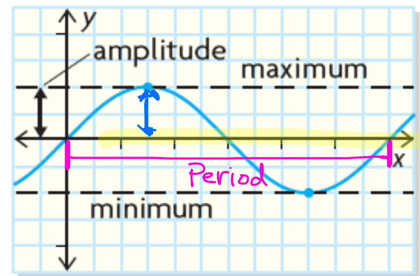
Graphing the Primary Trigonometric Functions

The graphs of the primary trigonometric functions are **periodic**. The sine and cosine functions have a distinct **wavelike** appearance (often referred to as a sinusoidal wave).

- the **period** is the interval of the independent variable needed for a repeating action to complete *one full cycle* (a cycle can begin at any point on the graph)
- the **equilibrium axis** is the equation of the horizontal line *halfway* between the maximum and the minimum value (calculated by finding $\frac{\max + \min}{2}$)
- the **amplitude** is the *distance* from the function's equilibrium axis to either the maximum or the minimum value (calculated by finding $\frac{\max - \min}{2}$)

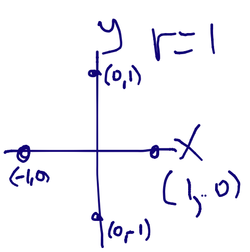


A. The Graph of $y = \sin x$

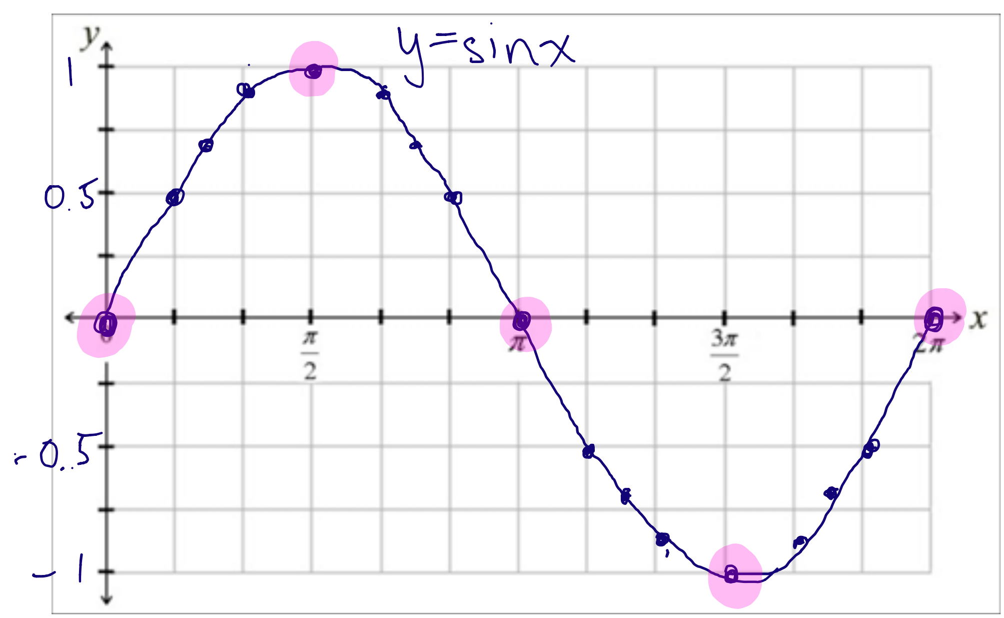
The **sine function** can be represented by the set of ordered pairs (x, sin$x$), where x is an angle in standard position measured in degrees or radians and $x \in R$. The equation of the sine function is written in the form $y = \sin x$ or $f(x) = \sin x$. Graph the equation $y = \sin x$, where x is an angle between 0 and 2π.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Exact value of sinx	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
Decimal value of sinx	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0

From special Δ's & unit circle



$\sin\theta = \frac{y}{r}$

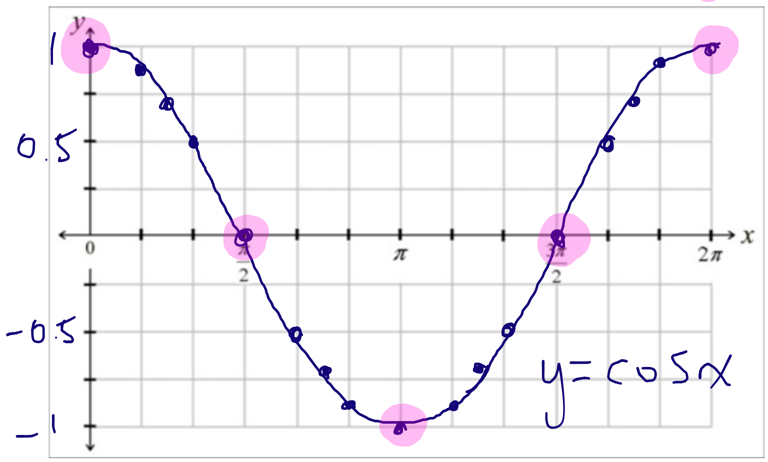


Key points
(0, 0)
($\frac{\pi}{2}$, 1)
(π, 0)
($\frac{3\pi}{2}$, -1)
(2π, 0)

B. The Graph of *y* = cos*x*

Graph the equation *y* = cos*x*, where *x* is an angle between 0 and 2π.

x	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Exact value of cos*x*	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Decimal value of cos*x*	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0	0.5	0.7	0.9	1



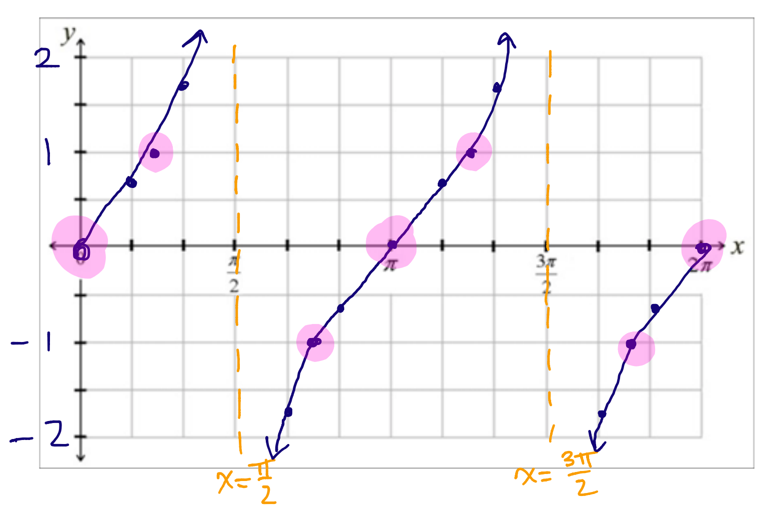
Key Points
(0, 1)
($\frac{\pi}{2}$, 0)
(π, -1)
($\frac{3\pi}{2}$, 0)
(2π, 1)

C. The Graph of *y* = tan*x*

$\tan\theta = \frac{y}{x}$

Graph the equation *y* = tan*x*, where *x* is an angle between 0 and 2π.

x	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Exact value of tan*x*	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{1}$	$\frac{\sqrt{3}}{1}$	$\frac{1}{0}$	$\frac{-\sqrt{3}}{1}$	$\frac{1}{-1}$	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	$\frac{-1}{-1}$	$\frac{\sqrt{3}}{1}$	$\frac{-1}{0}$	$\frac{-\sqrt{3}}{1}$	$\frac{-1}{1}$	$\frac{-1}{\sqrt{3}}$	0
Decimal value of tan*x*	0	0.6	1	1.7	DNE	-1.7	-1	-0.6	0	0.6	1	1.7	DNE	-1.7	-1	-0.6	0



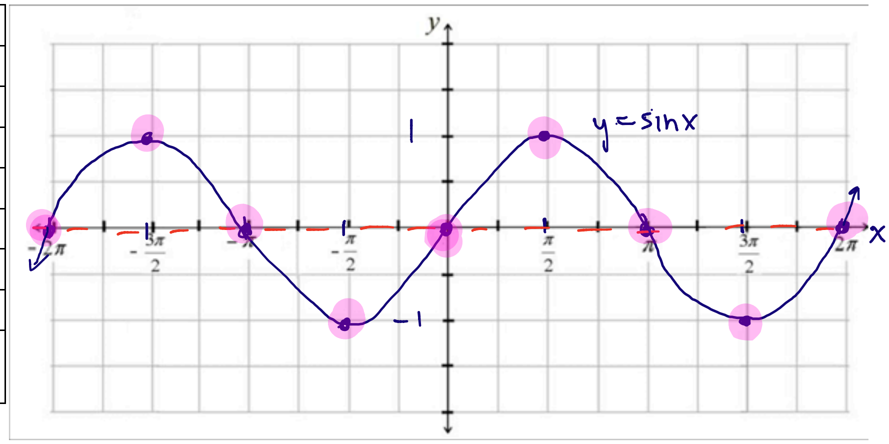
Keypoints
(0, 0)
($\frac{\pi}{4}$, 1)
asymptote
($\frac{3\pi}{4}$, -1)
(π, 0)
($\frac{5\pi}{4}$, 1)
asymptote
($\frac{7\pi}{4}$, -1)

(2π, 0)

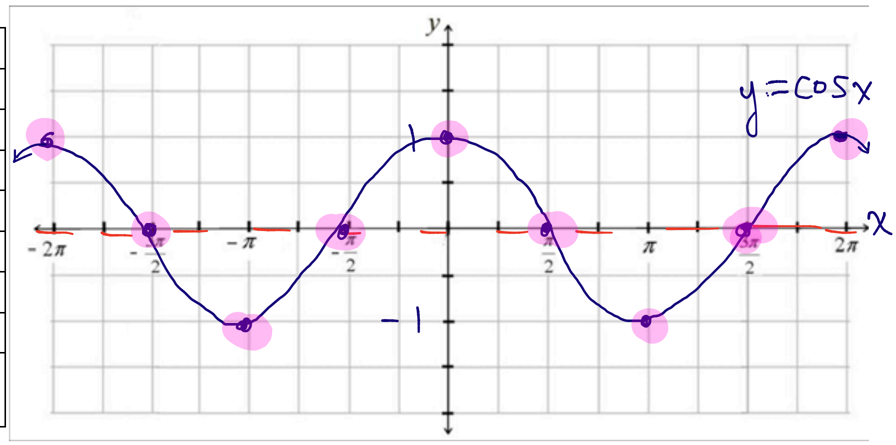
D. Key Features of Sinusoidal Functions

Sketch the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ and fill in the following tables on the interval $0 \le x \le 2\pi$. For this unit, we will use the y-values of 0, 1 and –1 to plot the key points for each curve.

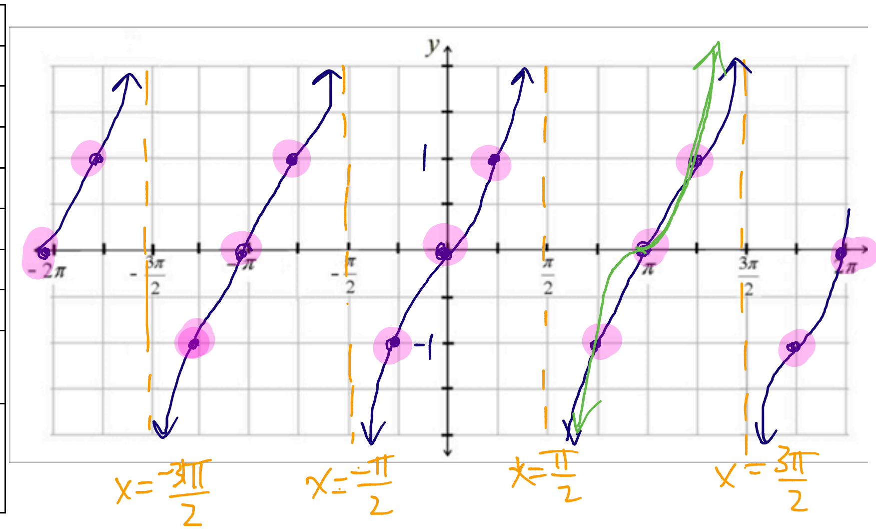
Key Features	**y = sinx**
Maximum value	1
Minimum value	−1
Amplitude	1
Period	2π
Domain	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \in \mathbb{R} \mid -1 \le y \le 1\}$
Zeros	$x = 0, \pi, 2\pi, \ldots$
Equation of the equilibrium axis	$y = 0$



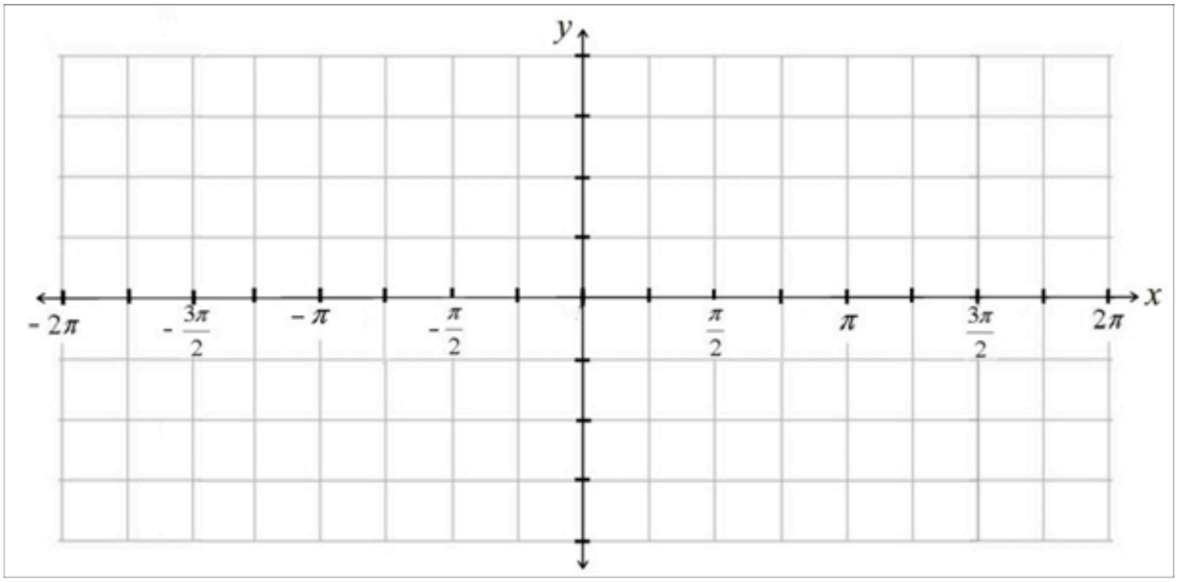
Key Features	**y = cosx**
Maximum value	1
Minimum value	−1
Amplitude	1
Period	2π
Domain	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \in \mathbb{R} \mid -1 \le y \le 1\}$
Zeros	$x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
Equation of the equilibrium axis	$y = 0$

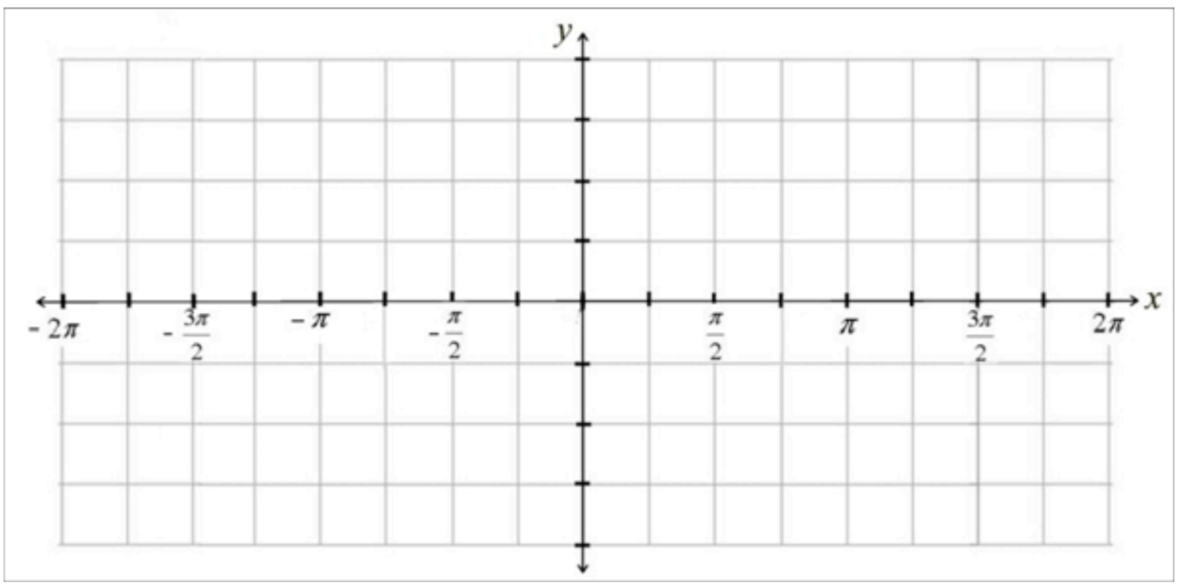


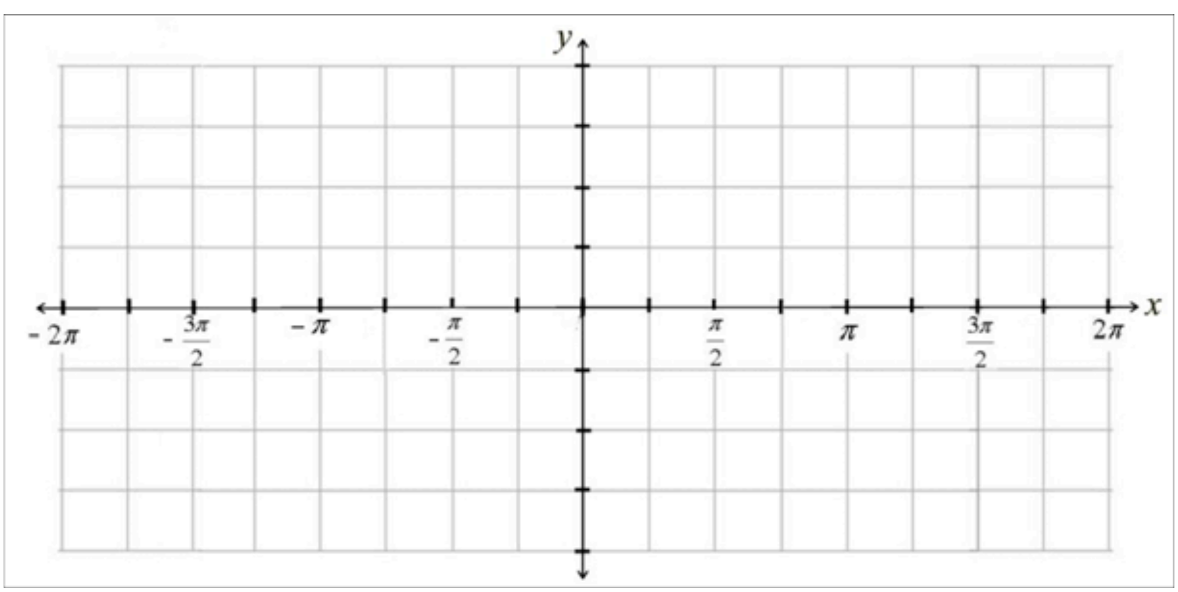
Key Features	**y = tanx**
Maximum value	∞
Minimum value	$-\infty$
Amplitude	n/a
Period	π
Domain	$\{x \in \mathbb{R} \mid x \ne \frac{\pi}{2} + \pi n, n \in I\}$
Range	$\{y \in \mathbb{R}\}$
Zeros	$x = 0, \pi, 2\pi, \ldots$
Equation of the equilibrium axis	n/a
Equation of vertical asymptotes	$x = \frac{\pi}{2}, x = \frac{3\pi}{2}, \ldots$



HW: Use the grids on the following page to sketch $y = \sin x$, $y = \cos x$, *and* $y = \tan x$ on the interval $-2\pi \le x \le 2\pi$. Where tanx is undefined, draw and label vertical asymptotes! Memorize these graphs!!


y
x
- 2π
- 3π/2
− π
- π/2
π/2
π
3π/2
2π


y
x
- 2π
- 3π/2
− π
- π/2
π/2
π
3π/2
2π


y
x
- 2π
- 3π/2
− π
- π/2
π/2
π
3π/2
2π

Stretches and Reflections of Periodic Functions

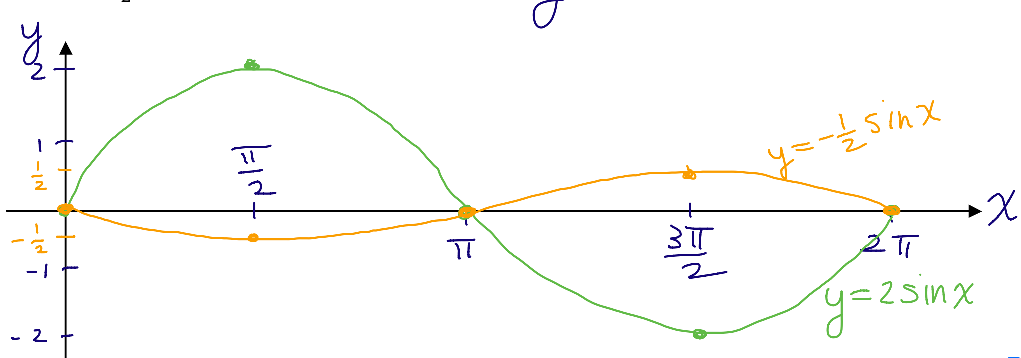
Like other functions, the stretches and reflections of sine and cosine functions can be summarized as follows:

Transformations	**Transformed Function**	**Effect on $y = \sin x$ or $y = \cos x$**
Vertical Reflection and Vertical Stretch	$y = a \sin x$ $y = a \cos x$	• If $a < 0$, the graph is **vertically reflected** in the ***x*-axis.** • If $\lvert a \rvert > 1$, the graph is **vertically expanded** by a factor of $\lvert a \rvert$. • If $0 < \lvert a \rvert < 1$, the graph is **vertically compressed** by a factor of $\lvert a \rvert$. • The point (x, y) on $y = f(x)$ becomes the point (x, ay) on $y = a f(x)$. • The **AMPLITUDE** of the function is $A = \lvert a \rvert$.
Horizontal Reflection and Horizontal Stretch	$y = \sin kx$ $y = \cos kx$	• If $k < 0$, the graph is **reflected** in the ***y*-axis.** • If $\lvert k \rvert > 1$, the graph is **horizontally compressed** by a factor of $\frac{1}{\lvert k \rvert}$. • If $0 < \lvert k \rvert < 1$, the graph is **horizontally expanded** by a factor of $\frac{1}{\lvert k \rvert}$. • The point (x, y) on $y = f(x)$ becomes the point $(\frac{1}{k}x, y)$ on $y = f(kx)$. • The **PERIOD** of the function is $P = \frac{2\pi}{\lvert k \rvert} = \frac{360°}{\lvert k \rvert}$.

A. Sketch the following using transformations on $y = \sin x$:

a) $y = 2 \sin x$ $(x, y) \rightarrow ($ x , $2y$ $)$ $A =$ 2 $P =$ 2π

b) $y = -\frac{1}{2} \sin x$ $(x, y) \rightarrow ($ x , $-\frac{1}{2}y$ $)$ $A =$ $\frac{1}{2}$ $P =$ 2π



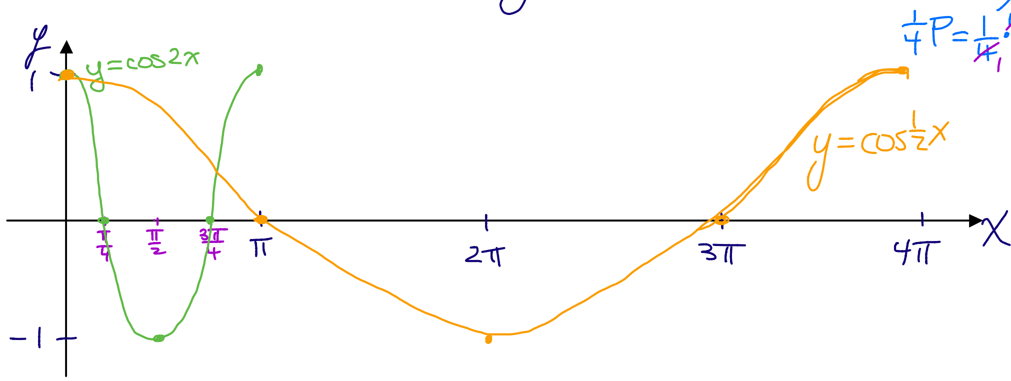
B. Sketch the following using transformations on $y = \cos x$:

a) $y = \cos 2x$ $(x, y) \rightarrow ($ $\frac{1}{2}x$, y $)$ $A =$ 1 $P =$ $\frac{2\pi}{k} = \frac{2\pi}{2} = \pi$

$\frac{1}{4} \cdot P = \frac{1}{4} \cdot \pi = \frac{\pi}{4}$ (distance between key points)

b) $y = \cos \frac{1}{2}x$ $(x, y) \rightarrow ($ $2x$, y $)$ $A =$ 1 $P =$ $\frac{2\pi}{(\frac{1}{2})} = 2\pi \cdot 2 = 4\pi$

$\frac{1}{4}P = \frac{1}{\cancel{4}} \cdot \cancel{4}\pi = \pi$ (distance between key points)



How to set the scale for horizontal stretches given *k*:

1) Find the period length, P, using: $P = \frac{2\pi}{\lvert k \rvert}$.

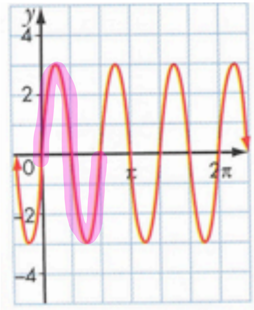
2) Get the number of radians between each key point by calculating $\frac{1}{4} \times P$.

3) Add $\frac{1}{4} \times P$ to the first point in the cycle, and to each subsequent point until the cycle(s) are complete.

4) Label the *x*-axis accordingly.

C. Determine the equations for the following a) *sine* and b) *cosine* functions:

a)

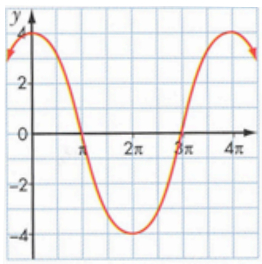


A = 3 P = $\frac{2\pi}{3}$

k = 3 $\because P = \frac{2\pi}{k}$

$\therefore y = 3\sin 3x$

b)



A = 4 P = 4π

k = $\frac{1}{2}$

$k = \frac{2\pi}{P} = \frac{2\pi}{4\pi} = \frac{1}{2}$

$\therefore y = 4\cos\frac{1}{2}x$

How to find k given the period length, P:

1) Rearrange the equation for period length to isolate $|k|$:

$$P = \frac{2\pi}{|k|}$$

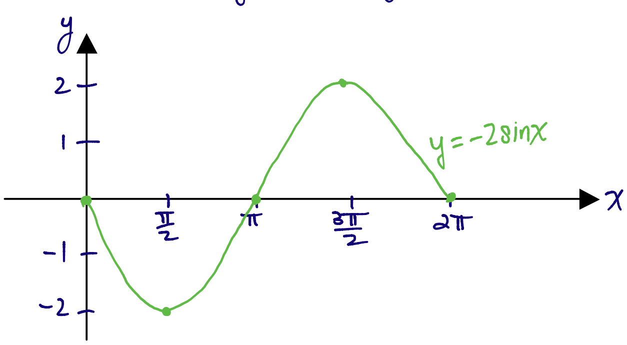
$$|k| = \frac{2\pi}{P}$$

2) Reduce the fraction to lowest terms.

D. Graph each function for one cycle and state the domain and range for your graph.

a) $y = -2\sin x$

$(x, y) \rightarrow (x, -2y)$



D = $\{x \in \mathbb{R} \mid 0 \le x \le 2\pi\}$

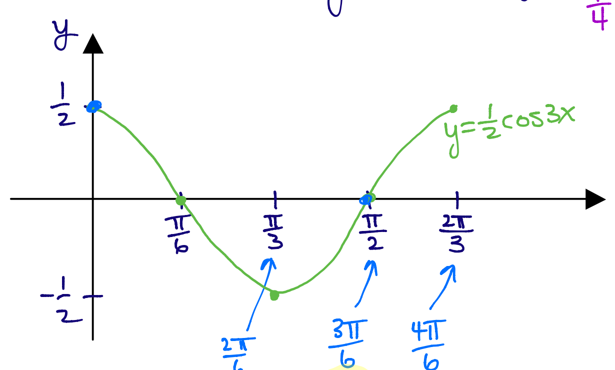
R = $\{y \in \mathbb{R} \mid -2 \le y \le 2\}$

b) $y = \frac{1}{2}\cos 3x$

$(x, y) \rightarrow (\frac{1}{3}x, \frac{1}{2}y)$

$\frac{1}{4} \cdot P = \frac{1}{\cancel{4}_2} \cdot \frac{\cancel{2}\pi}{3} = \frac{\pi}{6}$

x-scale



D = $\{x \in \mathbb{R} \mid 0 \le x \le \frac{2\pi}{3}\}$

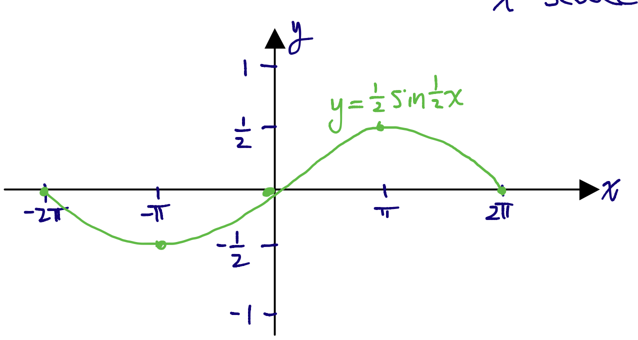
R = $\{y \in \mathbb{R} \mid -\frac{1}{2} \le y \le \frac{1}{2}\}$

E. Sketch the graph of the following functions.

a) $y = \frac{1}{2}\sin\frac{1}{2}x$ for $-2\pi \le x \le 2\pi$

$P = \frac{2\pi}{k} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

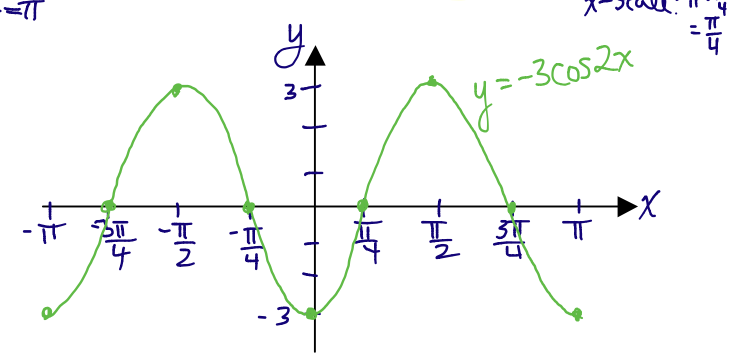
x-scale: $4\pi \cdot \frac{1}{4} = \pi$



b) $y = -3\cos 2x$ for $-\pi \le x \le \pi$

$P = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$

x-scale: $\pi \cdot \frac{1}{4} = \frac{\pi}{4}$



HW: p. 374-376 # 2, 4 and 5(equations only, no graphs), 6, 7b, 8a, 9cd, 12 (no graphs)

Translations of Periodic Functions

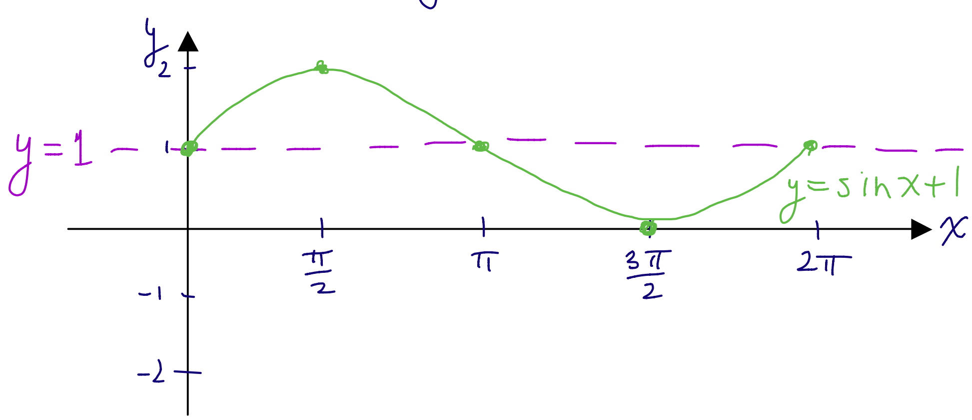
For the functions $y = \sin(x - d) + c$ and $y = \cos(x - d) + c$, **d** represents a phase shift (or horizontal translation) and **c** represents a vertical translation.

Transformation	Transformed Function	Effect on *y = sin x* or *y = cos x*
Horizontal Translation	$y = \sin(x - d)$ $y = \cos(x - d)$	If $d > 0$, the graph is horizontally translated **right** $\lvert d \rvert$ units. If $d < 0$, the graph is horizontally translated **left** $\lvert d \rvert$ units. The **PHASE SHIFT** of the function is: P.S. = $\lvert d \rvert$ units right if $d > 0$ ***or*** P.S. = $\lvert d \rvert$ units left if $d < 0$
Vertical Translation	$y = \sin x + c$ $y = \cos x + c$	If $c > 0$, the graph is **vertically translated up** $\lvert c \rvert$ units If $c < 0$, the graph is **vertically translated down** $\lvert c \rvert$ units The **VERTICAL TRANSLATION** of the function is: V.T. = $\lvert c \rvert$ units up if $c > 0$ ***or*** V.T. = $\lvert c \rvert$ units down if $c < 0$ The equation of the **equilibrium axis** is ***y = c*** **.**

A. Graph the function *y = sin x* for one cycle. Then graph the following using transformations on *y = sin x*:

$y = \sin x + 1$ $(x, y) \rightarrow ($ x , $y+1$ $)$ A = 1; P = 2π; P.S. = none; V.T = 1 unit up.

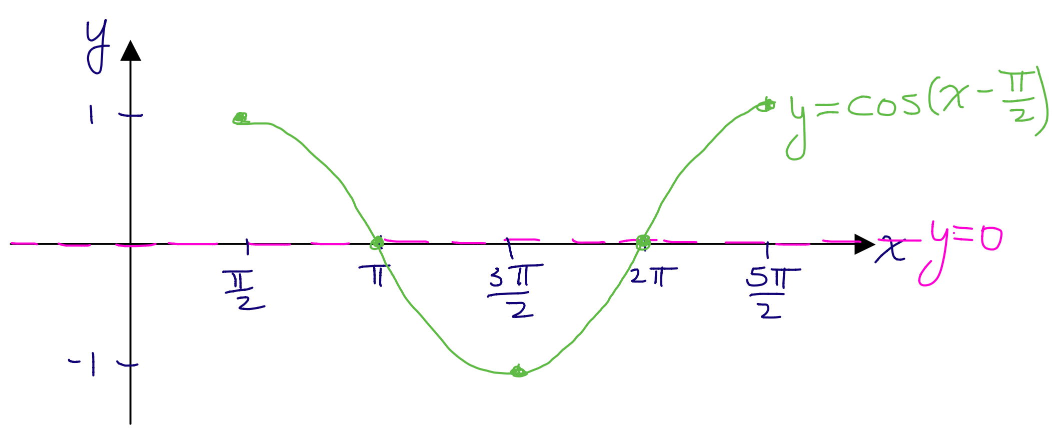
Equ. Axis: $y = 1$



B. Graph the function *y = cos x* for one cycle. Then graph the following using transformations on *y = cos x*:

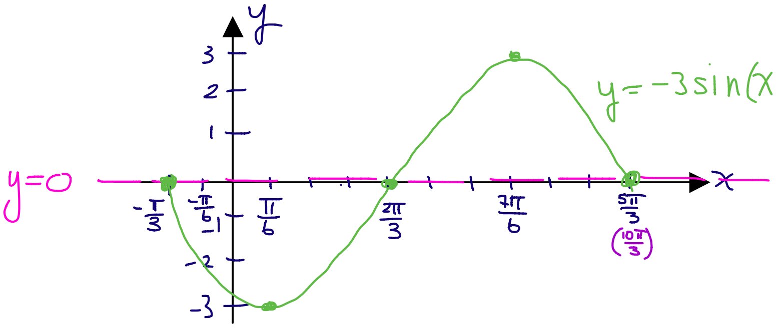
$y = \cos\left(x - \frac{\pi}{2}\right)$ $(x, y) \rightarrow ($ $x+\frac{\pi}{2}$, y $)$ A = 1; P = 2π; P.S. = $\frac{\pi}{2}$ rad/units right; V.T = none

$(y=0)$



C. Graph each function for one cycle. State the domain and range of the cycle.

a) $y = -3\sin(x + \frac{\pi}{3})$



V.R. across x-axis

A = 3

P = 2π (∵ k=1)

P.S. $\frac{\pi}{3}$ units left → $\frac{2\pi}{6}$ units left ② start point

$\frac{1}{4} \cdot P = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2} = \boxed{\frac{3\pi}{6}}$ ③ # of steps to next key point.

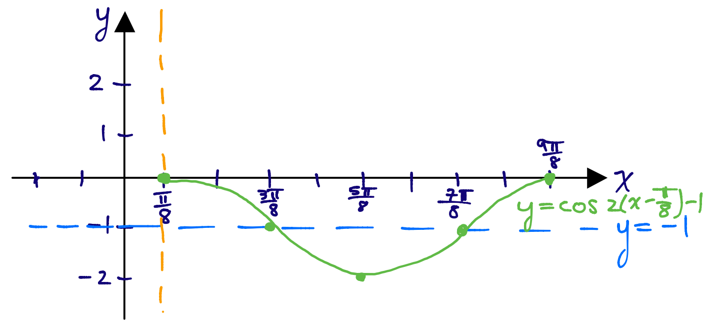
① x-scale $\frac{\pi}{6}$

④ $P = 2\pi \to \frac{12\pi}{6}$

D = $\{x \in \mathbb{R} \mid -\frac{\pi}{3} \le x \le \frac{5\pi}{3}\}$

R = $\{y \in \mathbb{R} \mid -3 \le y \le 3\}$

b) $y = \cos 2(x - \frac{\pi}{8}) - 1$



A = 1

VT: down 1 unit

P.S.: right $\frac{\pi}{8}$ ②

$P = \frac{2\pi}{2} = \pi$

↳ $\frac{1}{4} \cdot P = \frac{1}{4} \cdot \pi = \frac{\pi}{4} = \frac{2\pi}{8}$ ③

(x-scale is $\frac{\pi}{8}$) ①

④ $P = \pi = \frac{8\pi}{8}$

D = $\{x \in \mathbb{R} \mid \frac{\pi}{8} \le x \le \frac{9\pi}{8}\}$

R = $\{y \in \mathbb{R} \mid -2 \le y \le 0\}$

D. Determine the amplitude, period, phase shift and vertical translation for each function. Find the x-scale.

a) $y = -4\sin 2(x - \frac{\pi}{6})$

A = 4

$P = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi \leftarrow \pi \cdot \frac{1}{4} = \frac{\pi}{4}$

P.S. $\frac{\pi}{6}$ right

V.T. none

x-scale: $\frac{\pi}{12}$ is the x-scale

b) $y = \frac{1}{2}\cos\frac{2}{3}(x + \frac{\pi}{4}) - \frac{1}{2}$

$A = \frac{1}{2}$

$P = \frac{2\pi}{(2/3)} = 2\pi \cdot \frac{3}{2} = 3\pi \leftarrow 3\pi \cdot \frac{1}{4} = \frac{3\pi}{4}$

P.S. $\frac{\pi}{4}$ left

V.T. down $\frac{1}{2}$ unit

x-scale is $\frac{\pi}{4}$

E. Write an equation for the function in the form *y = a sin k(x – d) + c* or *y = a cos k(x – d) + c* $P = \frac{2\pi}{K} \longleftrightarrow K = \frac{2\pi}{P}$

a) *sine* function: A = 1 ; P = 4π ; P.S. = $\frac{\pi}{2}$ left ; V.T. = none

↳ a = 1

$k = \frac{2\pi}{P}$

$k = \frac{2\pi}{4\pi}$

$\therefore k = \frac{1}{2}$

$y = \sin\frac{1}{2}(x + \frac{\pi}{2})$

b) *cosine* function: A = 4 ; P = $\frac{\pi}{2}$; P.S. = none ; V.T. = up 3

↳ a = 4

$k = \frac{2\pi}{\frac{\pi}{2}}$

$k = 2\pi \cdot \frac{2}{\pi}$

$\therefore k = 4$

$y = 4\cos 4x + 3$

HW: p. 387-388 # 1 and 2 (even parts), 3ab, 4ade, 5, 6, 9