
↳ initial point
- clearly mark each key point with a dot
- label any intercepts
- draw an arrow showing the direction of opening

$$f(x) = \sqrt{x}$$

x	y
0	0
1	1
4	2
9	3

$$\begin{aligned} f(0) &= \sqrt{0} \\ f(1) &= \sqrt{1} \\ f(4) &= \sqrt{4} \\ f(9) &= \sqrt{9} \end{aligned}$$



$$D = \{x \mid x \in \mathbb{R}, x \geq 0\}$$

$$R = \{y \mid y \in \mathbb{R}, y \geq 0\}$$

The graph of the square root function is *half a parabola* opening to the side. Basic properties of the square root function $f(x) = \sqrt{x}$ are as follows:

- the expression under the radical sign is the **radicand** and takes on all *non-negative* real values
- the **domain** is $D = \{x \mid x \in \mathbb{R}, x \geq 0\}$
- the **range** is $R = \{y \mid y \in \mathbb{R}, y \geq 0\}$

2. Find the domain of the square root function by inspecting the **radicand**. It should only take on *non-negative* values.

a) $y = \sqrt{x+3}$ → value under the radical must be ≥ 0

$$\begin{aligned} x+3 &\geq 0 \\ x &\geq -3 \end{aligned}$$

$$\therefore D = \{x \mid x \in \mathbb{R}, x \geq -3\}$$

b) $y = \sqrt{3x+4}$

$$\begin{aligned} 3x+4 &\geq 0 \\ 3x &\geq -4 \\ x &\geq -\frac{4}{3} \end{aligned}$$

$$\therefore D = \{x \mid x \in \mathbb{R}, x \geq -\frac{4}{3}\}$$

↳ no real numbers can result when square-rooting a negative value

B. Transforming Square Root Functions

Square root functions of the form $y = \sqrt{x}$ can also be transformed according to: $y = a\sqrt{k(x-d)} + c$.

1. Sketch the following transformed functions on the grids below. List the transformations in order on the base function $y = \sqrt{x}$. State the domain and range.

a) $y + 3 = \sqrt{2 - 2x}$

rearrange
rearrange
factor 'k'

$$y = \sqrt{-2x + 2} - 3$$

$$y = \sqrt{-2(x-1)} - 3$$

Description:

- HR across y-axis
- HC by a factor of $\frac{1}{2}$
- HT 1 unit right
- VT 3 units down

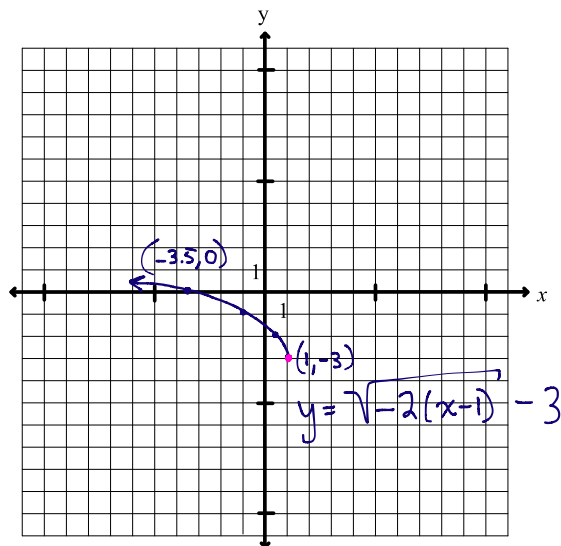
$$(x, y) \rightarrow \left(-\frac{1}{2}x + 1, y - 3\right)$$

* $(0, 0) \rightarrow (1, -3)$ * initial point

$$(1, 1) \rightarrow (0.5, -2)$$

$$(4, 2) \rightarrow (-1, -1)$$

$$(9, 3) \rightarrow (-3.5, 0)$$



$$D = \{x | x \in \mathbb{R}, x \leq 1\}$$

$$R = \{y | y \in \mathbb{R}, y \geq -3\}$$

b) $y = -3\sqrt{\frac{1}{2}x + 1} + 4$

factor 'k'

$$y = -3\sqrt{\frac{1}{2}(x+2)} + 4$$

Description:

- VR across the x-axis
- VE by a factor of 3
- HE by a factor of 2
- HT 2 units left
- VT 4 units up

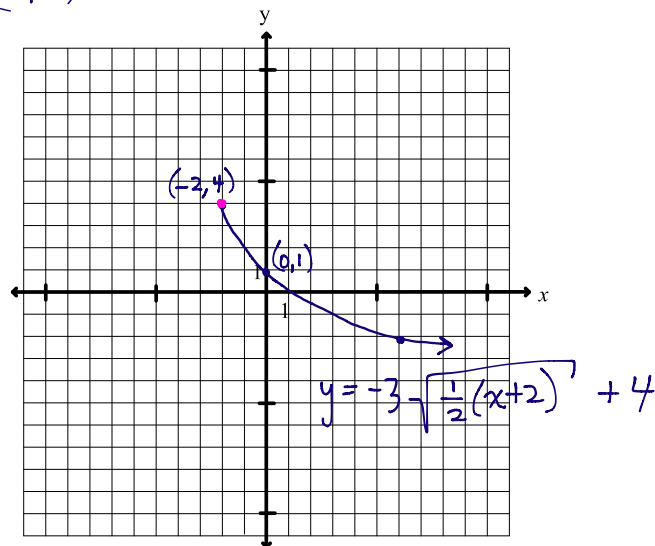
$$(x, y) \rightarrow (2x - 2, -3y + 4)$$

* $(0, 0) \rightarrow (-2, 4)$ * initial point

$$(1, 1) \rightarrow (0, 1)$$

$$(4, 2) \rightarrow (6, -2)$$

$$(9, 3) \rightarrow (16, -5)$$



$$D = \{x | x \in \mathbb{R}, x \geq -2\}$$

$$R = \{y | y \in \mathbb{R}, y \leq 4\}$$

2. Given $f(x) = \sqrt{x+5} - 4$, write equations for a) $y = -f(x)$ and b) $y = f(-x)$. Graph all three functions.

a) $y = -f(x)$

$$y = -[\sqrt{x+5} - 4]$$

$$y = -\sqrt{x+5} + 4$$

initial pt: $(-5, 4)$

b) $y = f(-x)$

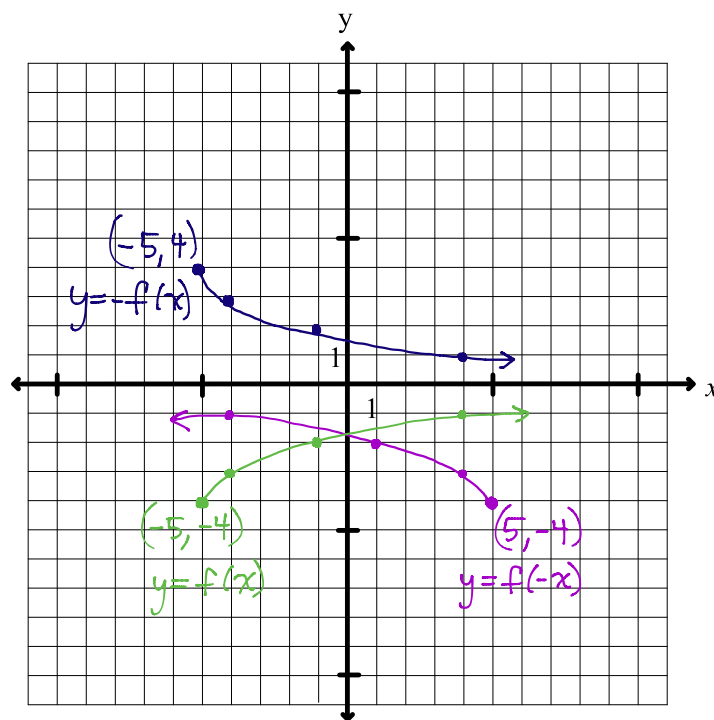
$$y = \sqrt{-x+5} - 4$$

$$y = \sqrt{-x+5} - 4$$

factor 'k'

$$y = \sqrt{-(x-5)} - 4$$

initial pt: $(5, -4)$



3. Write the resulting equation when the base function $y = \sqrt{x}$ is vertically reflected across the x-axis, horizontally compressed by a factor of $\frac{1}{2}$, horizontally translated 5 units right, and vertically translated 7 units up.

$$f(x) = -\sqrt{2(x-5)} + 7$$

Sketch:
 $(-9, -2)$

4. Given the function $f(x) = \sqrt{x+9} - 2$ determine the following:

a) Domain = $\{x | x \in \mathbb{R}, x \geq -9\}$

$$x+9 \geq 0 \quad \therefore x \geq -9$$

Range = $\{y | y \in \mathbb{R}, y \geq -2\}$

b) Find x , if $f(x) = 4$

$$4 = \sqrt{x+9} - 2$$

$$4 + 2 = \sqrt{x+9}$$

$$(6)^2 = (\sqrt{x+9})^2$$

$$36 = x+9$$

$$x = 27$$

$$\therefore f(27) = 4$$

c) Simplify $f(4a^2 - 9)$

$$f(4a^2 - 9) = \sqrt{(4a^2 - 9) + 9} - 2$$

$$= \sqrt{4a^2} - 2$$

$$= 2a - 2$$

$$\therefore f(4a^2 - 9) = 2a - 2$$