Solving Quadratic Equations, I: Three Methods

To find the *roots* (or *zeros* or *x-intercepts*) of a quadratic function:

1. Set $y = 0$ (and isolate the *0* on one side of the equation)
2. **Solve** the quadratic equation in terms of *x* using one of the following methods:
   - **a.** **Factor** the quadratic and find the values for *x* that give each factor a value of zero
   - **b.** Use the **Quadratic Formula** to calculate the value(s) for *x* when $ax^2 + bx + c = 0$
   - **c.** Solve by using **Inverse Operations** to isolate the variable when it appears only once

1. Solve by factoring, where $x \in R$. → no imaginary sol'ns

a) $-3(5x-1)(2x+7) = 0$

$5x - 1 = 0$ or $2x + 7 = 0$

$5x = 1$ $\quad 2x = -7$

$x = \frac{1}{5}$ $\quad x = -\frac{7}{2}$

$\boxed{\therefore x = -\frac{7}{2}, \frac{1}{5}}$

b) $16x^2 - 25 = 0$

$(4x+5)(4x-5) = 0$

$4x + 5 = 0$ or $4x - 5 = 0$

$4x = -5$ $\quad 4x = 5$

$x = \frac{-5}{4}$ $\quad x = \frac{5}{4}$

$\boxed{\therefore x = -\frac{5}{4}, \frac{5}{4}}$

c) $2x - 6x^2 = 0$

$2x(1 - 3x) = 0$

$2x = 0$ or $1 - 3x = 0$

$x = 0$ $\quad -3x = -1$

$x = \frac{1}{3}$

$\boxed{\therefore x = 0, \frac{1}{3}}$

2. Solve using the **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 + bx + c = 0$ *and* $x \in C$. → imaginary sol'ns are possible

a) $(x-1)^2 = 2x + 3$

$x^2 - 2x + 1 = 2x + 3$

$x^2 - 2x + 1 - 2x - 3 = 0$

$x^2 - 4x - 2 = 0$

$a = 1 \quad b = (-4) \quad c = (-2)$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-2)}}{2(1)}$

$x = \dfrac{4 \pm \sqrt{24}}{2}$

$x = \dfrac{4 \pm \sqrt{4}\sqrt{6}}{2}$

$x = \dfrac{4 \pm 2\sqrt{6}}{2}$

$x = \dfrac{\cancel{2}^{1}(2 \pm \sqrt{6})}{\cancel{2}_{1}}$

$\boxed{\therefore x = 2 \pm \sqrt{6}}$

b) $\left(\frac{x^2}{1} + \frac{x}{6} = -\frac{1}{2}\right) \times \frac{6}{1}$ ← eliminate fractions by multiplying each term by the LCD

$\frac{6x^2}{1} + \frac{6x}{6} = -\frac{6}{2}$

$6x^2 + x = -3$

$6x^2 + x + 3 = 0$

$a = 6 \quad b = 1 \quad c = 3$

$x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(6)(3)}}{2(6)}$

$x = \dfrac{-1 \pm \sqrt{-71}}{12}$

$x = \dfrac{-1 \pm \sqrt{-1}\sqrt{71}}{12}$

$\boxed{\therefore x = \dfrac{-1 \pm i\sqrt{71}}{12}}$

3. Solve using **inverse operations** to rearrange and isolate *x*, where $x \in C$.

a) $x^2 + 36 = 0$

$$\sqrt{x^2} = \pm\sqrt{-36}$$
$$x = \pm\sqrt{-36}$$
$$x = \pm\sqrt{-1}\sqrt{36}$$
$$\boxed{\therefore x = \pm 6i}$$

b) $2x^2 + 24 = 0$

$$2x^2 = -24$$
$$\frac{2x^2}{2} = \frac{-24}{2}$$
$$\sqrt{x^2} = \pm\sqrt{-12}$$
$$x = \pm\sqrt{-12}$$
$$x = \pm\sqrt{-1}\sqrt{4}\sqrt{3}$$
$$\boxed{\therefore x = \pm 2i\sqrt{3}}$$

c) $3(x+2)^2 - 27 = 0$

$$\frac{3(x+2)^2}{3} = \frac{27}{3}$$
$$\sqrt{(x+2)^2} = \pm\sqrt{9}$$
$$x + 2 = \pm 3 \quad ①$$
$$x = \pm 3 - 2 \quad ②$$

(two real sol'ns)

① $x = +3 - 2$, $x = 1$

② $x = -3 - 2$, $x = -5$

$$\boxed{\therefore x = -5, 1}$$

d) $\left(x + \frac{1}{3}\right)^2 + \frac{17}{9} = 0$

$$\sqrt{\left(x + \frac{1}{3}\right)^2} = \pm\sqrt{-\frac{17}{9}}$$
$$x + \frac{1}{3} = \pm\frac{\sqrt{-17}}{\sqrt{9}}$$
$$x + \frac{1}{3} = \pm\frac{i\sqrt{17}}{3}$$
$$x = \pm\frac{i\sqrt{17}}{3} - \frac{1}{3}$$
$$\boxed{\therefore x = \frac{-1 \pm i\sqrt{17}}{3}}$$

HW: p. 128-132 #2dfh, 8dfhj, 10fhjl, 11fh, 12dfln, 13bdfhj, 14bdh, 36ab, 38bcd

Solving Quadratic Equations, II: The Discriminant

Recall: *If* $ax^2+bx+c=0$, *and* $a\neq 0$, *then* $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ ⟵ ***Quadratic Formula***

Ex. 1: Solve for *x* using the quadratic formula, $x\in C$.

a) $4x^2-10x-25=0$

$a=4 \quad b=(-10) \quad c=(-25)$

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(4)(-25)}}{2(4)}$$

$$x=\frac{10\pm\sqrt{500}}{8}$$

$$x=\frac{10\pm\sqrt{100}\sqrt{5}}{8}$$

$$x=\frac{10\pm10\sqrt{5}}{8}$$

$$x=\frac{\cancel{2}^{1}(5\pm5\sqrt{5})}{\cancel{8}^{4}}$$

$$\boxed{\therefore\ x=\frac{5\pm5\sqrt{5}}{4}}$$

b) $9x^2+12x+4=0$

$a=9 \quad b=12 \quad c=4$

$$x=\frac{-(12)\pm\sqrt{(12)^2-4(9)(4)}}{2(9)}$$

$$x=\frac{-12\pm\sqrt{144-144}}{18}$$

$$x=\frac{-12\pm\sqrt{0}}{18}$$

$$x=\frac{-12+0}{18}\ \text{or}\ \frac{-12-0}{18}$$

$$\boxed{\therefore\ x=-\frac{2}{3},\ -\frac{2}{3}}$$

c) $3x^2+2x+6=0$

$a=3 \quad b=2 \quad c=6$

$$x=\frac{-(2)\pm\sqrt{(2)^2-4(3)(6)}}{2(3)}$$

$$x=\frac{-2\pm\sqrt{-68}}{6}$$

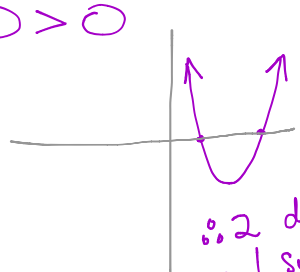
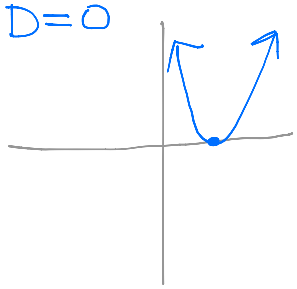
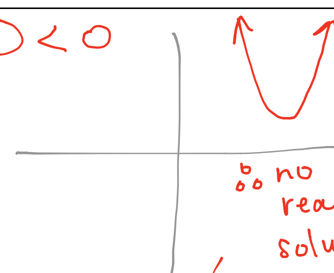
$$x=\frac{-2\pm\sqrt{-1}\sqrt{4}\sqrt{17}}{6}$$

$$x=\frac{-2\pm2i\sqrt{17}}{6}$$

$$x=\frac{\cancel{2}^{1}(-1\pm i\sqrt{17})}{\cancel{6}^{3}}$$

$$\boxed{\therefore\ x=\frac{-1\pm i\sqrt{17}}{3}}$$

What can you conclude about the value of b^2-4ac if the equation has:

distinct (different) real roots +	**equal (same) real roots** 0	**Imaginary (non-real) roots** −
$b^2-4ac>0$	$b^2-4ac=0$	$b^2-4ac<0$
$D>0$  ∴ 2 distinct real solutions	$D=0$  ∴ one distinct real solution	$D<0$  ∴ no real solutions (2 imaginary sol'ns)

The Discriminant: $D=b^2-4ac$

In the quadratic formula, the quantity under the radical sign is the ***discriminant, D***:

- the value (+, −, 0) of the discriminant determines the *nature* of the function's roots
- if the discriminant is a positive perfect square, the quadratic expression is *factorable*

Ex. 2: Use the ***discriminant*** to determine the nature of the roots (real or imaginary? distinct or equal?) for each of the following:

a) $4x^2 - 4x + 1 = 0$

$a = 4 \quad b = -4 \quad c = 1$

$D = b^2 - 4ac$

$= (-4)^2 - 4(4)(1)$

$= 16 - 16$

$= 0$

$\because D = 0 \therefore$ the equation has equal real roots

b) $1 - 2x - 3x^2 = 0$

$-3x^2 - 2x + 1 = 0$

$a = (-3) \quad b = (-2) \quad c = 1$

$D = b^2 - 4ac$

$= (-2)^2 - 4(-3)(1)$

$= 4 + 12$

$= 16$

$\because D > 0 \therefore$ the equation has distinct real roots

Ex. 3: Determine the value(s) of k that will give the indicated type of roots.

a) $kx^2 - 2x + 1 = 0$; imaginary roots $\hookrightarrow D < 0$

$a = k \quad b = (-2) \quad c = 1$

$b^2 - 4ac < 0$

$(-2)^2 - 4(k)(1) < 0$

$4 - 4k < 0$

$\frac{-4k}{-4} < \frac{-4}{-4}$

$\boxed{\therefore k > 1}$

b) $2x^2 - kx + k = 0$; equal real roots $\hookrightarrow D = 0$

$a = 2 \quad b = (-k) \quad c = k$

$b^2 - 4ac = 0$

$(-k)^2 - 4(2)(k) = 0$

$k^2 - 8k = 0$

$k(k - 8) = 0$

$k = 0 \qquad k - 8 = 0$

$k = 8$

$\boxed{\therefore k = 0, 8}$

c) $(1 - 3k)x^2 + 3x - 4 = 0$; distinct real roots $\hookrightarrow D > 0$

$a = (1 - 3k) \quad b = 3 \quad c = (-4)$

$b^2 - 4ac > 0$

$(3)^2 - 4(1 - 3k)(-4) > 0$

$9 + 16(1 - 3k) > 0$

$9 + 16 - 48k > 0$

$25 - 48k > 0$

$\frac{-48k}{-48} > \frac{-25}{-48}$

$\boxed{\therefore k < \frac{25}{48}}$

HW: WORKSHEET: "Solving Quadratic Equations, II: The Discriminant" #1adf, 2, 3a-h, 4ac, 6, 7

Solving Second-Degree (Quadratic) Inequalities

Quadratic inequalities are inequalities where the highest power of the unknown variable is two.
Examples: $3x^2 - 2x + 1 > 0$ and $4x^2 + 5x < 7$

Summary for solving a quadratic inequality:

- Rearrange to obtain 0 on the right side of the inequality.
- Redefine as a quadratic function.
- Find the zeros.
- Sketch the graph (scale the x-axis only).
- State the solution.

Solve the following inequalities. Present each solution using set notation, interval notation, and a number line.

filled-in circles

Ex. 1: $x^2 - 3x \leq 10$

$x^2 - 3x - 10 \leq 0$

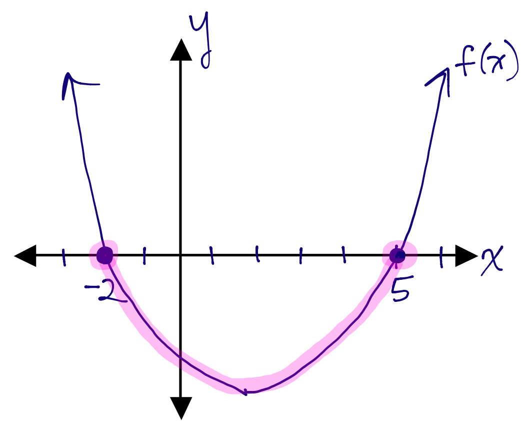
Let $f(x) = x^2 - 3x - 10$

$f(x) = (x-5)(x+2)$

$\therefore$ zeros are 5 and -2

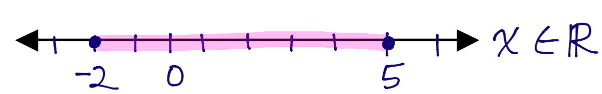
$\because a > 0$

$\therefore$ the parabola opens up



$\therefore$ the solution set is $\{x \mid x \in \mathbb{R}, -2 \leq x \leq 5\}$

and the interval is $x \in [-2, 5]$



open circles

Ex. 2: $x^2 > 16$

$x^2 - 16 > 0$

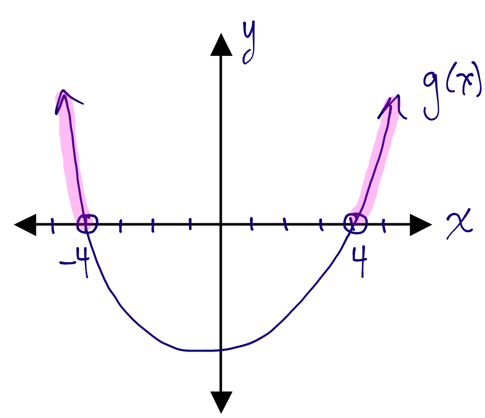
Let $g(x) = x^2 - 16$

$g(x) = (x-4)(x+4)$

$\therefore$ zeros are 4 and -4

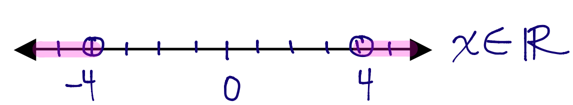
$\because a > 0$

$\therefore$ the parabola opens up



$\therefore$ the solution set is $\{x \mid x \in \mathbb{R}, x < -4, x > 4\}$

and the interval is $x \in (-\infty, -4)$ or $(4, \infty)$



Ex. 3: $-4x^2 - 8x \le -5$

$-4x^2 - 8x + 5 \le 0$

Let $h(x) = -4x^2 - 8x + 5$

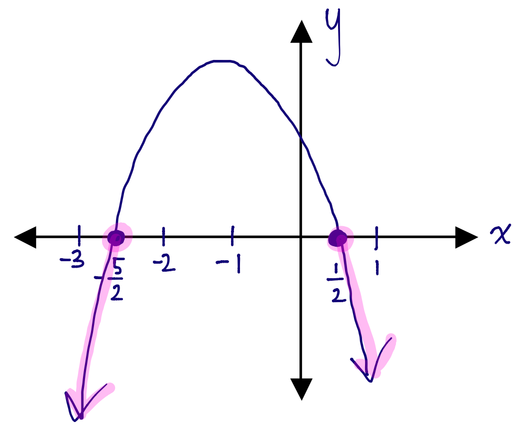
$h(x) = -(4x^2 + 8x - 5)$

$h(x) = -(2x-1)(2x+5)$

$\therefore$ zeros are $\frac{1}{2}$ and $-\frac{5}{2}$

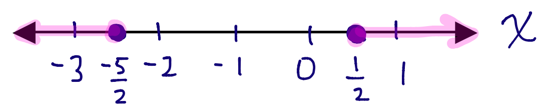
$\because a < 0$

$\therefore$ the parabola opens down



$\therefore$ the solution set is $\{x \mid x \in \mathbb{R},\ x \le -\frac{5}{2},\ x \ge \frac{1}{2}\}$

and the interval is $x \in (-\infty, -\frac{5}{2}]$ or $[\frac{1}{2}, \infty)$



Ex. 4: Determine the value(s) of k that will give the indicated type of roots.

$x^2 + kx + 2k = 0$; distinct real roots $\longrightarrow D > 0$

$a = 1 \quad b = k \quad c = 2k$

$b^2 - 4ac > 0$

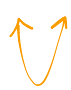
$(k)^2 - 4(1)(2k) > 0$

$k^2 - 8k > 0$

$k(k-8) > 0$

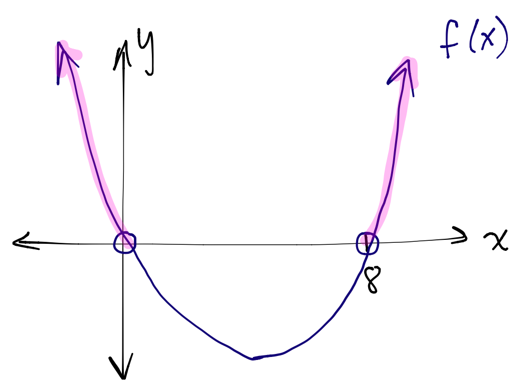
Let $f(x) = k(k-8)$

$\therefore$ zeros are 0 and 8



$\because a > 0$

$\therefore$ the parabola opens up



$\boxed{\therefore k \in (-\infty, 0) \text{ or } (8, +\infty)}$

HW: **i) Complete #4bde, #5 from the Lesson 5 Worksheet**

ii) Solve the following inequalities. Present your solution using set notation and a number line. Sketch the appropriate quadratic function, as demonstrated in the notes.

1. $x^2 - 4 > 0$
2. $x^2 - 6x < 16$
3. $x^2 - 6x \le 0$

iii) Solve the following inequalities. Present your solution using interval notation and a number line. Sketch the appropriate quadratic function, as demonstrated in the notes.

4. $2x^2 + 5x \le -2$
5. $5x^2 - 6x < -1$
6. $6x^2 > 14 - 17x$

Answers:

1. $\{x \mid x \in R,\ x < -2, x > 2\}$
2. $\{x \mid x \in R,\ -2 < x < 8\}$
3. $\{x \mid x \in R,\ 0 \le x \le 6\}$
4. $x \in \left[-2, -\frac{1}{2}\right]$
5. $x \in \left(\frac{1}{5}, 1\right)$
6. $x \in \left(-\infty, -\frac{7}{2}\right)$ *or* $\left(\frac{2}{3}, \infty\right)$