Transformations of Reciprocal Functions

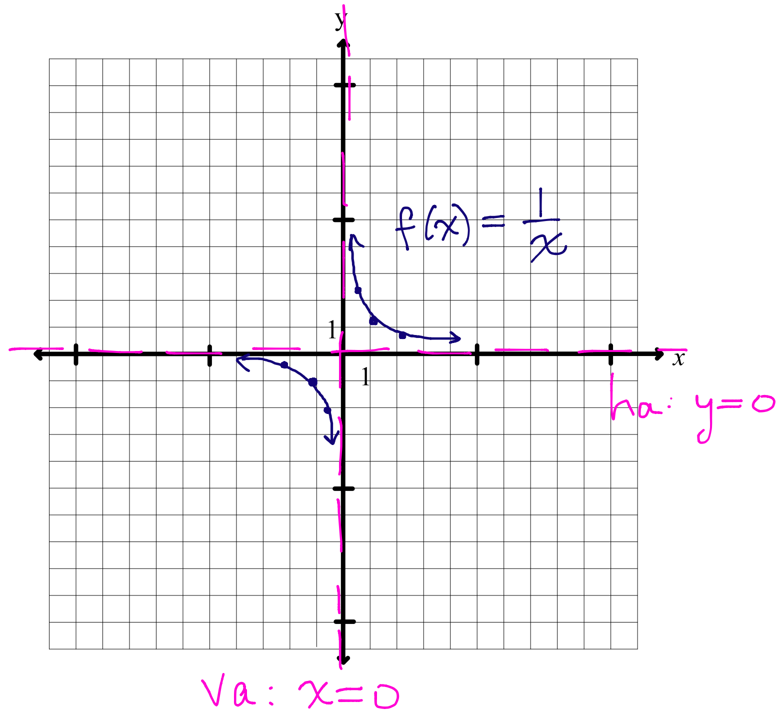
A. Characteristics of the Reciprocal Function *f(x) = 1/x*

1. Graph the function $f(x)=\frac{1}{x}$ on the grid below. State the domain and range.

f(x) = 1/x

x	y
-2	$-1/2$
-1	-1
-1/2	-2
0	undefined
1/2	2
1	1
2	$1/2$

$f(0) = 1/0$



D = $\{x \mid x \in \mathbb{R}, x \neq 0\}$

R = $\{y \mid y \in \mathbb{R}, y \neq 0\}$

The graph of the reciprocal function is *a hyperbola* found in quadrant I and quadrant III of the Cartesian plane. Basic properties of the reciprocal function ***f(x) = 1/x*** are as follows:

- the value 1/0 is **undefined**, so the expression in the *denominator cannot equal zero*
- any value of *x* that would yield 1/0 is a restriction on the function, and is visualized on the graph as a **vertical asymptote** (note: *in this course,* an ***asymptote*** is a line representing a value that the function will approach but *never reach*)
- if the numerator is a constant, no input value of *x* will result in an output value of zero, and is visualized on the graph as a **horizontal asymptote**
- the **domain** is any real number *except* the singularity: D = {x| x ∈ R, x ≠ 0} → v.a.: x = 0 ← d value
- the **range** is any real number *except* zero: R = {y| y ∈ R, y ≠ 0} → h.a.: y = 0 ← c value

2. When graphing reciprocal functions of the form $y=\frac{a}{k(x-d)}+c$, label the following on the graph:

- the equation of the function
- the equation of the vertical asymptote, *x = d*
- the equation of the horizontal asymptote, *y = c*
- clearly mark each key point with a dot
- label any intercepts

B. Transforming Reciprocal Functions

Reciprocal functions of the form $y = \frac{1}{x}$ can also be transformed according to: $y = \frac{a}{k(x-d)} + c$.

1. Sketch the following transformed functions on the grids below. List the transformations in order on the base function $y = \frac{1}{x}$. State the domain and range.

a) $y + 5 = -\frac{1}{x+2}$ rearrange

$y = -\left(\frac{1}{x+2}\right) - 5$

Description:

- VR across the x-axis
- HT 2 units left
- VT 5 units down

(x, y) → ($x - 2$, $-y - 5$)

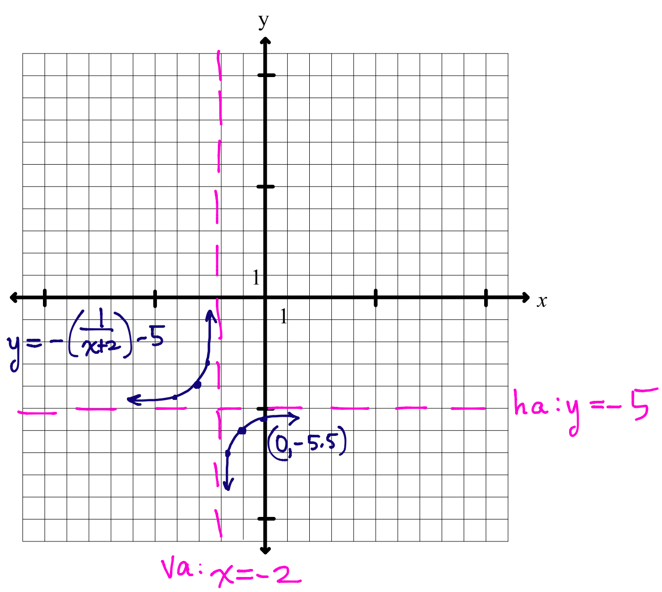
b) $y = \frac{2}{-x+3} + 4$ factor 'k'

$y = \frac{2}{-(x-3)} + 4$

Description:

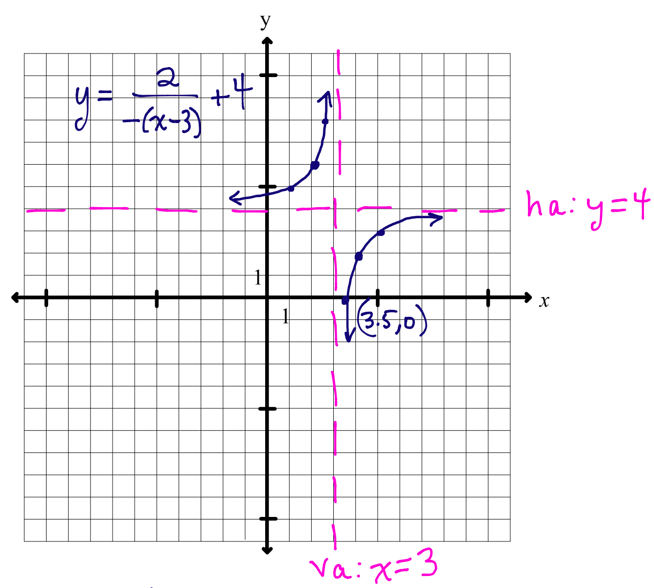
- VE by a factor of 2
- HR across the y-axis
- HT 3 units right
- VT 4 units up

(x, y) → ($-x + 3$, $2y + 4$)



D = $\{x \mid x \in \mathbb{R}, x \neq -2\}$

R = $\{y \mid y \in \mathbb{R}, y \neq -5\}$



D = $\{x \mid x \in \mathbb{R}, x \neq 3\}$

R = $\{y \mid y \in \mathbb{R}, y \neq 4\}$

HW: "WORKSHEET: Transformations of Reciprocal Functions"

Functions and Their Inverses

A. Introduction

We have seen reflections across the x-axis (i.e. across the line y = 0) and the y-axis (i.e. across the line x = 0), but the **inverse of a function** can be visualized as a reflection across the line *y = x.* To achieve this reflection, the coordinates for each point in the function (x, y) are inverted to create corresponding points (y, x).

The inverse of a function, $f(x)$, is denoted by $f^{-1}(x)$.

If we look at a table of values for a function and its inverse function, we see a pattern:

f(x) = x + 3	
x	***f(x)***
2	5
3	6
4	7
5	8

$f^{-1}(x) = x - 3$	
x	$f^{-1}(x)$
5	2
6	3
7	4
8	5

Notice that the input values for the first function become the output values for the second function. The inverse function "undoes" what the original function did! The **domain** for the original function becomes the range of the inverse, and the **range** of the original function becomes the domain of the inverse.

B. Finding the Inverse

There are 3 ways to find the inverse of a function: *1) interchanging* the coordinates of each point; *2) graphing* the function and its inverse; and *3)* finding the inverse *algebraically*.

I. Interchanging Coordinates

1. Given that $g = \{(-2, -8), (0, -2), (3, 4), (4, 7)\}$

a) $g^{-1} = \{(-8,-2), (-2,0), (4,3), (7,4)\}$

b) State the domain and range of g and g^{-1}.

$D_g = \{-2, 0, 3, 4\}$

$R_g = \{-8, -2, 4, 7\}$

$D_{g^{-1}} = \{-8, -2, 4, 7\}$

$R_{g^{-1}} = \{-2, 0, 3, 4\}$

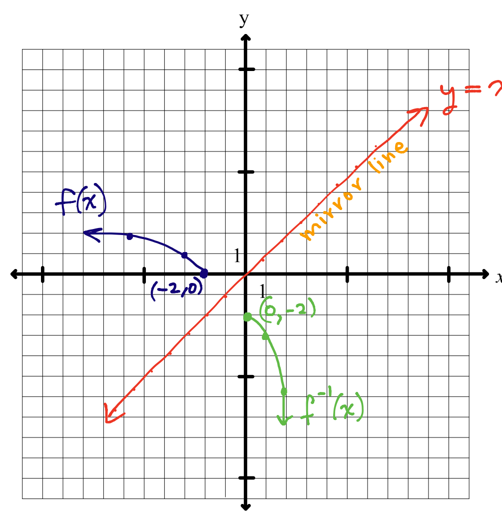
c) Are g and g^{-1} both functions? Explain.

Yes → no x-value is repeated ∴ they are both functions

II. Graphing the Inverse of a Function

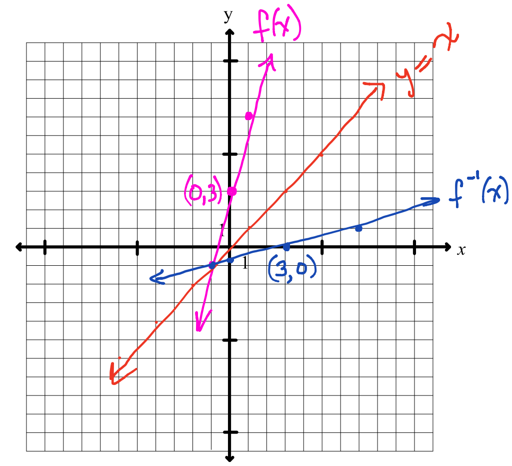
- Draw the line $y = x$
- Determine key points on *f(x)*
- Interchange *(x, y)* for *(y, x)*
- Plot the corresponding points as the inverse

1. Graph $f(x) = \sqrt{-(x+2)}$, and sketch its inverse.



III. Finding the Inverse of a Function **Algebraically**

- Replace $f(x)$ with y
- Interchange x and y in the equation
- Rearrange to isolate the 'new' y
- State whether or not the inverse is a function



1. Find the inverse of $f(x) = 4x + 3$. Graph both relations.

$y = 4x + 3$

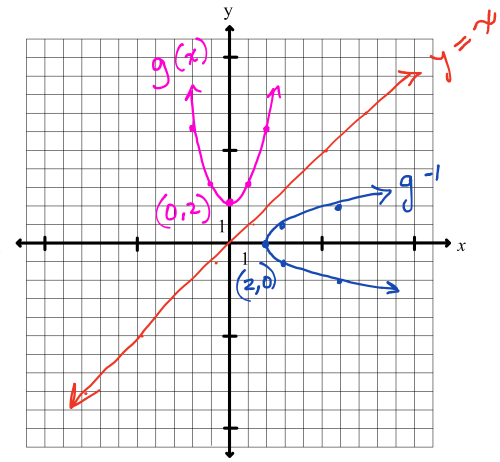
f^{-1}: $x = 4y + 3$

$x - 3 = 4y$

$\frac{x-3}{4} = y$

$y = \frac{1}{4}x - \frac{3}{4}$ } line with a slope of $\frac{1}{4}$

$\boxed{\therefore f^{-1}(x) = \frac{1}{4}x - \frac{3}{4}}$



2. a) Find the inverse of $g(x) = x^2 + 2$. Graph both relations.

$y = x^2 + 2$

f^{-1}: $x = y^2 + 2$

$x - 2 = y^2$

$\pm\sqrt{x-2} = \sqrt{y^2}$

$y = \pm\sqrt{x-2}$ → square root function and its vertical reflection ∴ NOT a function

$\boxed{\therefore g^{-1} = \pm\sqrt{x-2}}$

b) Is the inverse a function? no!

c) State the domain and range of g^{-1}.

$D_{g^{-1}} = \{x \mid x \in \mathbb{R},\ x \geq 2\}$

$R_{g^{-1}} = \{y \mid y \in \mathbb{R}\}$

d) Restrict the domain of $g(x)$ so that its inverse is a function.

$D_g = \{x \mid x \in \mathbb{R},\ x \geq 0\}$

OR $\{x \mid x \in \mathbb{R},\ x \leq 0\}$

3. Find the inverse of the following functions algebraically. For a) and b) use the range of the function to restrict the domain of the inverse, if necessary.

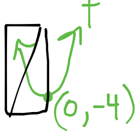
a) $f(x) = \sqrt{x+4}$ (-4,0) f(x)

$y = \sqrt{x+4}$

f^{-1}: $(x)^2 = (\sqrt{y+4})^2$

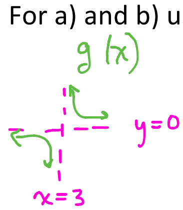
$x^2 = y + 4$

$x^2 - 4 = y$



$\boxed{\begin{array}{l} \therefore f^{-1}(x) = x^2 - 4 \\ D_{f^{-1}} = \{x \mid x \in \mathbb{R},\ x \geq 0\} \end{array}}$

b) $g(x) = \dfrac{1}{x-3}$



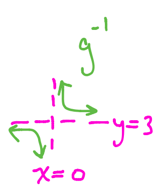
$y = \frac{1}{x-3}$

g^{-1}: $\frac{x}{1} = \frac{1}{y-3}$

$(y-3)x = 1$

$y - 3 = \frac{1}{x}$

$y = \frac{1}{x} + 3$



$\boxed{\therefore g^{-1}(x) = \frac{1}{x} + 3}$

c) $h(x) = \dfrac{2x+3}{x-1}$

$y = \frac{2x+3}{x-1}$

h^{-1}: $x = \frac{(2y+3)}{(y-1)}$

$(y-1)x = (2y+3)$

$xy - x = 2y + 3$

$xy - 2y = x + 3$

$y(x-2) = x + 3$

$y = \frac{x+3}{x-2}$

$\boxed{\therefore h^{-1}(x) = \frac{x+3}{x-2},\ x \neq 2}$

HW: A) p. 215-219 #2, 3bdg, 10(iv)abc & (v)abc, 11, 15ac, 16(iii)abc & (viii)abc, 30; B) Find the inverse of: $y = \dfrac{x+5}{x-2},\ x \neq 2$

Unit 1 Review

1. Complete the ***point-by-point transformations*** from the base function onto the following transformed functions:

a) $y = -3(2x-6)^2 - 1$

$\underbrace{}_{\text{factor 'k'}}$

$y = -3\left[2(x-3)\right]^2 - 1$

OR

$y = -3(2)^2(x-3)^2 - 1$

$y = -3(4)(x-3)^2 - 1$

$y = -12(x-3)^2 - 1$

$(x, y) \rightarrow (\frac{1}{2}x + 3, -3y - 1)$

OR $(x+3, -12y-1)$

b) $y = \dfrac{3}{2x-10} + 4$

factor 'k'

$y = \dfrac{3}{2(x-5)} + 4$

$(x, y) \rightarrow (\frac{1}{2}x + 5, 3y + 4)$

c) $y = 2\sqrt{-\frac{1}{3}x + 1} - 8$

factor 'k'

$y = 2\sqrt{-\frac{1}{3}(x-3)} - 8$

$(x, y) \rightarrow (-3x + 3, 2y - 8)$

2. Find the inverse of the following functions. For b) and c) use the range of the function to find the domain of the inverse.

a) $f(x) = 3(x+2)^2 - 1$

$y = 3(x+2)^2 - 1$

f^{-1}: $x = 3(y+2)^2 - 1$

$x + 1 = 3(y+2)^2$

$\pm\sqrt{\dfrac{(x+1)}{3}} = \sqrt{(y+2)^2}$

$\pm\sqrt{\frac{1}{3}(x+1)} = y + 2$

$y = \pm\sqrt{\frac{1}{3}(x+1)} - 2$

$\boxed{\therefore f^{-1} = \pm\sqrt{\frac{1}{3}(x+1)} - 2}$

note: f^{-1} is not a function

b) $f(x) = \sqrt{2x} + 1$

$(0,1)$ → $f(x)$

$y = \sqrt{2x} + 1$

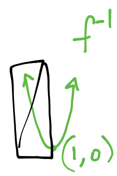
f^{-1}: $x = \sqrt{2y} + 1$

$(x-1)^2 = (\sqrt{2y})^2$

$(x-1)^2 = 2y$

$\dfrac{(x-1)^2}{2} = y$

$y = \frac{1}{2}(x-1)^2$



$\boxed{\begin{array}{l} \therefore f^{-1}(x) = \frac{1}{2}(x-1)^2 \\ D_{f^{-1}} = \{x \mid x \in R, x \geq 1\} \end{array}}$

c) $f(x) = \dfrac{2}{x-5} + 4$

$y = \dfrac{2}{x-5} + 4$

f^{-1}: $x = \dfrac{2}{y-5} + 4$

$(x-4) = \dfrac{2}{(y-5)}$

$(y-5)(x-4) = 2$

$y - 5 = \dfrac{2}{x-4}$

$y = \dfrac{2}{x-4} + 5$

$\boxed{\begin{array}{l} \therefore f^{-1}(x) = \dfrac{2}{x-4} + 5, \\ x \neq 4 \end{array}}$

3. Redo questions d., f., and i. from "WORKSHEET: Transformations of Reciprocal Functions".

4. Complete the following textbook questions:

Review: p. 246-253 #2, 4d, 6c, 23ab, 24a, 26a, 28, 29bdef, 32, ~~37~~, 41f, 42d, 43b, 44

Chapter Test: p. 254-256 #1ab, 2ab, 3ch, 6cd, 8b, 10, 11, 12a

Answers:

1. a) $(x, y) \rightarrow (x + 3, -12y - 1)$

   b) $(x, y) \rightarrow (\frac{1}{2}x + 5, 3y + 4)$

   c) $(x, y) \rightarrow (-3x + 3, 2y - 8)$

2. a) $f^{-1}(x) = \pm\sqrt{\frac{1}{3}(x+1)} - 2$; not a function

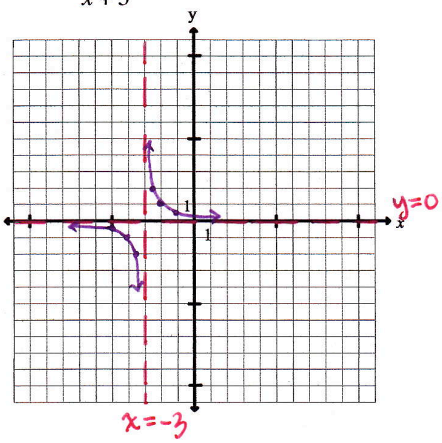
   b) $f^{-1}(x) = \frac{1}{2}(x-1)^2$; $D_{f^{-1}} = \{x \mid x \in R, x \geq 1\}$

   c) $f^{-1}(x) = \dfrac{2}{x-4} + 5$; $D_{f^{-1}} = \{x \mid x \in R, x \neq 4\}$

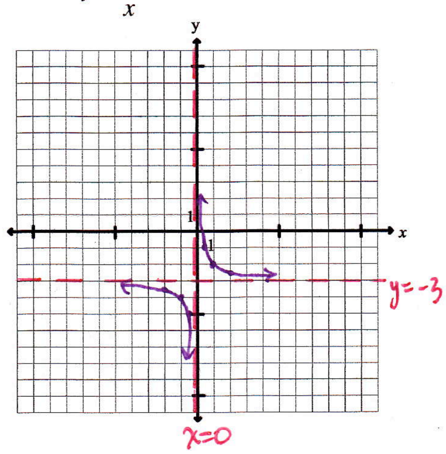
WORKSHEET: Transformations of Reciprocal Functions

Sketch a **graph** for each function below and state the **domain**, **range**, **vertical asymptote**, and **horizontal asymptote**. ***Show your work on a separate sheet of paper.***

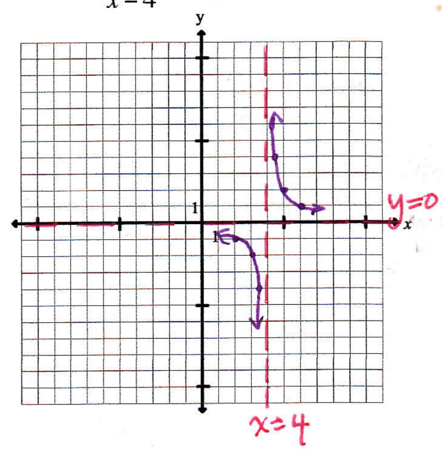
a. $y = \dfrac{1}{x+3}$



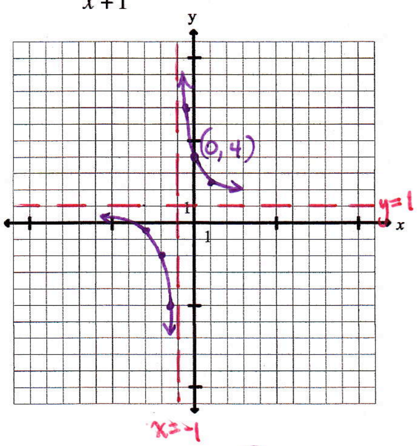
b. $y = \dfrac{1}{x} - 3$



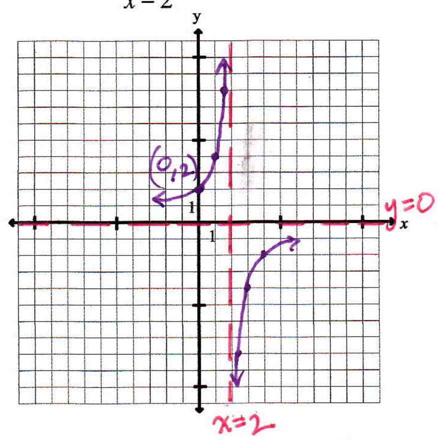
c. $y = \dfrac{2}{x-4}$



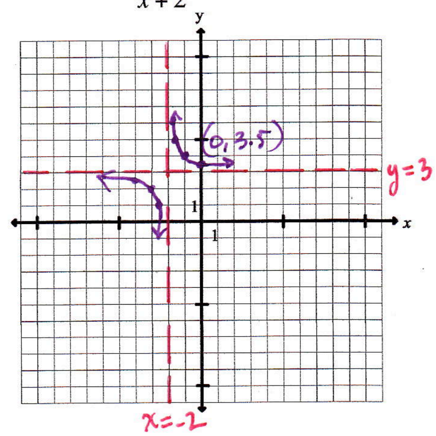
d. $y = \dfrac{3}{x+1} + 1$



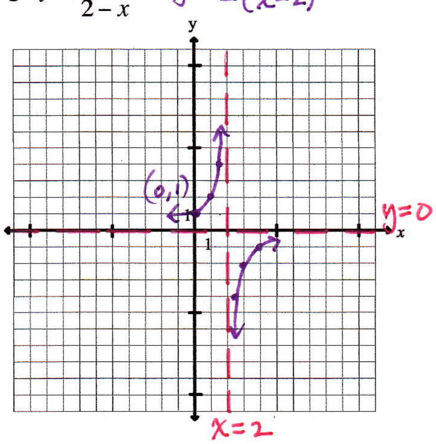
e. $y = \dfrac{-4}{x-2}$



f. $y = 3 + \dfrac{1}{x+2}$

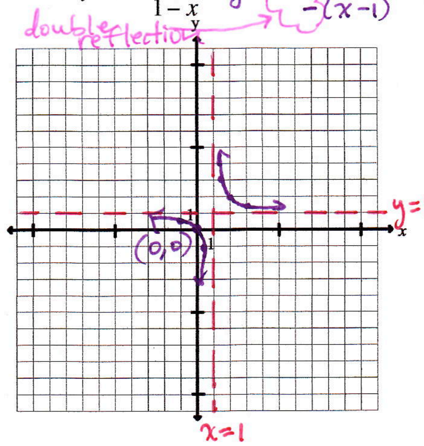


g. $y = \dfrac{2}{2-x}$ $\quad y = \dfrac{2}{-(x-2)}$

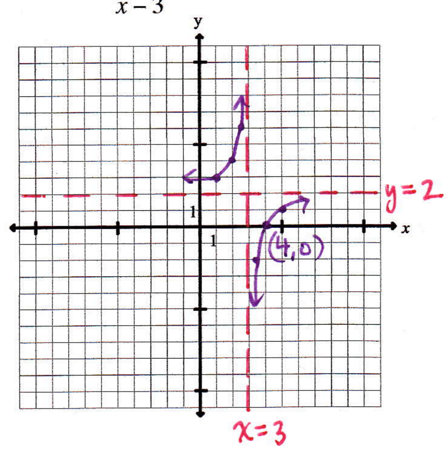


h. $y = 1 - \dfrac{1}{1-x}$ $\quad y = -\dfrac{1}{-(x-1)} + 1$

double reflection



i. $y = -\dfrac{2}{x-3} + 2$



Answers:

1. ii)	Domain $x \in R$	Range $y \in R$	Equation of the Asymptotes
a.	$x \neq -3$	$y \neq 0$	$x = -3, y = 0$
b.	$x \neq 0$	$y \neq -3$	$x = 0, y = -3$
c.	$x \neq 4$	$y \neq 0$	$x = 4, y = 0$
d.	$x \neq -1$	$y \neq 1$	$x = -1, y = 1$

e.	$x \neq 2$	$y \neq 0$	$x = 2, y = 0$
f.	$x \neq -2$	$y \neq 3$	$x = -2, y = 3$
g.	$x \neq 2$	$y \neq 0$	$x = 2, y = 0$
h.	$x \neq 1$	$y \neq 1$	$x = 1, y = 1$
i.	$x \neq 3$	$y \neq 2$	$x = 3, y = 2$