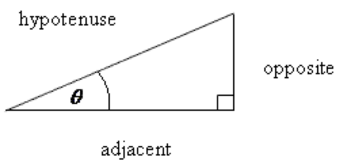
Reviewing the Trigonometry of Right Triangles

θ – theta
α – alpha
β – beta

A. Reviewing the Primary Trigonometric Ratios: SOH CAH TOA

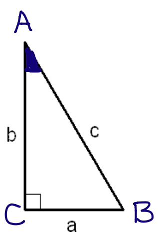
Recall that for any *right triangle*, we can write the three primary trigonometric ratios as equations in three unknowns, where the trig ratio of an angle is equal to one side divided by another side:



$\sin\theta = \dfrac{opp}{hyp}$ $\cos\theta = \dfrac{adj}{hyp}$ $\tan\theta = \dfrac{opp}{adj}$

Ex. 1: Write the primary trig ratios for i) ∠A and ii) ∠B in the triangle shown below:

i)

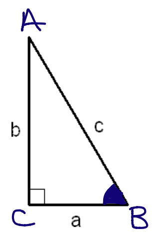


$\sin A = \dfrac{a}{c}$

$\cos A = \dfrac{b}{c}$

$\tan A = \dfrac{a}{b}$

ii)



$\sin B = \dfrac{b}{c}$

$\cos B = \dfrac{a}{c}$

$\tan B = \dfrac{b}{a}$

B. Solving for a Side Length

To solve for a **side length**, *clear the fraction* by multiplying both sides of the equation by the denominator or by cross-multiplying. Then, isolate the variable by *dividing out*, if necessary.	
$\tan 72° = \dfrac{y}{6.1}$ $6.1\tan 72° = y$ $\therefore y \doteq 18.8$ units	$\dfrac{\sin 58°}{1} = \dfrac{18.8}{x}$ $\dfrac{x\sin 58°}{\sin 58°} = \dfrac{18.8}{\sin 58°}$ $x = \dfrac{18.8}{\sin 58°}$ $x \doteq 22.2$ units

C. Solving for an Angle

To solve for an **angle** when the ratio is known, take the *inverse trig operation* of both sides:

$\cos C = \dfrac{7.9}{13.5}$

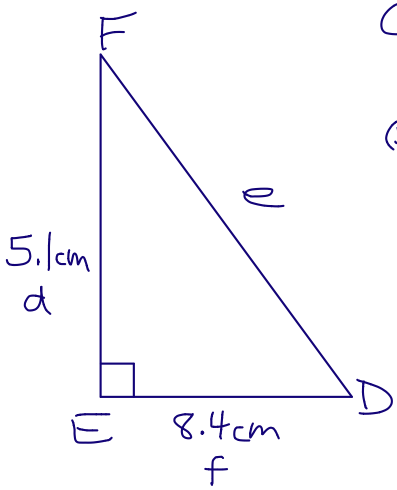
$\angle C = \cos^{-1}\left(\dfrac{7.9}{13.5}\right)$

$\angle C \doteq 54°$

D. Solving a Right Triangle

To ***solve*** a triangle means to find the value of *every side* and *every angle*. (Recall: the *Pythagorean Theorem* is $c^2 = a^2 + b^2$, where *c* is the hypoteneuse of a right triangle).

Ex. 2: Solve ΔDEF where $\angle E = 90°$, $d = 5.1$ cm and $f = 8.4$ cm. Round angles to the nearest degree and side lengths to one decimal place.



① Find e:

$d^2 + f^2 = e^2$

$(5.1)^2 + (8.4)^2 = e^2$

$e = \sqrt{(5.1)^2 + (8.4)^2}$

$e \doteq 9.8$ cm

② Find ∠D:

$\tan D = \frac{5.1}{8.4}$

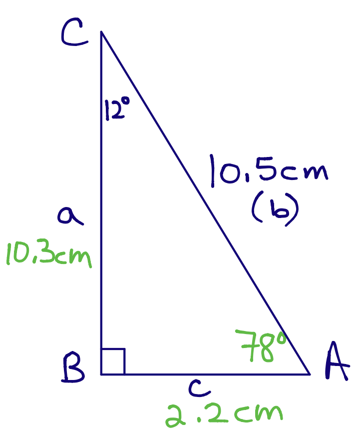
$\angle D = \tan^{-1}\left(\frac{5.1}{8.4}\right)$

$\angle D \doteq 31°$

③ $\angle F = 180° - 90° - 31°$

$\angle F = 59°$

Ex. 3: Solve ΔABC where $\angle B = 90°$, $\angle C = 12°$ and $b = 10.5$ cm. Round angles to the nearest degree and side lengths to one decimal place.



$\angle A = \underline{78°}$ $\quad a = \underline{10.3\text{ cm}}$

$\angle B = 90°$ $\quad b = 10.5$ cm

$\angle C = 12°$ $\quad c = \underline{2.2\text{ cm}}$

① $\angle A = 180° - 90° - 12°$

$\therefore \angle A = 78°$

② $\sin 12° = \frac{c}{10.5}$

$c = 10.5 \sin 12°$

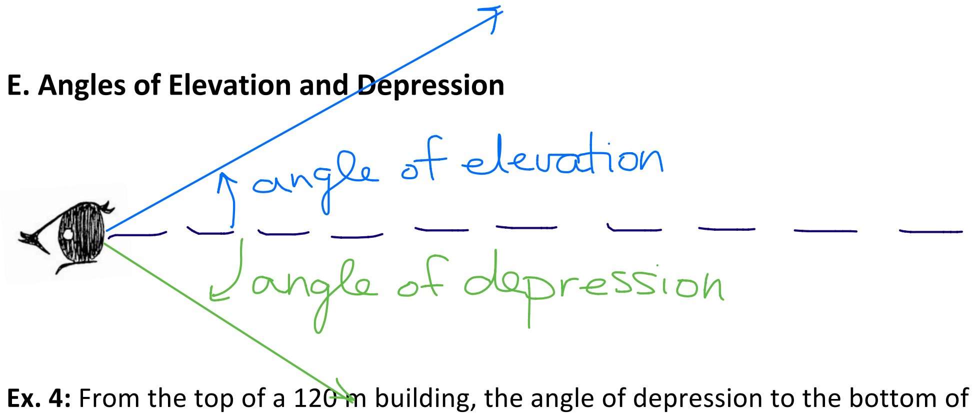
$\therefore c \doteq 2.2$ cm

③ $\cos 12° = \frac{a}{10.5}$

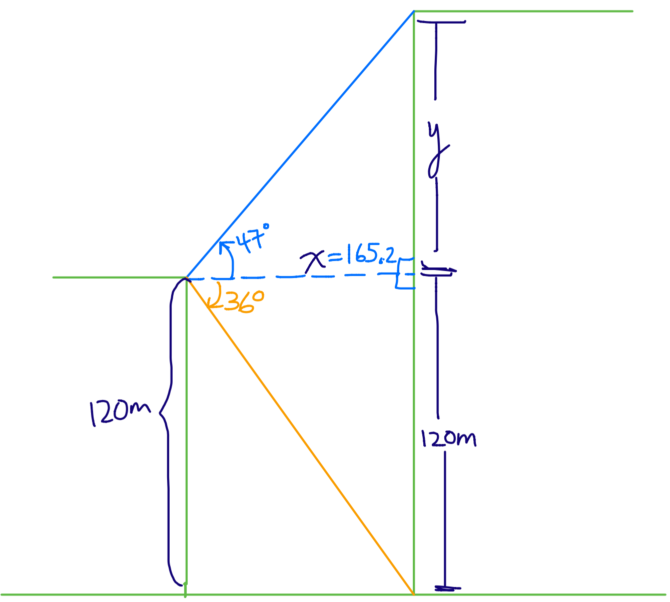
$a = 10.5 \cos 12°$

$\therefore a \doteq 10.3$ cm

E. Angles of Elevation and Depression



Ex. 4: From the top of a 120 m building, the angle of depression to the bottom of a second building is 36°, while the angle of elevation to the top of the second building is 47°. How far apart are the buildings? How tall is the second building?



Let x represent the distance between the buildings, in m.

Let $(y+120)$ represent the height of the second building, in m.

① Find x:

$$\tan 36^\circ = \frac{120}{x}$$

$$x \tan 36^\circ = 120$$

$$x = \frac{120}{\tan 36^\circ}$$

$$x \doteq 165.2$$

② Find y:

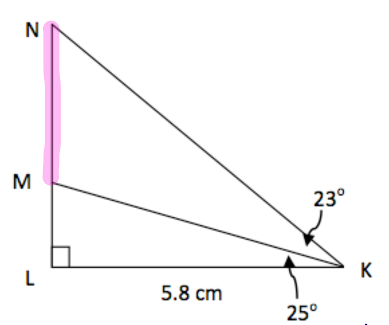
$$\tan 47^\circ = \frac{y}{165.2}$$

$$y = 165.2 \tan 47^\circ$$

$$y \doteq 177.2$$

∴ the buildings are 165.2 m apart and the second building is 297.2 m tall.

Ex. 5: Find MN.



① Find NL in △KLN

② Find ML in △KLM

③ MN = NL − ML

① Find ML:

$$\tan 25^\circ = \frac{ML}{5.8}$$

$$ML = 5.8 \tan 25^\circ$$

$$\boxed{\therefore ML \doteq 2.7 \text{ cm}}$$

② Find NL:

$$\tan 48^\circ = \frac{NL}{5.8}$$

$$NL = 5.8 \tan 48^\circ$$

$$\boxed{\therefore NL \doteq 6.4 \text{ cm}}$$

③ Find MN:

$$MN = 6.4 - 2.7$$

$$\boxed{\therefore MN = 3.7 \text{ cm}}$$

HW: p. 272-274 #1a, 2b, 3cd, 4a, 5- 8, 10, 11, 16, 18

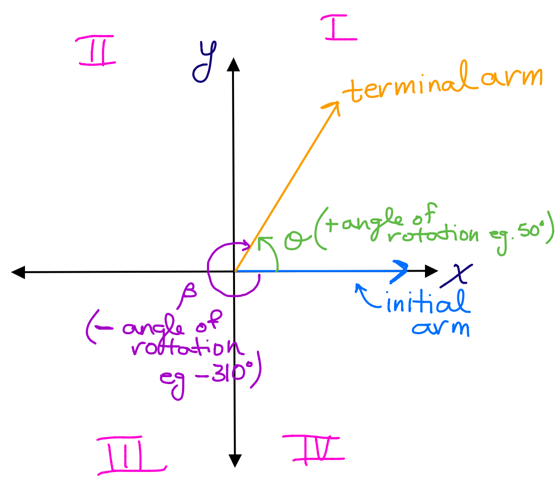
Angles in Standard Position and Radian Measure

Up until this point, we have studied triangle trigonometry – where any angle, θ, was defined as $\{0° < \theta < 180°\}$. We will now explore the trigonometric ratios for *any* size of angle, such that $\{\theta \in R\}$!

A. Angles in Standard Position

When drawing angles in on a coordinate plane in ***standard position***:

- The **vertex** is at the *origin.*
- The **initial arm** is on the *positive x-axis.*
- The **terminal arm** rotates through an angle about the origin:
  - if the direction of the rotation is counterclockwise, the measure of the angle is *positive*.
  - if the direction of the rotation is clockwise, the angle is *negative.*
- **Co-terminal Angles** are angles in standard position that have the *same terminal arm*.
  Note: To find co-terminal angles to θ, add *or* subtract 360° any number of times!

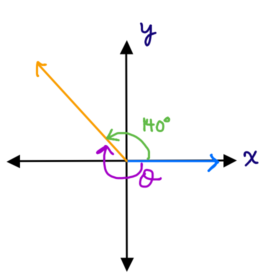


50° and −310° are co-terminal angles.

$$\begin{aligned} 50° - 360° &= -310° \\ \text{or } 50° + 360° &= 410° \\ 410° + 360° &= 770° \\ &\vdots \end{aligned}$$

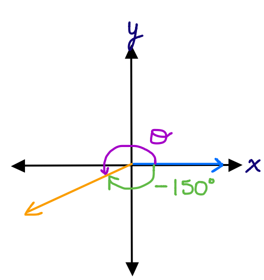
Ex. 1: Draw each angle in standard position. Name a **co-terminal** angle θ such that $-360° < \theta < 360°$.

a) 140°



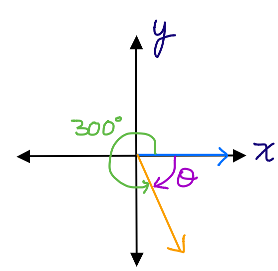
θ = $-220°$
$(140° - 360°)$

b) −150°



θ = $210°$
$(-150° + 360°)$

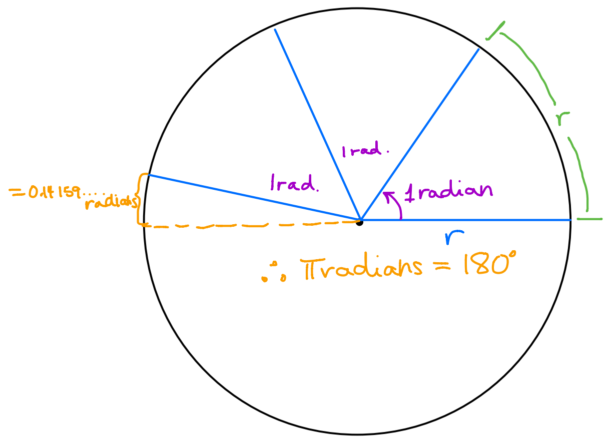
c) 300°



θ = $-60°$
$(300° - 360°)$

B. Radians

I) A **radian** is a unit, different from a degree, for measuring angles. One **radian** is "the measure of the *angle* subtended at the centre of a circle by an *arc equal in length to the radius* of the circle".



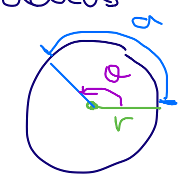
$\frac{\pi}{2}$ radians = 90°

$\boxed{\pi \text{ radians} = 180°}$

2π radians = 360°

*an angle θ in radians is found by dividing the arc length of a circle by its radius.

$\therefore \theta = \frac{a}{r}$



II) ***Converting Between Degrees and Radians:***

Since **180° = π radians**, $1° = \frac{\pi}{180} rad$ and $1 rad = \frac{180°}{\pi}$.

Therefore, we can use the conversion factors $\frac{180°}{\pi \ rad}$ and $\frac{\pi \ rad}{180°}$ to convert between units.

Ex. 2: Convert from degrees to radians. Give *exact* and *approximate* (*to 2 decimal places*) answers.

a) 45°

$= \cancel{45°}^{1} \times \frac{\pi}{\cancel{180°}_{4}}$

$= \frac{\pi}{4}$

$\doteq 0.76$

b) 72°

$= \cancel{72°}_{2} \times \frac{\pi}{\cancel{180°}_{5}}$

$= \frac{2\pi}{5}$

$\doteq 1.26$

c) 315°

$= \cancel{315°}_{7} \times \frac{\pi}{\cancel{180°}_{4}}$

$= \frac{7\pi}{4}$

$\doteq 5.50$

Ex. 3: Convert from radians to degrees. Give answers to one decimal place.

a) $\frac{\pi}{4}$ radians

$= \frac{\cancel{\pi}^{1}}{\cancel{4}_{1}} \times \frac{\overset{45°}{\cancel{180°}}}{\cancel{\pi}_{1}}$

$= 45°$

b) 4.3 radians

$= 4.3 \times \frac{180°}{\pi}$

$\doteq 246.4°$

c) $\frac{7\pi}{6}$

$= \frac{7\cancel{\pi}^{1}}{\cancel{6}_{1}} \times \frac{\overset{30°}{\cancel{180°}}}{\cancel{\pi}_{1}}$

$= 210°$

C. Problems Involving Arc Length

I) Recall: $\theta = \frac{a}{r}$, where θ is the measure of the angle in radians; a is the arc length; and r is the radius of the circle.

Ex. 4: Given a circle with the following measurements, find the unknown value.

a) $\theta = 2$ rad
$r = 6$ cm
$a = ?$

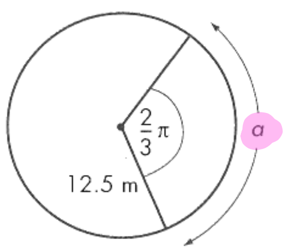
$$\theta = \frac{a}{r}$$
$$2 = \frac{a}{6}$$
$$a = 2(6)$$
$$\therefore a = 12\text{cm}$$

b) $\theta = ?$
$r = 10$ cm
$a = 45$ cm

$$\theta = \frac{a}{r}$$
$$\theta = \frac{45}{10}$$
$$\theta = 4.5 \text{ rad.}$$

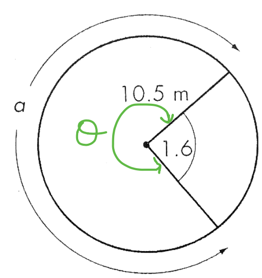
Ex. 5: Find the indicated quantity in each of the following diagrams.

a)



$$\theta = \frac{a}{r}$$
$$\frac{2\pi}{3} = \frac{a}{12.5}$$
$$a = 12.5\left(\frac{2\pi}{3}\right) \quad \text{or } \underbrace{a = \frac{25\pi}{3} \text{ m}}_{\text{exact value}}$$
$$a \doteq 26.2\text{m}$$

b)



Find θ:
$$\theta = 2\pi - 1.6$$
$$\theta \doteq 4.7 \text{ rad}$$

Find a:
$$\theta = \frac{a}{r}$$
$$4.7 = \frac{a}{10.5}$$
$$a = 10.5(4.7)$$
$$a \doteq 49.4 \text{ m}$$

II) *Angular Velocity:*

Ex. 6: An electric motor turns at 2000 revolutions/minute. Find the angular velocity in radians/second. Give an exact answer and an approximate answer.

$$2000 \frac{\cancel{\text{rev}}}{\cancel{\text{minute}}} \times \frac{2\pi}{1} \frac{\text{rad}}{\cancel{\text{rev}}} \times \frac{\cancel{1 \text{ min}}}{60\text{sec}} = \frac{2000(2\pi) \text{ radians}}{60 \text{ seconds}}$$
$$= \frac{200\pi}{3} \text{ rad/sec. (exact)}$$
$$\doteq 209.4 \text{ rad/sec.}$$

HW: p. 334-339 #1acej, 2aceg, 3ac, 4cegi, 5a, 6a, 7a, 8b, 9, 11, 13, 18, 20a, 22

Angles on a Coordinate Plane

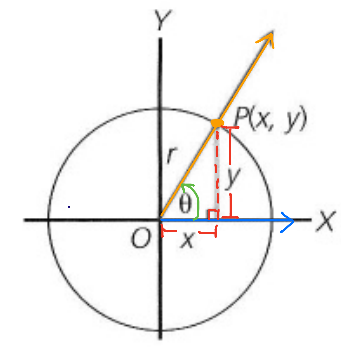
A. Primary Trigonometric Ratios for Any Angle in Standard Position

For an angle, θ, in standard position, with **point P(*x, y*)** on the terminal arm: $r=\sqrt{x^2+y^2}$, $r\geq 0$

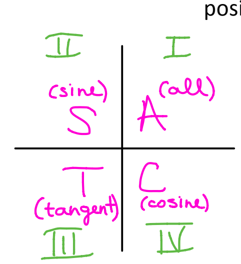
$$\sin\theta=\frac{y}{r}$$

$$\cos\theta=\frac{x}{r}$$

$$\tan\theta=\frac{y}{x}$$



The **CAST rule** gives the initials of the trigonometric ratios that are positive in each quadrant:



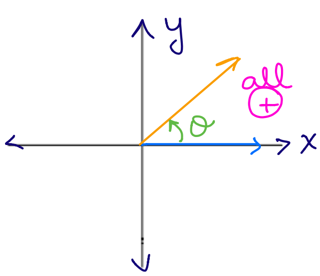
Function	Quadrant			
	I	II	III	IV
sine	+	+	–	–
cosine	+	–	–	+
tangent	+	–	+	–

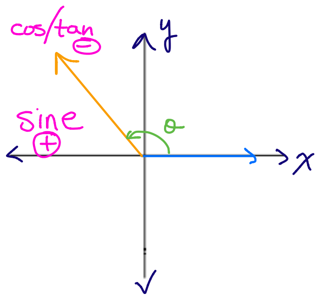
Note: The value of the "hypoteneuse", *r*, is always positive, as it is the **radius** of a circle that would have **P(*x, y*)** on its **circumference**.

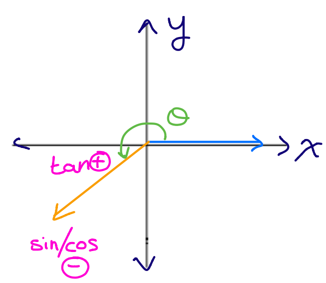
B. The CAST Rule

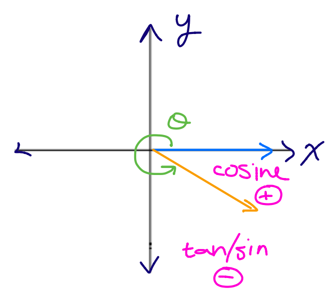
The coordinate plane can be divided into four **quadrants** using the *x*- and *y*-axes. The three primary trig ratios take on *different signs* depending on which quadrant the terminal arm is found, as summarized by the **CAST Rule.**

QI = {0° < θ < 90°} QII = {90° < θ < 180°} QIII = {180° < θ < 270°} QIV = {270° < θ < 360°}







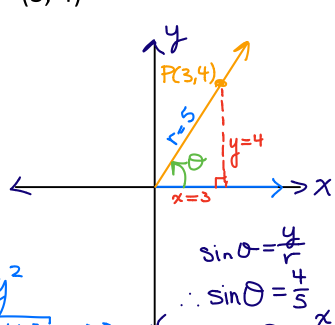


Ex. 1: Use the CAST rule to determine which quadrant(s) the terminal arm could be found for each trigonometric ratio.

a) sin θ = 0.5 Q: I, II b) cos θ = – 0.6432 Q: II, III c) tan θ = – 1.2 Q: II, IV

Ex. 2: Find the exact primary trig ratios for each point on the terminal arm of θ, an angle in standard position.

a) (3, 4)



$r^2=x^2+y^2$

$r=\sqrt{x^2+y^2}$, $r\geq 0$

$r=\sqrt{(3)^2+(4)^2}$

$r=\sqrt{25}$

$r=5$

$\sin\theta=\frac{y}{r}$

$\therefore \sin\theta=\frac{4}{5}$

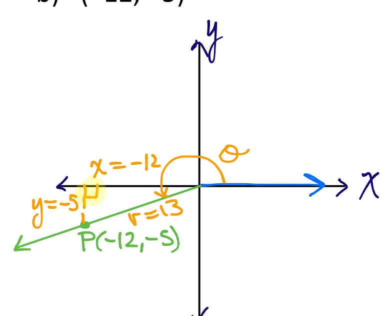
$\cos\theta=\frac{x}{r}$

$\therefore \cos\theta=\frac{3}{5}$

$\tan\theta=\frac{y}{x}$

$\therefore \tan\theta=\frac{4}{3}$

b) (–12, –5)



$r=\sqrt{x^2+y^2}$

$r=\sqrt{(-12)^2+(-5)^2}$

$r=13$

$\sin\theta=\frac{y}{r}$

$\therefore \sin\theta=\frac{-5}{13}$

$\cos\theta=\frac{x}{r}$

$\therefore \cos\theta=-\frac{12}{13}$

$\tan\theta=\frac{y}{x}$

$\therefore \tan\theta=+\frac{5}{12}$

Positive rotation
single rotation
working in radians

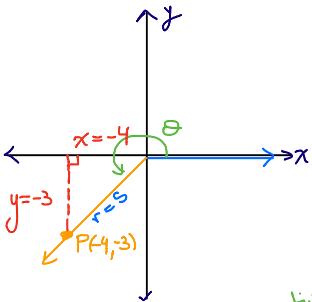
Ex. 3: For each example below, $\angle\theta$ is given in standard position, and $0 \le \theta \le 2\pi$. A trigonometric ratio is given. Find all possible *exact values* of the other two trigonometric ratios.

a) $\sin\theta = -\frac{3}{5}$ and the terminal arm of $\angle\theta$ is in quadrant III

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\therefore \cos\theta = -\frac{4}{5}$



$\tan\theta = \frac{y}{x}$

$\therefore \tan\theta = +\frac{3}{4}$

$x^2+y^2=r^2$

x is negative in QIII.

$x = -\sqrt{r^2-y^2}$

$x = -\sqrt{(5)^2-(-3)^2}$

$x = -4$

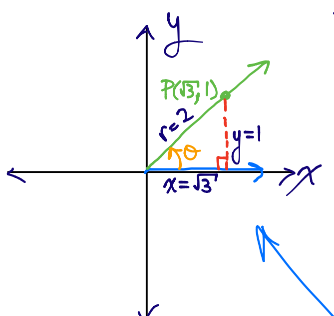
b) $\tan\theta = \frac{1}{\sqrt{3}} \rightarrow$ QI or III, $\tan\theta = \frac{y}{x}$

QI:

$x = \sqrt{3}$

$y = 1$

$r = 2$

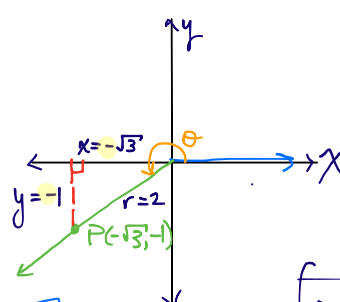


QIII:

$x = -\sqrt{3}$

$y = -1$

$r = 2$



$r = \sqrt{(\sqrt{3})^2 + (1)^2}$

$r = \sqrt{4}$

$r = 2$

$\boxed{\begin{array}{l} \therefore \text{In QI:} \\ \sin\theta = \frac{1}{2} \\ \cos\theta = \frac{\sqrt{3}}{2} \end{array}}$

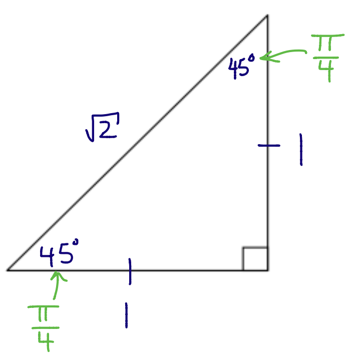
$\boxed{\begin{array}{l} \therefore \text{In QIII:} \\ \sin\theta = -\frac{1}{2} \\ \cos\theta = -\frac{\sqrt{3}}{2} \end{array}}$

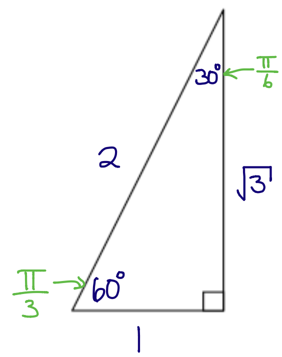
HW: p. 348-349 #1, 2, 6, 7 (Sketch the diagrams!) for all questions.

Special Angles: Special Triangles and the Unit Circle

We have started to investigate the trigonometric ratios for any angle. We can further our discussion by examining the **special angles** using *special triangles* and the *unit circle*.

A. Special Triangles:

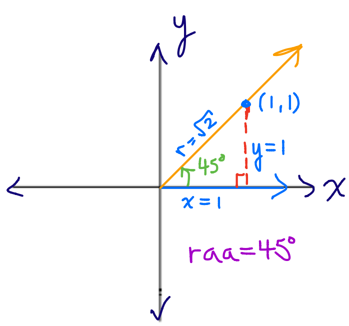




The **reference angle** (or *related acute angle* ***r.a.a.)*** is the acute angle made **between** the <u>*x*-axis and the terminal arm</u> of an angle in standard position. When the reference angle is 30°, 45°, or 60°, we can use the special triangles to find the **exact values** (i.e. fractions – no decimals!) of the trig ratios. ☺

Ex. 1: Use special triangles and reference angles to find the **exact values** of the trig ratios for the following angles.

a) $\theta = 45°$

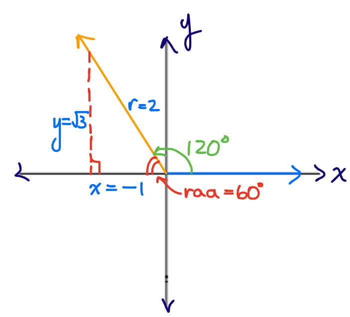


$\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 45° = 1$

b) $\theta = 120°$



$\sin 120° = \frac{\sqrt{3}}{2}$

$\cos 120° = -\frac{1}{2}$

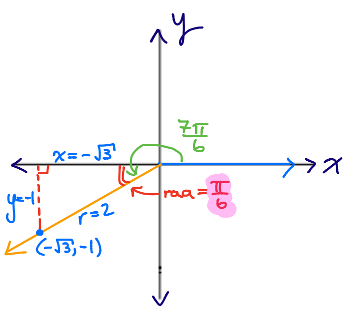
$\tan 120° = -\sqrt{3}$ $\left(\frac{\sqrt{3}}{-1}\right)$

c) $\theta = \frac{7\pi}{6}$

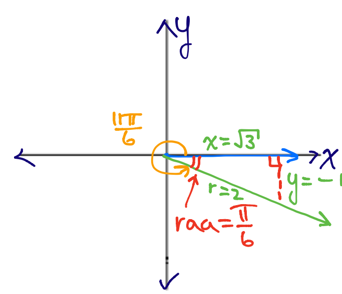
$\sin \frac{7\pi}{6} = -\frac{1}{2}$

$\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$\tan \frac{7\pi}{6} = +\frac{1}{\sqrt{3}}$ $\left(\frac{-1}{-\sqrt{3}}\right)$



d) $\theta = \frac{11\pi}{6}$



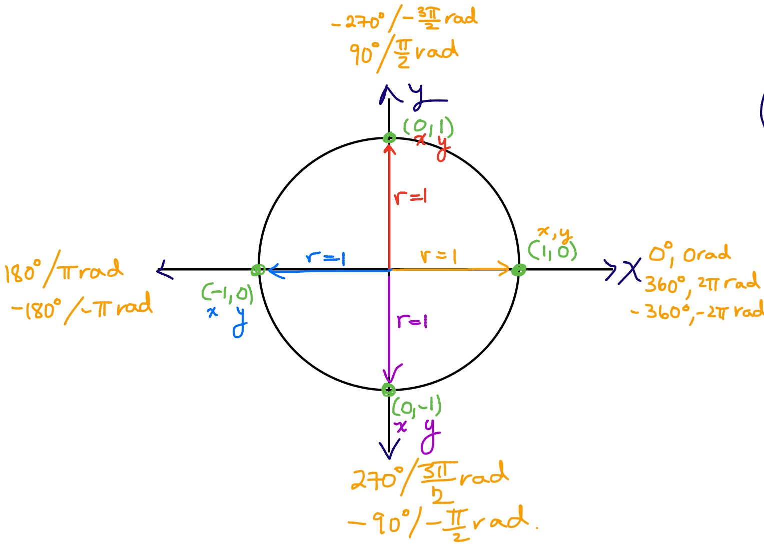
$\sin \frac{11\pi}{6} = -\frac{1}{2}$

$\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan \frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

B. The Unit Circle (a circle with radius = 1)

Previously, we have investigated angles on the coordinate plane *in the four quadrants*, but what about angles that lie _on the axes_? Enter the unit circle!



i) $\sin 90° = \frac{1}{1}$

$\left(\sin\theta = \frac{y}{r}\right)$

$\therefore \sin 90° = 1$

ii) $\tan\frac{3\pi}{2} = \frac{-1}{0}$

$\left(\tan\theta = \frac{y}{x}\right)$

$\therefore \tan\frac{3\pi}{2}$ is undefined.

iii) $\cos\pi = \frac{-1}{1}$

$\left(\cos\theta = \frac{x}{r}\right)$

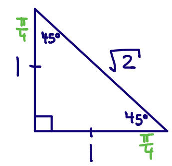
$\therefore \cos\pi = -1$

iv) $\cos(-270°) = \frac{0}{1}$

$\left(\cos\theta = \frac{x}{r}\right)$

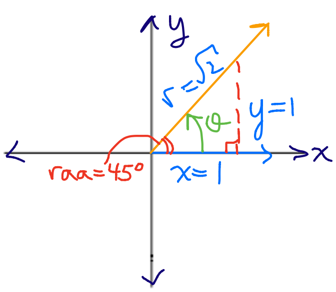
$\therefore \cos(-270°) = 0$

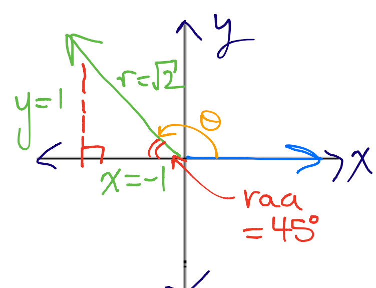
Ex. 2: If $\sin\theta = \frac{1}{\sqrt{2}}$, $0° \le \theta \le 360°$, find the values of $\cos\theta$ and θ



QI, II

raa = 45°





In QI:

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \text{raa}$

$\therefore \theta = 45°$

In QII:

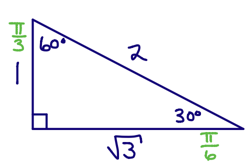
$\cos\theta = \frac{-1}{\sqrt{2}}$

$\theta = 180° - \text{raa}$

$\theta = 180° - 45°$

$\therefore \theta = 135°$

Ex. 3: Find the *exact* value of 2sin30°cos30°.



$2\sin 30°\cos 30°$

$= \cancel{2}\left(\frac{1}{\cancel{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$

$\boxed{\therefore 2\sin 30°\cos 30° = \frac{\sqrt{3}}{2}}$

HW: p. 348-349 #3abc, 4aef, 11-14, 18ace