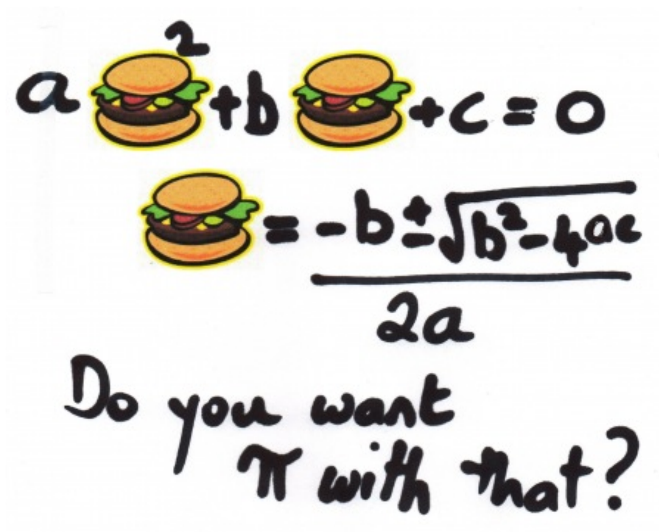
"Just a darn minute — yesterday you said that X equals **two**!"



MCR3UI

Unit 3: Polynomial Functions & Equations

Maximum/Minimum of a Quadratic Function, I

Recall:

- A quadratic function in **standard form** is $f(x) = ax^2 + bx + c$, *where a ≠ 0*
- A quadratic function in **vertex form** is $f(x) = a(x-h)^2 + k$, where the point *(h, k)* is the **vertex***, x = h* is the equation of the **axis of symmetry,** and *a ≠ 0*
- When ***a* > 0** (when *a* is positive), the parabola opens **up**, and the vertex is a *minimum*
- When ***a* < 0** (when *a* is negative), the parabola opens **down**, and the vertex is a *maximum*
- To find the vertex given standard form, rewrite the equation in *vertex form* by **completing the square**

Ex. 1. Find the maximum or minimum value of each function and when it occurs.

(output/dependent value of vertex = k) *(input/independent value of vertex = h)*

a) $A(w) = -4w^2 - 32w + 6$

b) $h(t) = 0.3t^2 - 1.2t + 4.5$

c) $F(g) = \frac{1}{3}g^2 + 4g - 6$

d) $d(t) = -3t^2 + 5t$

Ex. 2. A flare is fired vertically from the top of a platform so that its height *h*, in metres, after time *t*, in seconds, is given by $h(t) = 4 + 30t - 5t^2$. What is the maximum height that the flare reaches?

Ex. 3. A swimming pool is treated periodically to control the growth of bacteria. The concentration of bacteria *t* days after treatment is $C(t) = 30t^2 - 240t + 500$, where *C* is the concentration of bacteria per cm^3. What is the lowest concentration of bacteria during the first week of treatment? On what day does this value occur?

HW: p. 116-118 #2afhl, 3egjn, 15, 16, 17

Maximum/Minimum of a Quadratic Function, II

Note: To find the vertex, (h, k), given standard form, $f(x) = ax^2 + bx + c$:

- rewrite the equation in *vertex form,* $f(x) = a(x - h)^2 + k$, by **completing the square** *OR*
- use the short-cut formula $\left[h,\ k\right] = \left[\left(\frac{-b}{2a}\right),\ f(h)\right]$

Ex. 1. Use the short-cut formula above to find the vertex for the following quadratics:

a) $h(t) = 0.3t^2 - 1.2t + 4.5$ **b)** $F(g) = \frac{1}{3}g^2 + 4g - 6$ **c)** $d(t) = -3t^2 + 5t$

Ex. 2. Determine two numbers whose difference is 12 and whose product is a minimum.

Ex. 3. Find two positive quantities whose sum is 18, if the sum of their squares is a minimum.

Ex. 4. A magazine producer can sell 600 of her magazines at $6.00 each. A marketing survey shows her that for every $0.50 she increases the price, she will lose 30 sales. What price should she set to obtain the greatest revenue?

HW: p. 116-118 #5, 6, 9, 12, 19

Solving First-Degree Inequalities

Inequalities can have an **infinite** number of solutions among a **set of numbers**, whereas *equations* have a *finite* number of solutions (real or imaginary!) Inequalities can be solved in a manner very similar to equations, with a few notable exceptions…

Ex. 1: Take the inequality 9 > 6:

Operation:	**Add 3**	**Subtract 3**	**Multiply by 3**	**Divide by 3**	**Multiply by -3**	**Divide by -3**
Resulting Inequality:						
✔/✖:						

Rule: When **multiplying or dividing an inequality by a negative**, *change the*__________________________ *of the inequality sign!!*

Solutions for inequalities can be presented in a variety of ways:

Inequality	**Graph on a Number Line**	**Solution Set**	**Interval Notation**
$x < 4$			
$x \geq -2$			

Ex. 2: Solve the following. Present each solution using a *number line*, *set notation*, and/or *interval notation.*

a) $4x - 1 < 11$

Graph on a Number Line:

Interval Notation:

b) $3(2-x)+1 \le 13$

Graph on a Number Line:

Solution Set:

c) $\frac{3}{4}x+\frac{1}{2}x > 5$

Interval Notation:

Solution Set:

d) $\frac{z-1}{5} \ge \frac{z+2}{4}-1$

Graph on a Number Line:

Solution Set:

Interval Notation:

Practice Interval Notation! Go to Unit 1 Lesson 2 and include interval notation for Ex. 2-5 in the note.
HW: p. 78-80 #3ceg, 4dh, 5bdf, 6e, 7ad, 8, 10, 15ab (<u>Note</u>: Graph #3 and #5 only. No formal checks required.)

Solving Quadratic Equations, I: Three Methods

To find the *roots* (or *zeros* or *x-intercepts*) of a quadratic function:

1. Set *y = 0* (and isolate the *0* on one side of the equation)
2. **<u>Solve</u>** the quadratic equation in terms of *x* using one of the following methods:

   a. **Factor** the quadratic and find the values for *x* that give each factor a value of zero

   b. Use the **Quadratic Formula** to calculate the value(s) for *x* when $ax^2 + bx + c = 0$

   c. Solve by using **Inverse Operations** to isolate the variable when it appears only <u>once</u>

1. Solve by factoring, where $x \in R$.

a) $-3(5x-1)(2x+7) = 0$

b) $16x^2 - 25 = 0$

c) $2x - 6x^2 = 0$

2. Solve using the **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 + bx + c = 0$ *and* $x \in C$.

a) $(x-1)^2 = 2x + 3$

b) $x^2 + \dfrac{x}{6} = -\dfrac{1}{2}$

3. Solve using **inverse operations** to rearrange and isolate *x*, where $x \in C$.

a) $x^2 + 36 = 0$

b) $2x^2 + 24 = 0$

c) $3(x+2)^2 - 27 = 0$

d) $\left(x + \frac{1}{3}\right)^2 + \frac{17}{9} = 0$

HW: p. 128-132 #2dfh, 8dfhj, 10fhjl, 11fh, 12dfln, 13bdfhj, 14bdh, 36ab, 38bcd

Solving Quadratic Equations, II: The Discriminant

Recall: *If* $ax^2 + bx + c = 0$, *and* $a \neq 0$, *then* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ⟵ ***Quadratic Formula***

Ex. 1: Solve for *x* using the quadratic formula, $x \in C$.

a) $4x^2 - 10x - 25 = 0$

b) $9x^2 + 12x + 4 = 0$

c) $3x^2 + 2x + 6 = 0$

What can you conclude about the value of $b^2 - 4ac$ if the equation has:

distinct (different) real roots	equal (same) real roots	Imaginary (non-real) roots

The Discriminant: $D = b^2 - 4ac$

In the quadratic formula, the quantity under the radical sign is the ***discriminant, D***:

- the value (+, −, 0) of the discriminant determines the *nature* of the function's roots
- if the discriminant is a positive perfect square, the quadratic expression is *factorable*

Ex. 2: Use the ***discriminant*** to determine the nature of the roots (real or imaginary? distinct or equal?) for each of the following:

a) $4x^2 - 4x + 1 = 0$

b) $1 - 2x - 3x^2 = 0$

Ex. 3: Determine the value(s) of k that will give the indicated type of roots.

a) $kx^2 - 2x + 1 = 0$**;** imaginary roots

b) $2x^2 - kx + k = 0$**;** equal real roots

c) $(1 - 3k)x^2 + 3x - 4 = 0$**;** distinct real roots

HW: WORKSHEET: "Solving Quadratic Equations, II: The Discriminant" #1adf, 2, 3a-h, 4ac, 6, 7

WORKSHEET: Solving Quadratic Equations, II: The Discriminant

PART A

1. State the value of the discriminant for each of the following equations.

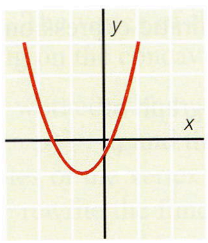
a. $x^2 - 4x + 1 = 0$ **b.** $x^2 + x - 2 = 0$ **c.** $x^2 - 5x = 0$

d. $x^2 + 4x + 4 = 0$ **e.** $x^2 - 10 = 0$ **f.** $2x^2 - x - 3 = 0$

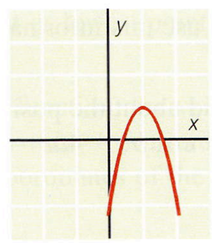
g. $3x^2 + 6x + 3 = 0$ **h.** $1 - x - 2x^2 = 0$ **i.** $4x^2 + 3x - 2 = 0$

2. For each of the graphs below, state whether the discriminant of the corresponding quadratic equation is greater than 0, equal to 0, or less than 0.

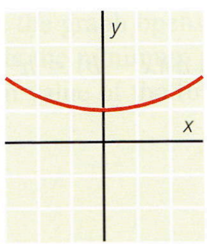
a.



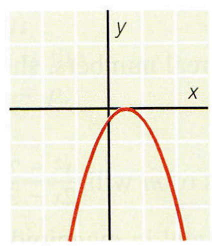
b.



c.



d.



PART B

3. Determine the nature of the roots for each of the following quadratic equations.

a. $x^2 - 8x + 12 = 0$ **b.** $x^2 + 4x + 5 = 0$

c. $x^2 - 10x + 5 = 0$ **d.** $4x^2 - 4x + 1 = 0$

e. $1 - 2x - 3x^2 = 0$ **f.** $2x^2 + 2 = 0$

g. $3x^2 - 4x = 0$ **h.** $4 - 5x = x^2$

i. $5 - x^2 = 4x$ **j.** $x^2 - \sqrt{8}x + 2 = 0$

k. $2 - 2x - 0.25x^2 = 0$ **l.** $\sqrt{2}x^2 - 2x - \sqrt{2} = 0$

4. Determine the value(s) of k that will give the indicated type of roots.

a. $x^2 - 6x + k = 0$; equal roots

b. $x^2 + kx - 16 = 0$; real distinct roots

c. $kx^2 - 2x + 1 = 0$; non-real roots

d. $2x^2 - kx + k = 0$; equal roots

e. $2kx^2 + 3x + 2k = 0$; real distinct roots

5. For what values of k will the graph of $y = 9x^2 + 3kx + k$ have no x-intercepts?

6. For what values of $b^2 - 4ac$ will a quadratic equation have rational roots? How could you use this information when factoring quadratic expressions?

7. What can be said about the position of the vertex of the graph of $y = ax^2 + bx + c$ if $b^2 - 4ac = 0$?

***Answers*:**

1. a. 12 **b.** 9 **c.** 25 **d.** 0 **e.** 40 **f.** 25 **g.** 0 **h.** 9 **i.** 41 **2. a.** $D > 0$ **b.** $D > 0$ **c.** $D < 0$ **d.** $D = 0$ **3. a.** real, unequal **b.** non-real **c.** real, unequal **d.** real, equal **e.** real, unequal **f.** non-real **g.** real, unequal **h.** real, unequal **i.** real, unequal **j.** real, equal **k.** real, unequal **l.** real, unequal **4. a.** $k = 9$ **b.** $k \in R$ **c.** $k > 1$ **d.** $k = 0$ or $k = 8$ **e.** $\frac{-3}{4} < k < \frac{3}{4}$ **5.** $0 < k < 4$ **6.** $b^2 - 4ac$ is a square **7.** on the x-axis

Solving Second-Degree (Quadratic) Inequalities

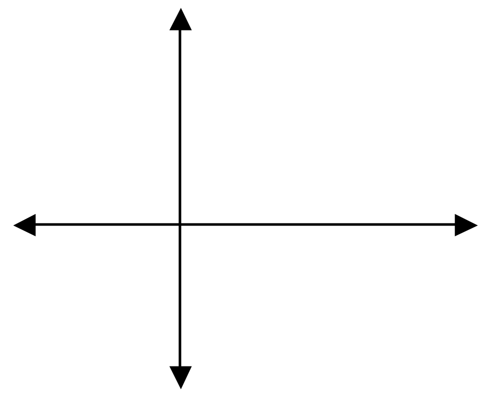
Quadratic inequalities are inequalities where the highest power of the unknown variable is two.
Examples: $3x^2 - 2x + 1 > 0$ and $4x^2 + 5x < 7$

Summary for solving a quadratic inequality:

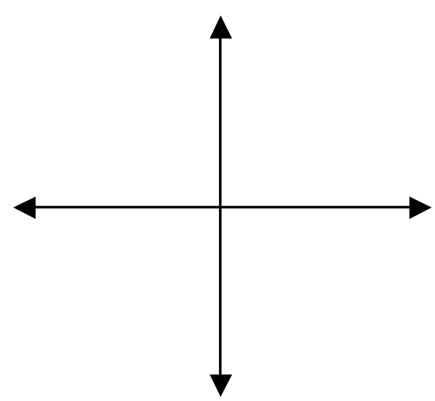
- Rearrange to obtain 0 on the right side of the inequality.
- Redefine as a quadratic function.
- Find the zeros.
- Sketch the graph (scale the x-axis only).
- State the solution.

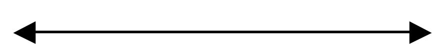
Solve the following inequalities. Present each solution using set notation, interval notation, and a number line.

Ex. 1: $x^2 - 3x \le 10$

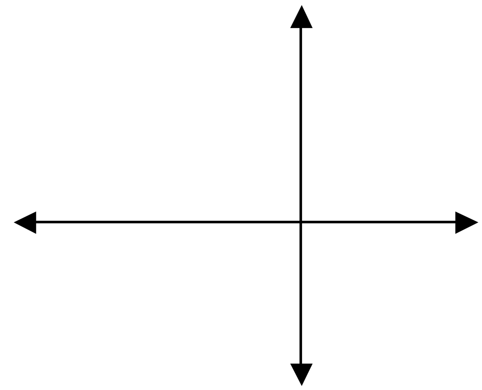


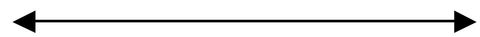
Ex. 2: $x^2 > 16$





Ex. 3: $-4x^2 - 8x \le -5$





Ex. 4: Determine the value(s) of k that will give the indicated type of roots.

$x^2 + kx + 2k = 0$; distinct real roots

HW: **i) Complete #4bde, #5 from the Lesson 5 Worksheet**

ii) Solve the following inequalities. Present your solution using set notation and a number line. Sketch the appropriate quadratic function, as demonstrated in the notes.

1. $x^2 - 4 > 0$
2. $x^2 - 6x < 16$
3. $x^2 - 6x \le 0$

iii) Solve the following inequalities. Present your solution using interval notation and a number line. Sketch the appropriate quadratic function, as demonstrated in the notes.

4. $2x^2 + 5x \le -2$
5. $5x^2 - 6x < -1$
6. $6x^2 > 14 - 17x$

Answers:

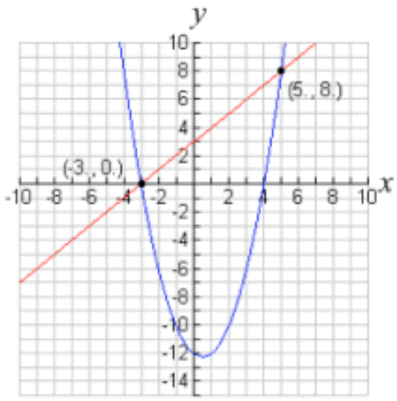
1. $\{x | x \in R,\ x < -2, x > 2\}$
2. $\{x | x \in R,\ -2 < x < 8\}$
3. $\{x | x \in R,\ 0 \le x \le 6\}$
4. $x \in \left[-2, -\frac{1}{2}\right]$
5. $x \in \left(\frac{1}{5},\ 1\right)$
6. $x \in \left(-\infty,\ -\frac{7}{2}\right)\ or\ \left(\frac{2}{3},\ \infty\right)$

Solving Systems of Equations

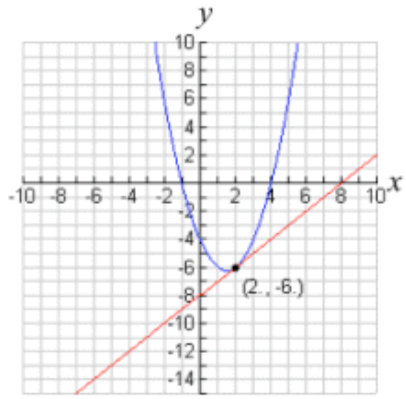
A. Linear-Quadratic Systems

Graphing a straight line and a parabola on the same set of axes yields three possible scenarios:

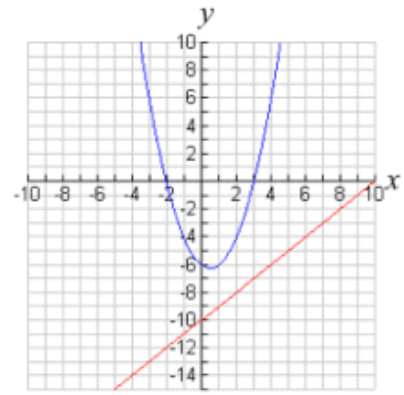
1. The equations will intersect in **two locations**. *Two distinct real solutions.*



2. The equations will intersect in **one location**. *Two identical real solutions.*



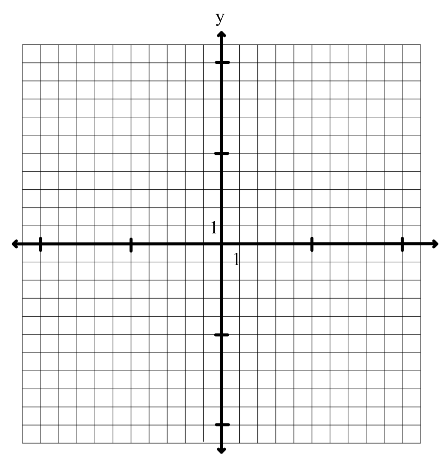
3. The equations will **not** intersect. *No real solutions.*



Ex. 1: Solve the following linear-quadratic system <u>algebraically</u> and <u>graph</u> the solution.

$$y = -x^2 - 6x - 7$$

$$y = \frac{3}{2}x + 4$$



Ex. 2: i) Determine the nature of the solutions <u>without solving or graphing</u>. ii) Interpret your results **geometrically**.

a)
$$y = x^2 - 2x + 1$$
$$3 = x - y$$

b)
$$y = (x-2)^2 - 3$$
$$y - 1 = -4x$$

B. Other Systems of Equations

Ex. 3: Solve the following systems of equations.

a) $x^2 + y^2 = 25$
$4y = 3x$

b) $y = 2x^2 - 2x + 3$
$y = x^2 + 5x - 7$

Homework:

A. Solve the following systems of equations **algebraically** and illustrate the results **graphically**. Round solutions to 1 decimal place, where necessary. Follow example 1 from the note.

1. $y = x^2$
   $y = 2x + 3$

2. $y = x^2 + 3$
   $x - 2y = 2$

3. $y = x^2 - 4x + 3$
   $y = -1$

4. $y = 3x - 1$
   $y = -x^2 + 4x + 1$

5. $y = \frac{1}{2}x + 2$
   $y = x^2 + 2x$

6. $y + 1 = x$
   $y = -x^2 + 2x + 5$

B. **Without solving or graphing**, determine the nature of the solutions for the following systems of equations and give a **geometric** interpretation. Follow example 2 from the note.

7. $y = 3x^2 + 2x + 5$
   $y = -x - 2$

8. $y = -2x^2 + 2x + 7$
   $y = x + 3$

9. $y = 2x^2 - 12x + 16$
   $y = -x^2 + 6x - 11$

Answers:

1. (3, 9), (-1, 1) 2. no real solutions 3. (2, -1) 4. (2, 5), (-1, -4) 5. (-2.4, 1.6), (0.9, 4.9)

6. (3, 2), (-2, -3) 7. no real solutions; the line will never intersect the parabola 8. two distinct real solutions; the line intersects the parabola at two distinct points 9. two identical real solutions; the two parabolas intersect at one point

Solving Quadratic Equations, III: Applications

Ex. 1: The following equation represents the height, *h*, in metres of an object thrown off the top of a cliff after *t* seconds: $h = -5t^2 + 20t + 80$.

a) How high is the cliff?

b) When does the object hit the ground?

c) Does the object ever reach a height of 110 m? Use two methods…

Ex. 2: A dining room measures 5 *m* by 4 *m*. A strip of uniform width is added to two adjacent sides to increase the area by 5 m^2. Find the width of the strip to 1 decimal place.

Ex. 3: A uniform-width boardwalk is built around the inside edge of a rectangular parkland that is 10 m by 15 m. If the boardwalk takes up 20% of the lot, how wide is the boardwalk to the nearest centimetre.

HW: p. 130-132 #15, 16, 18, 26, 28c, 29, 34, 39

Solving Cubic and Quartic Equations

Note: *A polynomial equation of the n^{th} degree has n roots.*

Ex. 1: Solve for *x* in each of the following, $x \in C$.

a) $-4x^3 - 18x^2 + 10x = 0$

b) $x^4 - 24x^2 - 25 = 0$

c) $3x^3 + x^2 + 24x + 8 = 0$

d) $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24 = 0$

e) $\left(x + \frac{1}{x}\right)^2 - 7\left(x + \frac{1}{x}\right) + 12 = 0$

HW: "WORKSHEET: Solving Cubic and Quartic Equations"

WORKSHEET: Solving Cubic and Quartic Equations

Note: *A polynomial equation of the n^th degree has n roots.*

Solve for *x* in each of the following, $x \in C$.

a) $4x^3 - 9x = 0$

b) $-2x^3 + 3x^2 - x = 0$

c) $12x^3 - 40x^2 - 52x = 0$

d) $x^4 - 5x^2 + 4 = 0$

e) $x^2 + 4 = \frac{32}{x^2}$

f) $20x^4 - 25x^2 - 45 = 0$

g) $x^3 - 3x^2 + 4x - 12 = 0$

h) $4x^3 + 8x^2 - x - 2 = 0$

i) $2x^4 - 4x^3 - 8x^2 + 16x = 0$

j) $(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$

k) $(2x^2 + 5x)^2 - 10(2x^2 + 5x) - 24 = 0$

l) $(x^2 + 2x)^2 - (x^2 + 2x) - 12 = 0$

m) $\left(x + \frac{6}{x}\right)^2 - 2\left(x + \frac{6}{x}\right) - 35 = 0$

n) $\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 6 = 0$

o) $\left(x - \frac{1}{x}\right)^2 - \frac{77}{12}\left(x - \frac{1}{x}\right) + 10 = 0$

Answers:

a. $0, -\frac{3}{2}, \frac{3}{2}$ **b.** $0, \frac{1}{2}, 1$ **c.** $0, \frac{13}{3}, -1$ **d.** $-2, 2, -1, 1$ **e.** $-2, 2, -2i\sqrt{2}, 2i\sqrt{2}$ **f.** $-i, i, -\frac{3}{2}, \frac{3}{2}$

g. $-2i, 2i, 3$ **h.** $-2, -\frac{1}{2}, \frac{1}{2}$ **i.** $0, -2, 2, 2$ **j.** $4, -2, 3, -1$ **k.** $-\frac{1}{2}, -2, \frac{3}{2}, -4$

l. $-1-\sqrt{5}, -1+\sqrt{5}, -1-i\sqrt{2}, -1+i\sqrt{2}$ **m.** $-3, -2, 1, 6$ **n.** $1, 1, \frac{3 \pm \sqrt{5}}{2}$ **o.** $-\frac{1}{3}, -\frac{1}{4}, 3, 4$

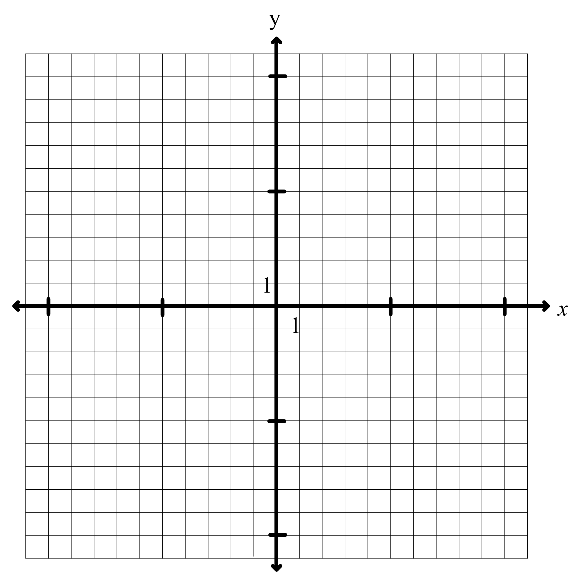
Unit 3 Review

1. Determine the maximum or minimum value of $y = \frac{1}{2}x^2 - 2x + 3$ and when it occurs.

2. Solve the following linear-quadratic system of equations and illustrate your solution graphically.

$y = -2(x+2)^2 + 8$

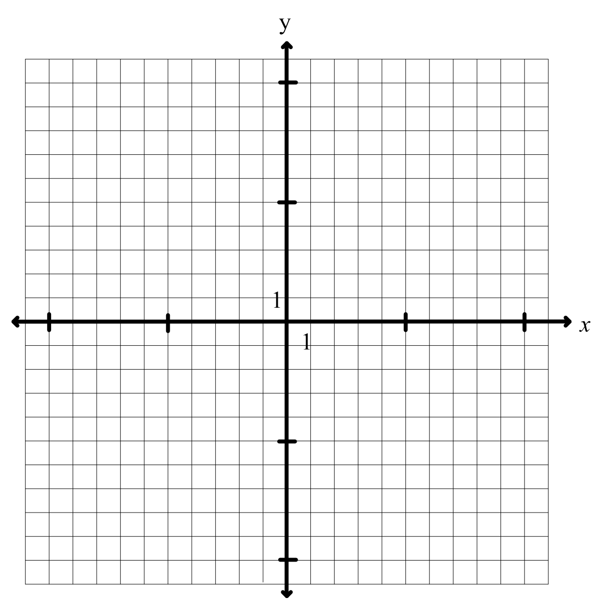
$y = -2x$



3. **Without solving**, determine the number of solutions to the following linear-quadratic system and illustrate your results graphically.

$y = \frac{1}{2}x^2 - 2x + 3$

$2x - 3y - 6 = 0$



4. Determine the value(s) of k that will give the indicated type of roots.

a) $(2k-3)x^2-6x+1=0$; ***two distinct real roots***

b) $x^2+kx+(8-k)=0$; ***two equal real roots***

5. Solve for $x \in C$.

a) $2(x-3)^2-32=0$

b) $\frac{2}{1-x}=\frac{3}{x+2}-1$

c) $x^4-x^2-72=0$

d) $2x^3-4x^2-50x+100=0$

Textbook Review: p. 128-130 #2dg, 14h, 33, 36c, 37a; p. 152 #20acfj; p. 155 #12-15; p. 158-159 #3-9; p. 693 #23bg