Maximum/Minimum of a Quadratic Function, I

Recall:

- A quadratic function in **standard form** is $f(x) = ax^2 + bx + c$, *where a ≠ 0*
- A quadratic function in **vertex form** is $f(x) = a(x-h)^2 + k$, where the point *(h, k)* is the **vertex**, *x = h* is the equation of the **axis of symmetry,** and *a ≠ 0*
- When ***a* > 0** (when *a* is positive), the parabola opens **up**, and the vertex is a *minimum*
- When ***a* < 0** (when *a* is negative), the parabola opens **down**, and the vertex is a *maximum*
- To find the vertex given standard form, rewrite the equation in *vertex form* by **completing the square**

1) Factor "a" out of variable terms
2) $\pm\left(\frac{b}{2}\right)^2$ within the brackets

Ex. 1. Find the <u>maximum or minimum value</u> of each function and <u>when</u> it occurs.

(output/dependent value of vertex = k) *(input/independent value of vertex = h)*

a) $A(w) = -4w^2 - 32w + 6$

$A(w) = -4(w^2 + 8w) + 6$

$A(w) = -4\left[w^2 + 8w + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2\right] + 6$

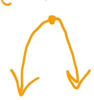
$A(w) = -4(w^2 + 8w + 16 - 16) + 6$

$A(w) = -4(w^2 + 8w + 16) + 64 + 6$

$A(w) = -4(w+4)^2 + 70$

$w = -4$ @ max

(−4, 70)



∴ Vertex is (−4, 70)

∴ the maximum value for A is 70 and it occurs when $w = -4$

b) $h(t) = 0.3t^2 - 1.2t + 4.5$

$h(t) = 0.3(t^2 - 4t) + 4.5$

$h(t) = 0.3\left[t^2 - 4t + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] + 4.5$

$h(t) = 0.3(t^2 - 4t + 4 - 4) + 4.5$

$h(t) = 0.3(t^2 - 4t + 4) - 1.2 + 4.5$

$h(t) = 0.3(t - 2)^2 + 3.3$

$t = 2$ @ min

(2, 3.3)

∴ Vertex is (2, 3.3)

∴ the minimum value for h is 3.3 and it occurs when $t = 2$

c) $F(g) = \frac{1}{3}g^2 + 4g - 6$

$F(g) = \frac{1}{3}(g^2 + 12g) - 6$

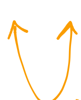
$F(g) = \frac{1}{3}\left[g^2 + 12g + \left(\frac{12}{2}\right)^2 - \left(\frac{12}{2}\right)^2\right] - 6$

$F(g) = \frac{1}{3}(g^2 + 12g + 36 - 36) - 6$

$F(g) = \frac{1}{3}(g^2 + 12g + 36) - 12 - 6$

$F(g) = \frac{1}{3}(g + 6)^2 - 18$

$g = -6$ @ min



(−6, −18)

∴ vertex is (−6, −18)

∴ the minimum value for F is −18 and it occurs when $g = -6$

d) $d(t) = -3t^2 + 5t$

$d(t) = -3\left(t^2 - \frac{5}{3}t\right)$

$d(t) = -3\left[t^2 - \frac{5}{3}t + \left(\frac{5}{2(3)}\right)^2 - \left(\frac{5}{2(3)}\right)^2\right]$

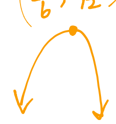
$d(t) = -3\left[t^2 - \frac{5}{3}t + \frac{25}{36} - \frac{25}{36}\right]$

$d(t) = -3\left[t^2 - \frac{5}{3}t + \frac{25}{36}\right] + \frac{75}{36}$ ÷3 ÷3

$d(t) = -3\left(t - \frac{5}{6}\right)^2 + \frac{25}{12}$

$t = \frac{5}{6}$ @ max

$\left(\frac{5}{6}, \frac{25}{12}\right)$



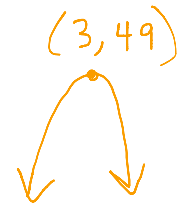
∴ vertex is $\left(\frac{5}{6}, \frac{25}{12}\right)$

∴ the maximum value for d is $\frac{25}{12}$ and it occurs when $t = \frac{5}{6}$

Ex. 2. A flare is fired vertically from the top of a platform so that its height *h*, in metres, after time *t*, in seconds, is given by $h(t) = 4 + 30t - 5t^2$. What is the maximum height that the flare reaches?

$$h(t) = -5t^2 + 30t + 4$$
$$h(t) = -5(t^2 - 6t) + 4$$
$$h(t) = -5(t^2 - 6t + 9 - 9) + 4$$
$$h(t) = -5(t^2 - 6t + 9) + 45 + 4$$
$$h(t) = -5\underbrace{(t-3)^2}_{t=3}\ \underbrace{+49}_{\text{max}}$$

$t = 3$ @ max



$\therefore$ vertex is at $(3, 49)$

$\therefore$ the maximum height is 49 m

Ex. 3. A swimming pool is treated periodically to control the growth of bacteria. The concentration of bacteria *t* days after treatment is $C(t) = 30t^2 - 240t + 500$, where *C* is the concentration of bacteria per cm^3. What is the lowest concentration of bacteria during the first week of treatment? On what day does this value occur?

$$C(t) = 30t^2 - 240t + 500$$
$$C(t) = 30(t^2 - 8t) + 500$$
$$C(t) = 30(t^2 - 8t + 16 - 16) + 500$$
$$C(t) = 30(t^2 - 8t + 16) - 480 + 500$$
$$C(t) = 30\underbrace{(t-4)^2}_{t=4}\ \underbrace{+20}_{\text{min}}$$

$t = 4$ @ min

$\therefore$ vertex is at $(4, 20)$

$\therefore$ the minimum concentration is 20 bacteria per cm^3 on the 4^{th} day

HW: p. 116-118 #2afhl, 3egjn, 15, 16, 17

Maximum/Minimum of a Quadratic Function, II

Note: To find the vertex, (h, k), given standard form, $f(x) = ax^2 + bx + c$:

- rewrite the equation in *vertex form*, $f(x) = a(x - h)^2 + k$, by **completing the square** *OR*
- use the short-cut formula $[h,\ k] = \left[\left(\frac{-b}{2a}\right),\ f(h)\right]$

"x-value at the vertex" $\rightarrow [x_v,\ f(x_v)] \leftarrow$ "y-value at the vertex"

These are a few examples from Lesson 1 :)

Ex. 1. Use the short-cut formula above to find the vertex for the following quadratics:

a) $h(t) = 0.3t^2 - 1.2t + 4.5$

$a = 0.3 \quad b = -1.2 \quad c = 4.5$

Find t-value at vertex:

$t_v = \frac{-b}{2a}$

$t_v = \frac{-(-1.2)}{2(0.3)}$

$t_v = \frac{1.2}{0.6}$

$t_v = 2$

Find $h(t_v)$:

$h(2) = 0.3(2)^2 - 1.2(2) + 4.5$

$h(2) = 3.3$

∴ the vertex is $(2, 3.3)$

b) $F(g) = \frac{1}{3}g^2 + 4g - 6$

$a = \frac{1}{3} \quad b = 4 \quad c = -6$

Find g-value at vertex:

$g_v = \frac{-b}{2a}$

$g_v = \frac{-(4)}{2\left(\frac{1}{3}\right)}$

$g_v = \frac{-4}{\left(2/3\right)} \rightarrow = \frac{-4}{1} \cdot \frac{3}{2}$ (K.C.I.)

$g_v = -6$

Find $F(g_v)$:

$F(-6) = \frac{1}{3}(-6)^2 + 4(-6) - 6$

$F(-6) = -18$

∴ the vertex is $(-6, -18)$

c) $d(t) = -3t^2 + 5t$

$a = -3 \quad b = 5 \quad c = 0$

Find t-value at vertex:

$t_v = \frac{-b}{2a}$

$t_v = \frac{-(5)}{2(-3)}$

$t_v = \frac{5}{6}$

Find $d(t_v)$:

$d\left(\frac{5}{6}\right) = -3\left(\frac{5}{6}\right)^2 + 5\left(\frac{5}{6}\right)$

$= -3\left(\frac{25}{36}\right) + \frac{25}{6}$

$= -\frac{25}{12} + \frac{50}{12}$

$= \frac{25}{12}$

∴ the vertex is $\left(\frac{5}{6}, \frac{25}{12}\right)$

Ex. 2. Determine two numbers whose difference is 12 and whose product is a minimum.

Let P represent the product.

Let m and n represent the two numbers.

① $m - n = 12 \rightarrow$ rearrange: $m = n + 12$ ③

② $P = m \cdot n$

Substitute ③ into ②:

$P = (n+12)n$

$P = n^2 + 12n$

$a = 1 \quad b = 12 \quad c = 0$

Find n-value at vertex:

$n_v = \frac{-(12)}{2(1)}$

$n_v = -6$

Find m (the other number):

∴ $n = -6$ ∴ $m = n + 12$

$m = -6 + 12$

$m = 6$

$\boxed{\text{∴ the two numbers are 6 and } -6}$

Ex. 3. Find two positive quantities whose sum is 18, if the sum of their squares is a minimum.

Let S represent the sum of the squares.

Let p and q represent the two positive quantities.

① : $p + q = 18$ $\longrightarrow$ Rearrange: $p = 18 - q$ ③

② : $S = p^2 + q^2$

Sub ③ into ② :

$$S = (18-q)^2 + q^2$$
$$S = (18-q)(18-q) + q^2$$
$$S = 324 - 18q - 18q + q^2 + q^2 \quad \text{(expand)}$$
$$S = 2q^2 - 36q + 324 \quad \text{(rearrange)}$$

$a = 2 \qquad b = -36 \qquad c = 324$

Find q-value at vertex:

$$q_v = \frac{-(-36)}{2(2)} \longrightarrow q_v = 9$$

Find p (the other positive quantity):

$\therefore q = 9$

$\therefore p = 18 - q$

$p = 18 - 9$

$p = 9$

$\therefore$ the two positive quantities are 9 and 9

Ex. 4. A magazine producer can sell 600 of her magazines at \$6.00 each. A marketing survey shows her that for every \$0.50 she increases the price, she will lose 30 sales. What price should she set to obtain the greatest revenue?

Let R represent the total revenue, in dollars.

Let n represent the number of price increases.

$R(0) = (6.00)(600) \longrightarrow$ no price increases

$R(1) = (6.00 + 0.50)(600 - 30) \longrightarrow$ 1 price increase

$\therefore R(n) = (6.00 + 0.50n)(600 - 30n) \longrightarrow$ n price increases

$R(n) = 3600 - 180n + 300n - 15n^2$ (expand)

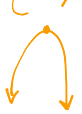
$R(n) = -15n^2 + 120n + 3600$ (rearrange)

$a = -15 \qquad b = 120 \qquad c = 3600$

① Find n-value of vertex:

$$n_v = \frac{-b}{2a}$$
$$n_v = \frac{-(120)}{2(-15)}$$
$$n_v = 4$$

(4, ?)



② New price $= 6.00 + 0.50n$

$= 6.00 + 0.50(4)$

$= 8.00$

$\therefore$ to obtain the maximum revenue she should set her price at \$8.00

HW: p. 116-118 #5, 6, 9, 12, 19

Solving First-Degree Inequalities

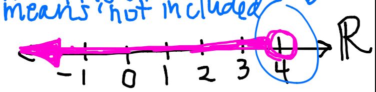
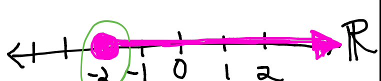
Inequalities can have an **infinite** number of solutions among a **set of numbers**, whereas *equations* have a *finite* number of solutions (real or imaginary!) Inequalities can be solved in a manner very similar to equations, with a few notable exceptions…

Ex. 1: Take the inequality 9 > 6:

Operation:	Add 3	Subtract 3	Multiply by 3	Divide by 3	Multiply by -3	Divide by -3
Resulting Inequality:	$9+3>6+3$ $12>9$	$9-3>6-3$ $6>3$	$9(3)>6(3)$ $27>18$	$\frac{9}{3}>\frac{6}{3}$ $3>2$	$9(-3)>6(-3)$ $-27>-18$	$\frac{9}{-3}>\frac{6}{-3}$ $-3>-2$
✔/✖:	✓	✓	✓	✓	✗	✗

Rule: When **multiplying or dividing an inequality by a negative**, *change the* direction *of the inequality sign!!*

Solutions for inequalities can be presented in a variety of ways:

Inequality	Graph on a Number Line	Solution Set	Interval Notation
$x<4$	not filled in means 'not included' 	$\{x \mid x \in \mathbb{R}, x<4\}$	$(-\infty, 4)$ 'not included' = round parenthesis
$x \geq -2$	 filled in means 'included'	$\{x \mid x \in \mathbb{R}, x \geq -2\}$	$[-2, \infty)$ 'included' = square bracket

Ex. 2: Solve the following. Present each solution using a *number line*, *set notation*, and/or *interval notation.*

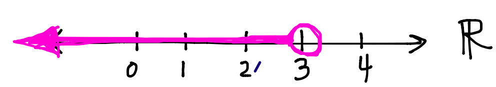
a) $4x-1<11$

$4x<11+1$

$\frac{4x}{4}<\frac{12}{4}$

$\boxed{\therefore x<3}$

Graph on a Number Line:



Interval Notation:

$(-\infty, 3)$

b) $3(2-x)+1 \le 13$

$6-3x+1 \le 13$

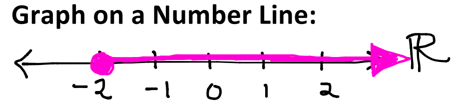
$-3x+7 \le 13$

$-3x \le 13-7$

$\frac{-3x}{-3} \le \frac{6}{-3}$

$\boxed{\therefore x \ge -2}$

Graph on a Number Line:



Solution Set:

$\{x \mid x \in \mathbb{R},\ x \ge -2\}$

c) $\left(\frac{3}{4}x+\frac{1}{2}x>\frac{5}{1}\right)\frac{4}{1}$ multiply by LCD = 4

$\frac{3(\cancel{4})^{1}}{\cancel{4}}x + \frac{1(\cancel{4})^{2}}{\cancel{2}}x > \frac{5}{1}(4)$

$3x+2x>20$

$\frac{5x}{5}>\frac{20}{5}$

$\boxed{\therefore x > 4}$

Interval Notation:

$(4, \infty)$

Solution Set:

$\{x \mid x \in \mathbb{R},\ x > 4\}$

d) $\left(\frac{z-1}{5} \ge \frac{z+2}{4} - \frac{1}{1}\right)\frac{20}{1}$ multiply by LCD = 20

$\frac{\cancel{20}^{4}(z-1)}{\cancel{5}^{1}} \ge \frac{\cancel{20}^{5}(z+2)}{\cancel{4}^{1}} - \frac{20(1)}{1}$

$4(z-1) \ge 5(z+2)-20$

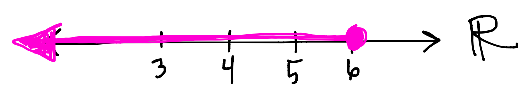
$4z-4 \ge 5z+10-20$

$4z-5z \ge 10-20+4$

$\frac{-z}{-1} \ge \frac{-6}{-1}$

$\boxed{\therefore z \le 6}$

Graph on a Number Line:



Solution Set:

$\{z \mid z \in \mathbb{R},\ z \le 6\}$

Interval Notation:

$(-\infty, 6]$

Practice Interval Notation! Go to Unit 1 Lesson 2 and include interval notation for Ex. 2-5 in the note.
HW: p. 78-80 #3ceg, 4dh, 5bdf, 6e, 7ad, 8, 10, 15ab (<u>Note</u>: Graph #3 and #5 only. No formal checks required.)