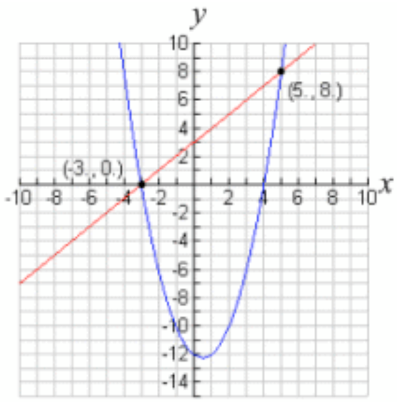
Solving Systems of Equations

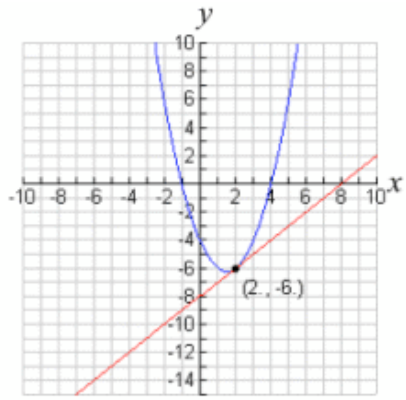
A. Linear-Quadratic Systems

Graphing a straight line and a parabola on the same set of axes yields three possible scenarios:

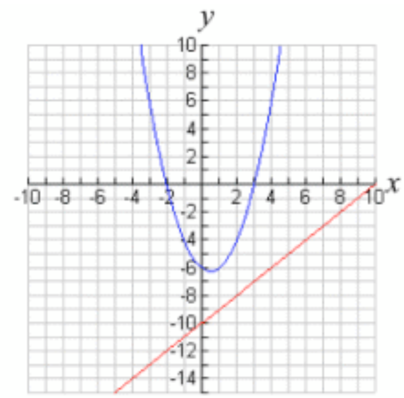
1. The equations will intersect in **two locations**. *Two distinct real solutions.*



2. The equations will intersect in **one location**. *Two identical real solutions.*



3. The equations will **not** intersect. *No real solutions.*



Ex. 1: Solve the following linear-quadratic system <u>algebraically</u> and <u>graph</u> the solution.

① $y = -x^2 - 6x - 7$ complete the square: $\rightarrow$ $y = -(x^2+6x) - 7$

$y = -(x^2+6x+9-9) - 7$

$y = -(x^2+6x+9) + 9 - 7$

$y = -(x+3)^2 + 2$

② $y = \frac{3}{2}x + 4 \longrightarrow m = \frac{3}{2}$ rise / run, $b = 4$ y-int

Sub ② into ①:

$\frac{3}{2}x + 4 = -x^2 - 6x - 7 \quad \times \frac{2}{1}$

$3x + 8 = -2x^2 - 12x - 14$

$2x^2 + 12x + 3x + 14 + 8 = 0$

$2x^2 + 15x + 22 = 0$

$(2x+11)(x+2) = 0$

$x = -\frac{11}{2}$ or $x = -2$

$\therefore x = -5\frac{1}{2}, -2$

Sub $x = -\frac{11}{2}$ and $x = -2$ into ②:

$y = \frac{3}{2}\left(-\frac{11}{2}\right) + \frac{4}{1}$

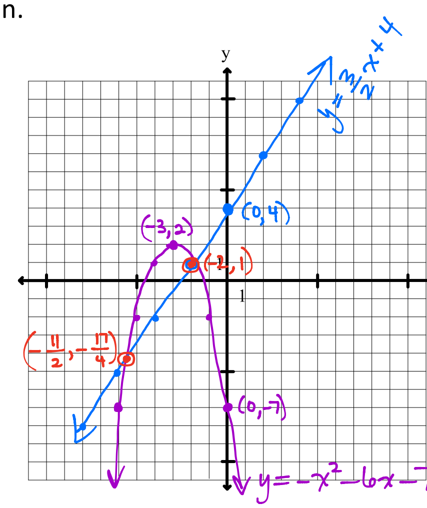
$y = -\frac{33}{4} + \frac{16}{4}$

$y = -\frac{17}{4}$

$y = \frac{3}{\cancel{2}_1}(\cancel{-2}^{-1}) + 4$

$y = -3 + 4$

$y = 1$



$\therefore$ the solution is $\left(-\frac{11}{2}, -\frac{17}{4}\right)$ and $(-2, 1)$.

Ex. 2: i) Determine the <u>nature</u> of the solutions <u>without solving or graphing</u>. ii) Interpret your results **geometrically**.

$\rightarrow D = b^2 - 4ac$

a) ① $y = x^2 - 2x + 1$

② $3 = x - y \longrightarrow y = x - 3$

Sub ② into ①:

$x - 3 = x^2 - 2x + 1$

$0 = x^2 - 2x - x + 1 + 3$

$0 = x^2 - 3x + 4$

$a = 1 \quad b = -3 \quad c = 4$

$D = b^2 - 4ac$

$= (-3)^2 - 4(1)(4)$

$= 9 - 16$

$= -7$

$\therefore D < 0$

$\therefore$ imaginary solutions

b) ① $y = (x-2)^2 - 3$

② $y - 1 = -4x \longrightarrow y = -4x + 1$

Sub ② into ①:

$-4x + 1 = (x-2)^2 - 3$

$-4x + 1 = x^2 - 4x + 4 - 3$

$0 = x^2 - 4x + 4x + 1 - 1$

$0 = x^2$

$a = 1 \quad b = 0 \quad c = 0$

$D = b^2 - 4ac$

$= (0)^2 - 4(1)(0)$

$= 0$

$\therefore D = 0$

$\therefore$ equal real solutions

B. Other Systems of Equations

Ex. 3: Solve the following systems of equations.

a)
① $x^2 + y^2 = 25$
② $4y = 3x \longrightarrow y = \frac{3}{4}x$

Sub ② into ①:

$$x^2 + \left(\frac{3}{4}x\right)^2 = 25$$
$$\left(x^2 + \frac{9}{16}x^2 = 25\right) \times \frac{16}{1}$$
$$16x^2 + \frac{\cancel{16}(9)}{\cancel{16}}x^2 = 16(25)$$
$$16x^2 + 9x^2 = 400$$
$$\frac{25x^2}{25} = \frac{400}{25}$$
$$x^2 = 16$$
$$x = \pm\sqrt{16}$$
$$x = \pm 4$$

Sub $x=4$ and $x=-4$ into ②:

$y = \frac{3}{4}(4)$ $\qquad$ $y = \frac{3}{4}(-4)$

$y = 3$ $\qquad$ $y = -3$

$\boxed{\therefore \text{the points of intersection are } (4,3) \text{ \& } (-4,-3)}$

b)
$y = 2x^2 - 2x + 3$ ①
$y = x^2 + 5x - 7$ ②

Sub ② into ①:

$$x^2 + 5x - 7 = 2x^2 - 2x + 3$$
$$0 = 2x^2 - x^2 - 2x - 5x + 3 + 7$$
$$0 = x^2 - 7x + 10$$
$$0 = (x-5)(x-2)$$
$$\therefore x = 5, 2$$

Sub $x=5$ and $x=2$ into ②:

$y = (5)^2 + 5(5) - 7$ $\qquad$ $y = (2)^2 + 5(2) - 7$

$y = 25 + 25 - 7$ $\qquad$ $y = 4 + 10 - 7$

$y = 43$ $\qquad$ $y = 7$

$\boxed{\therefore \text{the points of intersection are } (5,43) \text{ \& } (2,7)}$

Homework:

A. Solve the following systems of equations **algebraically** and illustrate the results **graphically**. Round solutions to 1 decimal place, where necessary. Follow example 1 from the note.

1. $y = x^2$
   $y = 2x + 3$

2. $y = x^2 + 3$
   $x - 2y = 2$

3. $y = x^2 - 4x + 3$
   $y = -1$

4. $y = 3x - 1$
   $y = -x^2 + 4x + 1$

5. $y = \frac{1}{2}x + 2$
   $y = x^2 + 2x$

6. $y + 1 = x$
   $y = -x^2 + 2x + 5$

B. Without solving or graphing, determine the nature of the solutions for the following systems of equations and give a **geometric** interpretation. Follow example 2 from the note.

7. $y = 3x^2 + 2x + 5$
   $y = -x - 2$

8. $y = -2x^2 + 2x + 7$
   $y = x + 3$

9. $y = 2x^2 - 12x + 16$
   $y = -x^2 + 6x - 11$

Answers:

1. (3, 9), (-1, 1) 2. no real solutions 3. (2, -1) 4. (2, 5), (-1, -4) 5. (-2.4, 1.6), (0.9, 4.9) 6. (3, 2), (-2, -3) 7. no real solutions; the line will never intersect the parabola 8. two distinct real solutions; the line intersects the parabola at two distinct points 9. two identical real solutions; the two parabolas intersect at one point

Solving Quadratic Equations, III: Applications

Ex. 1: The following equation represents the height, *h*, in metres of an object thrown off the top of a cliff after *t* seconds: $h = -5t^2 + 20t + 80$.

a) How high is the cliff? Sub in $t=0$

$h = -5(0)^2 + 20(0) + 80$

$h = 80$

$\boxed{\therefore \text{ the cliff is 80 metres high}}$

b) When does the object hit the ground? $\rightarrow t = ?$ $\rightarrow h = 0$

$0 = -5t^2 + 20t + 80$

$0 = -5(t^2 - 4t - 16)$

$0 = -5(t + ?)(t - ?)$ * not factorable any further! ∴ Q.F.

$a = 1 \quad b = -4 \quad c = -16$

$t = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-16)}}{2(1)}$

$t = \dfrac{4 \pm \sqrt{16 + 64}}{2}$

$t = \dfrac{4 \pm \sqrt{80}}{2}$

$t = \dfrac{4 \pm \sqrt{16}\sqrt{5}}{2}$

$t = \dfrac{4 \pm 4\sqrt{5}}{2}$

$t = \dfrac{\cancel{2}^{1}(2 \pm 2\sqrt{5})}{\cancel{2}_{1}}$

$\boxed{t = 2 + 2\sqrt{5}}$ or $\cancel{2 - 2\sqrt{5}}$ inadmissable

$\boxed{\therefore \text{ the object hits the ground at } 2 + 2\sqrt{5} \text{ seconds}}$

c) Does the object ever reach a height of 110 m? Use two methods…

A: Use the discriminant to check sol'ns…

$110 = -5t^2 + 20t + 80$

$0 = -5t^2 + 20t - 30$

$a = (-5), \quad b = 20, \quad c = (-30)$

$D = b^2 - 4ac$

$D = (20)^2 - 4(-5)(-30)$

$D = -200$

$\because D < 0$

∴ there are no real solutions for t given $h = 110$

$\boxed{\therefore \text{ the object does not reach 110 m}}$

B: Find the vertex and compare…

$y = -5t^2 + 20t + 80$

$y = -5(t^2 - 4t) + 80$

$y = -5(t^2 - 4t + 4 - 4) + 80$

$y = -5(t^2 - 4t + 4) + 20 + 80$

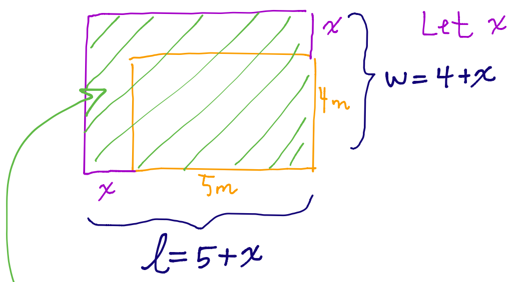
$y = -5\underbrace{(t-2)^2}_{t=2} + \underbrace{100}_{\text{max}}$

$t=2$ @ max

$\because$ 100 m is the object's maximum height

$\boxed{\therefore \text{ the object does not reach 110 m}}$

Ex. 2: A dining room measures 5 *m* by 4 *m*. A strip of uniform width is added to two adjacent sides to increase the area by 5 m^2. Find the width of the strip to 1 decimal place.



Let x represent the width of the strip, in m.

$A_{total} = 25 m^2$

$A_{total} = lw$

$25 = (5+x)(4+x)$

$25 = 20 + 9x + x^2$

$0 = x^2 + 9x - 5$

$a = 1 \quad b = 9 \quad c = (-5)$

$$x = \frac{-9 \pm \sqrt{9^2 - 4(1)(-5)}}{2(1)}$$

$$x = \frac{-9 \pm \sqrt{101}}{2}$$

$$x = \frac{-9 \pm 10.05}{2}$$

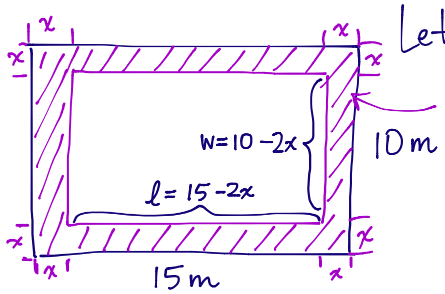
$$x = \frac{-9 + 10.05}{2} \quad \text{or} \quad x = \frac{-9 - 10.05}{2}$$

$\boxed{x \doteq 0.5}$ or $\cancel{x \doteq -9.5}$

inadmissable

∴ the width of the uniform strip is 0.5 m

Ex. 3: A uniform-width boardwalk is built around the inside edge of a rectangular parkland that is 10 m by 15 m. If the boardwalk takes up 20% of the lot, how wide is the boardwalk to the nearest centimetre.



Let x represent the uniform width of the boardwalk, in m.

Boardwalk Area = 20% of $(10 \times 15) m^2$

$A_B = 0.2\,(10 \times 15)\, m^2$

$A_B = 30 m^2$

$A_{inner} = 150 m^2 - 30 m^2 = 120 m^2$

$A = lw$

$120 = (15-2x)(10-2x)$

$120 = 150 - 30x - 20x + 4x^2$

$0 = 4x^2 - 50x + 30$

$a = 4 \quad b = (-50) \quad c = 30$

$$x = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(4)(30)}}{2(4)}$$

$$x = \frac{50 \pm \sqrt{2020}}{8}$$

$$x \doteq \frac{50 \pm 44.94}{8}$$

$\cancel{x \doteq 11.87}$ or $x \doteq 0.63$

inadmissable (too large)

∴ the boardwalk is 0.63m or 63 cm wide

HW: p. 130-132 #15, 16, 18, 26, 28c, 29, 34, 39

Solving Cubic and Quartic Equations

$y=x$ $y=-x$
$y=x^2$ $y=-x^2$
$y=x^3$ $y=-x^3$
$y=x^4$ $y=-x^4$

Note: *A polynomial equation of the n^{th} degree has n roots.*

Ex. 1: Solve for *x* in each of the following, $x \in C$.

a) $-4x^3 - 18x^2 + 10x = 0$

$-2x(2x^2+9x-5)=0$

$-2x(2x-1)(x+5)=0$

$x=0$ or $2x-1=0$ or $x+5=0$

$\boxed{\therefore x = 0, \frac{1}{2}, -5}$

b) $x^4 - 24x^2 - 25 = 0$

$(x^2-25)(x^2+1)=0$

$(x+5)(x-5)(x^2+1)=0$

$x+5=0$ or $x-5=0$ or $x^2+1=0$

$x=-5$ $\quad$ $x=5$ $\quad$ $x^2=-1$

$x=\pm\sqrt{-1}$

$x=\pm i$

$\boxed{\therefore x = \pm 5, \pm i}$

c) $3x^3 + x^2 + 24x + 8 = 0$

$x^2(3x+1)+8(3x+1)=0$

$(3x+1)(x^2+8)=0$

$3x+1=0$ or $x^2+8=0$

$x=-\frac{1}{3}$ $\quad$ $x^2=-8$

$x=\pm\sqrt{-8}$

$x=\pm 2i\sqrt{2}$

$\boxed{\therefore x = -\frac{1}{3}, \pm 2i\sqrt{2}}$

d) $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24 = 0$

Let $a=(x^2-5x)$

$a^2-2a-24=0$

$(a-6)(a+4)=0$

$(x^2-5x-6)(x^2-5x+4)=0$

$(x-6)(x+1)(x-1)(x-4)=0$

$\boxed{\therefore x = \pm 1, 4, 6}$

e) $\left(x+\frac{1}{x}\right)^2 - 7\left(x+\frac{1}{x}\right) + 12 = 0$ $\qquad$ Let $y=\left(x+\frac{1}{x}\right)$

$y^2-7y+12=0$

$(y-4)(y-3)=0$

$\left(x+\frac{1}{x}-4\right)\left(x+\frac{1}{x}-3\right)=0$

$x\left[x+\frac{1}{x}-4=0\right]$ or $\left[x+\frac{1}{x}-3=0\right]x$ $\Leftarrow$ Multiply each equation by the LCD to clear the fractions ☺

$x^2+1-4x=0$ or $x^2+1-3x=0$

① $x^2-4x+1=0$ or ② $x^2-3x+1=0$

$a=1$ $\quad b=-3$ $\quad c=1$

$x = \dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}$

$\therefore x = \dfrac{3\pm\sqrt{5}}{2}$

$a=1$ $\quad b=-4$ $\quad c=1$

$x = \dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$

$x = \dfrac{4\pm\sqrt{12}}{2}$ $\rightarrow$ $x=\dfrac{4\pm 2\sqrt{3}}{2}$

$\therefore x = 2\pm\sqrt{3}$

$\boxed{\therefore x = \dfrac{3\pm\sqrt{5}}{2}, 2\pm\sqrt{3}}$

HW: "WORKSHEET: Solving Cubic and Quartic Equations"

Unit 3 Review

1. Determine the maximum or minimum value of $y = \frac{1}{2}x^2 - 2x + 3$ and when it occurs.

$y = \frac{1}{2}(x^2 - 4x) + 3$

$y = \frac{1}{2}(x^2 - 4x + 4 - 4) + 3$

$y = \frac{1}{2}(x^2 - 4x + 4) - 2 + 3$

$y = \frac{1}{2}(x-2)^2 + 1$

$\therefore$ the minimum value is 1 and it occurs when $x = 2$.

2. Solve the following linear-quadratic system of equations and illustrate your solution graphically.

① $y = -2(x+2)^2 + 8$

② $y = -2x$

A. Sub ② into ① and solve:

$-2x = -2(x+2)^2 + 8$

$-2x = -2(x^2 + 4x + 4) + 8$

$-2x = -2x^2 - 8x \cancel{-8} \cancel{+8}$

$0 = -2x^2 - 6x$

$0 = -2x(x+3)$

$\boxed{\therefore x = 0 \quad \text{or} \quad x = -3}$

B. Sub $x = 0$ into ②:

$y = -2(0)$

$y = 0$

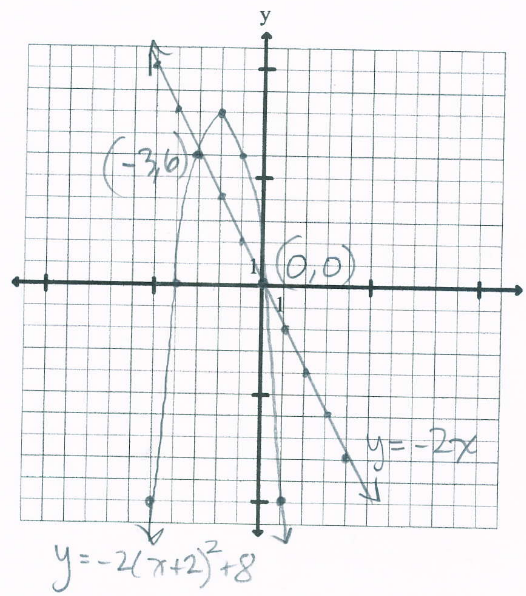
$\boxed{\therefore P_1(0,0)}$

C. Sub $x = -3$ into ②:

$y = -2(-3)$

$y = 6$

$\boxed{\therefore P_2(-3, 6)}$



3. **Without solving**, determine the number of solutions to the following linear-quadratic system and illustrate your results graphically.

① $y = \frac{1}{2}x^2 - 2x + 3 \Rightarrow y = \frac{1}{2}(x-2)^2 + 1$

② $2x - 3y - 6 = 0 \Rightarrow y = \frac{2}{3}x - 2$

A. Sub ① into ②:

$2x - 3\left(\frac{1}{2}x^2 - 2x + 3\right) - 6 = 0$

$2x - \frac{3}{2}x^2 + 6x - 9 - 6 = 0$

$\left[-\frac{3}{2}x^2 + 8x - 15 = 0\right] \times 2$

$-3x^2 + 16x - 30 = 0$

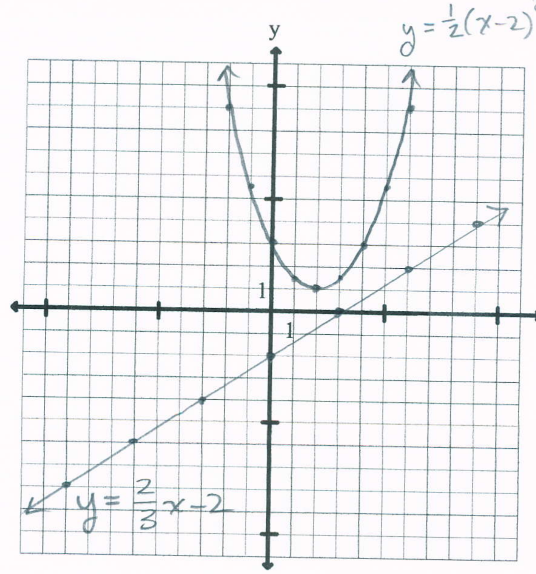
B. Sub into $D = b^2 - 4ac$

$D = (16)^2 - 4(-3)(-30)$

$D = 256 - 360$

$D = -104$

$\therefore$ there are no real solutions!



4. Determine the value(s) of k that will give the indicated type of roots.

a) $(2k-3)x^2 - 6x + 1 = 0$; ***two distinct real roots*** $\hookrightarrow \therefore D > 0$

$b^2 - 4ac > 0$

$(-6)^2 - 4(2k-3)(1) > 0$

$36 - 8k + 12 > 0$

$-8k > -48$

$\frac{-8k}{-8} < \frac{-48}{-8}$

$\boxed{\therefore k < 6}$

b) $x^2 + kx + (8-k) = 0$; ***two equal real roots*** $\hookrightarrow \therefore D = 0$

$b^2 - 4ac = 0$

$(k)^2 - 4(1)(8-k) = 0$

$k^2 - 32 + 4k = 0$

$k^2 + 4k - 32 = 0$

$(k+8)(k-4) = 0$

$\boxed{\therefore k = -8 \text{ or } k = 4}$

5. Solve for $x \in C$.

a) $2(x-3)^2 - 32 = 0$

$\frac{2(x-3)^2}{2} = \frac{32}{2}$

$(x-3)^2 = 16$

$x - 3 = \pm\sqrt{16}$

$x - 3 = \pm 4$

$x = \pm 4 + 3$

$\boxed{\therefore x = 7 \text{ or } x = -1}$

b) $\frac{2}{1-x} = \frac{3}{x+2} - 1$ LCD: $(1-x)(x+2)$

$2(x+2) = 3(1-x) - [(1-x)(x+2)]$

$2x + 4 = 3 - 3x - [x + 2 - x^2 - 2x]$

$0 = -1 - 5x - [-x^2 - x + 2]$

$0 = -1 - 5x + x^2 + x - 2$

$0 = x^2 - 4x - 3$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

$x = \frac{4 \pm \sqrt{28}}{2}$

$x = \frac{4 \pm 2\sqrt{7}}{2}$

$x = \frac{\cancel{2}(2 \pm \sqrt{7})}{\cancel{2}}$

$\boxed{\therefore x = 2 \pm \sqrt{7}}$

c) $x^4 - x^2 - 72 = 0$

$(x^2 + 8)(x^2 - 9) = 0$

$(x^2 + 8)(x + 3)(x - 3) = 0$

$x^2 + 8 = 0$ $\therefore x = -3$ $\therefore x = 3$

$x^2 = -8$

$x = \pm\sqrt{-8}$

$x = \pm\sqrt{4}\sqrt{2}\sqrt{-1}$

$\therefore x = \pm 2i\sqrt{2}$

$\boxed{\therefore x = \pm 3, \pm 2i\sqrt{2}}$

d) $2x^3 - 4x^2 - 50x + 100 = 0$

$2x^2(x-2) - 50(x-2) = 0$

$(x-2)(2x^2 - 50) = 0$

$2(x-2)(x^2 - 25) = 0$

$2(x-2)(x+5)(x-5) = 0$

$\boxed{\therefore x = 2, -5, 5}$

Textbook Review: p. 128-130 #2dg, 14h, 33, 36c, 37a; p. 152 #20acfj; p. 155 #12-15; p. 158-159 #3-9; p. 693 #23bg