Trigonometric Ratios for Positive and Negative Angles of Rotation

Find the <u>exact</u> values of the primary trig ratios for the following angles in standard position.

Steps: *1) Draw the angle in standard position*

2) If the terminal arm is on one of the axes, use the unit circle to find the exact values for the required ratios

3) If the terminal arm is in one of the four quadrants:

i) add or subtract 360° or 2π to determine the angle between the terminal arm and the x-axis

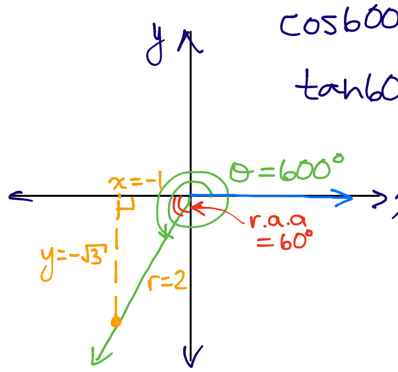
ii) Use the related acute angle and the CAST rule to determine the exact values for the required ratios

a) $\theta = 600°$

$\sin 600° = -\frac{\sqrt{3}}{2}$

$\cos 600° = -\frac{1}{2}$

$\tan 600° = \sqrt{3}$

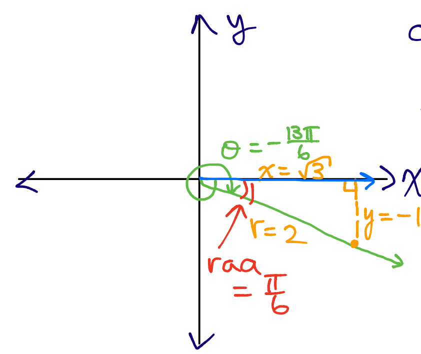


b) $\theta = -\frac{13\pi}{6}$

$\sin\left(-\frac{13\pi}{6}\right) = -\frac{1}{2}$

$\cos\left(-\frac{13\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\tan\left(-\frac{13\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

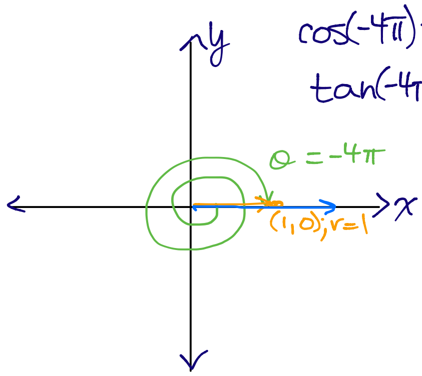


c) $\theta = -4\pi$

$\sin(-4\pi) = 0$

$\cos(-4\pi) = 1$

$\tan(-4\pi) = 0$

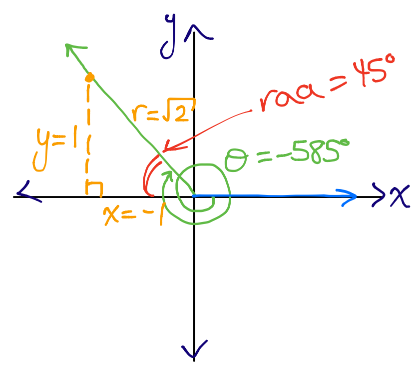


d) $\theta = -585°$

$\sin(-585°) = \frac{1}{\sqrt{2}}$

$\cos(-585°) = -\frac{1}{\sqrt{2}}$

$\tan(-585°) = -1$



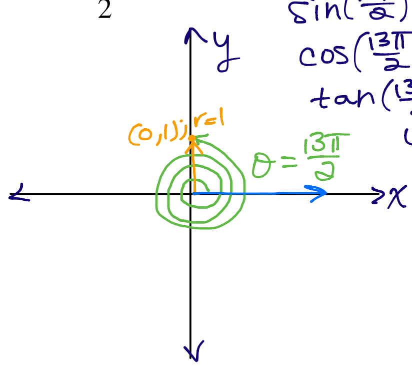
e) $\theta = \frac{13\pi}{2}$

$\sin\left(\frac{13\pi}{2}\right) = 1$

$\cos\left(\frac{13\pi}{2}\right) = 0$

$\tan\left(\frac{13\pi}{2}\right) = \frac{1}{0}$

↳ undefined.



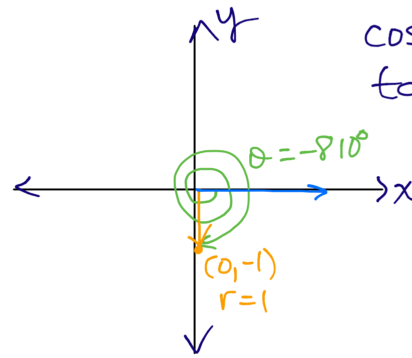
f) $\theta = -810°$

$\sin(-810°) = -1$

$\cos(-810°) = 0$

$\tan(-810°) = -\frac{1}{0}$

↳ undefined



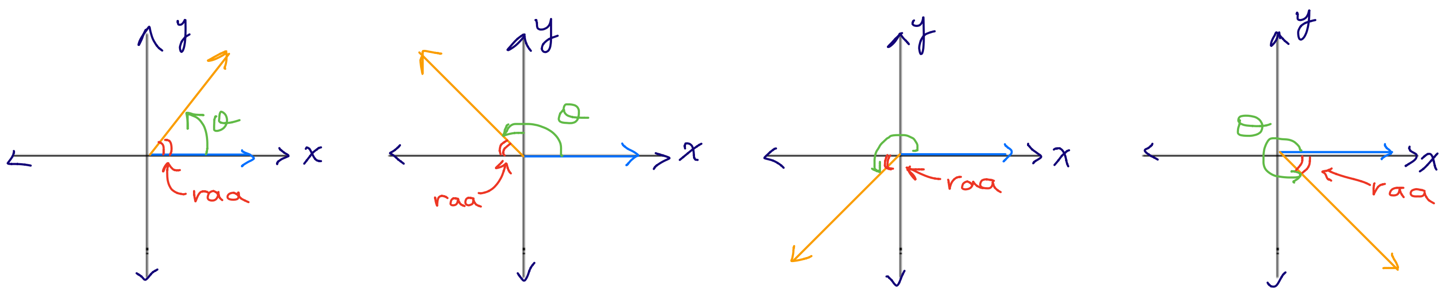
HW: p. 353 # 1, 2 and p. 354 # 3, 4

Combining the CAST Rule, Related Acute Angles, and Special Triangles

Steps for finding the possible value(s) of θ:

- determine the possible *quadrants* for the terminal arm using the **CAST rule**
- sketch the *terminal arm* in the appropriate quadrant(s)
- find the ***r.a.a.*** for θ using ***special angles*** or *inverse trig operations* on the value of the ratio (do not include the negative sign for this calculation – we are looking for the related *acute* angle!)
- add or subtract the *r.a.a.* from 180° (π) or 360° (2π) to get θ, the positive angle between the initial arm and the terminal arm for each quadrant
- provide a therefore statement for each quadrant

Given any trig ratio, we can find all angles that will satisfy the equation for $0° \le \theta \le 360°$ or $0 \le \theta \le 2\pi$.



QI: θ = raa

QII: θ = π − raa
$\theta = 180° -$ raa

QIII: θ = π + raa
$\theta = 180° +$ raa

QIV: θ = 2π − raa
$\theta = 360° -$ ra

A. Find θ, an angle in standard position, for $0° \le \theta \le 360°$. Round to the nearest whole degree, where necessary.

1. $\cos\theta = \frac{1}{2}$, raa = 60°
QI, QIV

QI:
$\theta = 60°$

QIV:
$\theta = 360° - 60°$
$\theta = 300°$

2. $\tan\theta = -2.3546$ QII, QIV
raa $= \tan^{-1}(+2.3546)$
$\therefore$ raa $\doteq 67°$

QII:
$\theta = 180° - 67°$
$\theta = 113°$

QIV:
$\theta = 360° - 67°$
$\theta = 293°$

3. $\tan\theta = -1$

4. $\sin\theta = 0.1357$

5. $\cos\theta = -0.8436$

6. $\sin\theta = \frac{\sqrt{3}}{2}$

B. Find θ, an angle in standard position, for $0 \le \theta \le 2\pi$. Give the exact solution, where possible. Otherwise, round to the nearest hundredth of a radian.

1. $\sin\theta = -\frac{1}{2}$, QIII, QIV, raa $= \frac{\pi}{6}$

   QIII:
   $\theta = \pi + \frac{\pi}{6}$
   $\theta = \frac{6\pi}{6} + \frac{\pi}{6}$
   $\theta = \frac{7\pi}{6}$

   QIV:
   $\theta = 2\pi - \frac{\pi}{6}$
   $\theta = \frac{12\pi}{6} - \frac{\pi}{6}$
   $\theta = \frac{11\pi}{6}$

2. $\tan\theta = 3.4$, QI, QIII
   raa $= \tan^{-1}(3.4)$
   raa $\doteq 1.28$

   QI:
   $\theta \doteq 1.28$ rad

   QIII:
   $\theta = \pi + 1.28$
   $\theta \doteq 4.42$ rad

3. $\cos\theta = \frac{1}{\sqrt{2}}$

4. $\tan\theta = -\sqrt{3}$

5. $\cos\theta = -0.9205$

6. $\sin\theta = \frac{1}{\sqrt{3}}$

C. Combine the CAST Rule, related acute angles, and special triangles ***or*** use the unit circle to find exact values of the following. **Rationalize the denominator when necessary.*

1. $(\sin 135°)(\cos 240°)$ QII, raa=45° QIII, raa=60°

   $= \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right)$

   $= -\frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

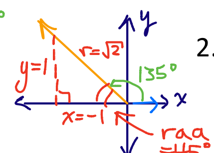
   $= \frac{-\sqrt{2}}{2\sqrt{4}}$

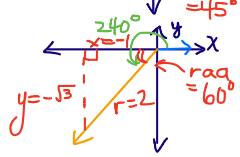
   $= -\frac{\sqrt{2}}{4}$

   or $= (+\sin 45°)(-\cos 60°)$

   $= \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right)$

   ⋮





2. $\tan\frac{11\pi}{6} - \cos\left(-\frac{3\pi}{4}\right)$ QIV, raa=$\frac{\pi}{6}$ QIII, raa $\frac{\pi}{4}$

   $= -\tan\frac{\pi}{6} - \left(-\cos\frac{\pi}{4}\right)$

   $= -\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{2}}\right)$

   $= -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

   $= \frac{-\sqrt{3}\,(2)}{3\,(2)} + \frac{\sqrt{2}\,(3)}{2\,(3)}$

   $= \frac{-2\sqrt{3} + 3\sqrt{2}}{6}$

3. $\left(\cos\frac{11\pi}{6}\right)^2 + \left(\sin\frac{3\pi}{2}\right)^2$

4. $\tan(-300°) \cdot \cos(-210°)$

5. $2(\cos 2\pi) + \frac{1}{\left(\cos\frac{7\pi}{3}\right)}$

6. $\left(\sin\frac{5\pi}{4}\right)^2 + \left(\cos\frac{5\pi}{4}\right)^2$

7. $\tan(-510°) + (\sin 600° + \tan 540°)$

8. $\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) + \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$

9. $\frac{\tan\frac{\pi}{3} + \tan\frac{7\pi}{4}}{1 - \left(\tan\frac{\pi}{3}\right)\left(\tan\frac{7\pi}{4}\right)}$

Answers:

Part A:

1. $\cos\theta = \dfrac{1}{2}$

Q: I and IV, r.a.a. $= 60^o$

$\therefore \theta = 60° \text{ or } 300°$

2. $\tan\theta = -2.3546$

Q: III and IV, r.a.a. $\doteq 67^o$

$\therefore \theta = 113° \text{ or } 293°$

3. $\tan\theta = -1$

Q: II and IV, r.a.a. $= 45^o$

$\therefore \theta = 135° \text{ or } 315°$

4. $\sin\theta = 0.1357$

Q: I and II, r.a.a. $\doteq 8^o$

$\therefore \theta = 8° \text{ or } 172°$

5. $\cos\theta = -0.8436$

Q: II and III, r.a.a. $\doteq 32^o$

$\therefore \theta = 148° \text{ or } 212°$

6. $\sin\theta = \dfrac{\sqrt{3}}{2}$

Q: I and II, r.a.a. $= 60^o$

$\therefore \theta = 60° \text{ or } 120°$

Part B:

1. $\sin\theta = -\dfrac{1}{2}$

Q: III and IV, r.a.a. $= \dfrac{\pi}{6}$

$\therefore \theta = \dfrac{7\pi}{6} \text{ or } \dfrac{11\pi}{6}$

2. $\tan\theta = 3.4$

Q: I and III, r.a.a. $\doteq 1.28$

$\therefore \theta \doteq 1.28 \text{ or } 4.42$

3. $\cos\theta = \dfrac{1}{\sqrt{2}}$

Q: I and IV, r.a.a. $= \dfrac{\pi}{4}$

$\therefore \theta = \dfrac{\pi}{4} \text{ or } \dfrac{7\pi}{4}$

4. $\tan\theta = -\sqrt{3}$

Q: II and IV, r.a.a. $= \dfrac{\pi}{3}$

$\therefore \theta = \dfrac{2\pi}{3} \text{ or } \dfrac{5\pi}{3}$

5. $\cos\theta = -0.9205$

Q: II and III, r.a.a. $\doteq 0.40$

$\therefore \theta \doteq 2.74 \text{ or } 3.54$

6. $\sin\theta = \dfrac{1}{\sqrt{3}}$

Q: I and II, r.a.a. $\doteq 0.62$

$\therefore \theta \doteq 0.62 \text{ or } 2.52$

Part C:

1. $(\sin 135°)(\cos 240°)$

$= -\dfrac{\sqrt{2}}{4}$

2. $\tan\dfrac{11\pi}{6} - \cos\left(-\dfrac{3\pi}{4}\right)$

$= \dfrac{3\sqrt{2} - 2\sqrt{3}}{6}$

3. $\left(\cos\dfrac{11\pi}{6}\right)^2 + \left(\sin\dfrac{3\pi}{2}\right)^2$

$= \dfrac{7}{4}$

4. $\tan(-300°)\cdot\cos(-210°)$

$= -\dfrac{3}{2}$

5. $2(\cos 2\pi) + \dfrac{1}{\left(\cos\dfrac{7\pi}{3}\right)}$

$= 4$

6. $\left(\sin\dfrac{5\pi}{4}\right)^2 + \left(\cos\dfrac{5\pi}{4}\right)^2$

$= 1$

7. $\tan(-510°) + (\sin 600° + \tan 540°)$

$= -\dfrac{\sqrt{3}}{6}$

8. $\left(\cos\dfrac{\pi}{3}\right)\left(\cos\dfrac{\pi}{4}\right) + \left(\sin\dfrac{\pi}{3}\right)\left(\sin\dfrac{\pi}{4}\right)$

$= \dfrac{\sqrt{2} + \sqrt{6}}{4}$

9. $\dfrac{\tan\dfrac{\pi}{3} + \tan\dfrac{7\pi}{4}}{1 - \left(\tan\dfrac{\pi}{3}\right)\left(\tan\dfrac{7\pi}{4}\right)}$

$= 2 - \sqrt{3}$

The Cosine Law

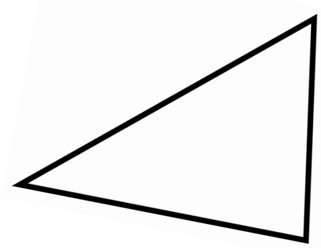
The Cosine Law is another expansion of a fundamental trig ratio into a law that can be used on any triangle (non-right triangles included)! It represents a general version of the Pythagorean Theorem, adapted to non-right triangles. It can also be rearranged to solve for the unknown angle.

The Cosine Law	***Rearranged to Solve for the Angle***
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

When do we use the *Cosine Law*? when you do not have a side/angle pair.

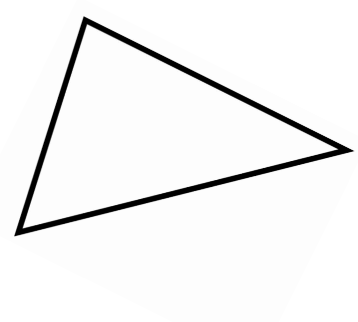
1. When we are given two sides and a contained angle, and we want to find the side opposite the angle.

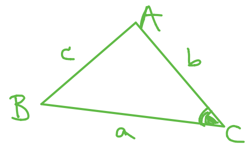
   SAS



2. When we are given three sides, and we want to find any other angle.
   (When given a choice, find the largest angle first!)

   SSS



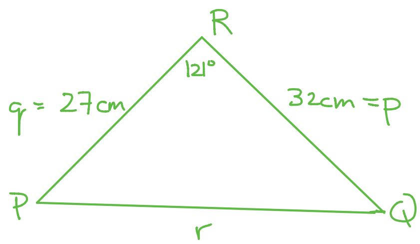


$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$c^2 = a^2+b^2-2ab\cos C$$

1. Two Sides and a Contained Angle

Solve ΔPQR, where $\angle R = 121^{\circ}$, p = 32 cm, and q = 27 cm



① Find r:

$$r^2 = p^2+q^2-2pq\cos R$$

$$r^2 = (32)^2+(27)^2-2(32)(27)\cos 121^{\circ}$$

$$r^2 =$$

$$r = \sqrt{32^2+27^2-2(32)(27)\cos 121^{\circ}}$$

$$\boxed{r \doteq 51.4 \text{ cm}}$$

② Find ∠P:

$$\cos P = \frac{q^2+r^2-p^2}{2qr}$$

$$\cos P = \frac{(27)^2+(51.4)^2-(32)^2}{2(27)(51.4)}$$

$$\cos P =$$

$$\angle P = \cos^{-1}\left[\frac{27^2+51.4^2-32^2}{2(27)(51.4)}\right]$$

$$\boxed{\angle P \doteq 32^{\circ}}$$

③ Find ∠Q

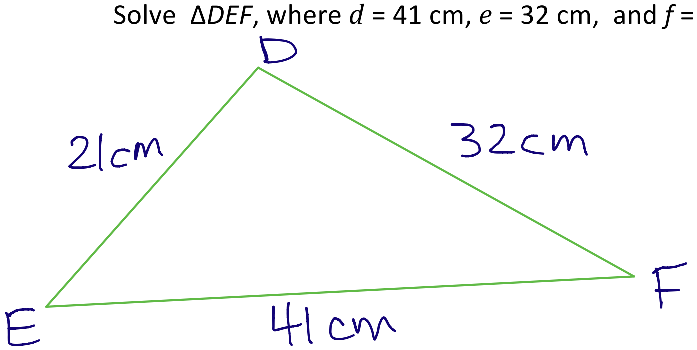
$$\angle Q = 180^{\circ}-32^{\circ}-121^{\circ}$$

$$\boxed{\angle Q = 27^{\circ}}$$

2. Three Sides

Solve ΔDEF, where d = 41 cm, e = 32 cm, and f = 21 cm

* Find the largest angle first!



① Find ∠D first:

$$\cos D = \frac{e^2+f^2-d^2}{2ef}$$

$$\cos D = \frac{(32)^2+(21)^2-(41)^2}{2(32)(21)}$$

$$\angle D = \cos^{-1}\left(\frac{-216}{1344}\right)$$

$$\boxed{\angle D \doteq 99^{\circ}}$$

② Find ∠E:

$$\cos E = \frac{f^2+d^2-e^2}{2fd}$$

$$\cos E = \frac{(41)^2+(21)^2-(32)^2}{2(41)(21)}$$

$$\angle E = \cos^{-1}\left(\frac{1098}{1722}\right)$$

$$\boxed{\angle E \doteq 50^{\circ}}$$

③ Find ∠F:

$$\angle F = 180^{\circ}-99^{\circ}-50^{\circ}$$

$$\boxed{\angle F = 31^{\circ}}$$

HW: p. 290-293 #1bf, 2ce, 3cef, 4b, 5bd, 8, 10, 12

The Sine Law

The Sine Law is an expansion of a fundamental trig ratio into a law that can be used on <u>any</u> triangle (non-right triangles included)! It states that the sines of the angles of a triangle are proportional to the lengths of their opposite sides.

The Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

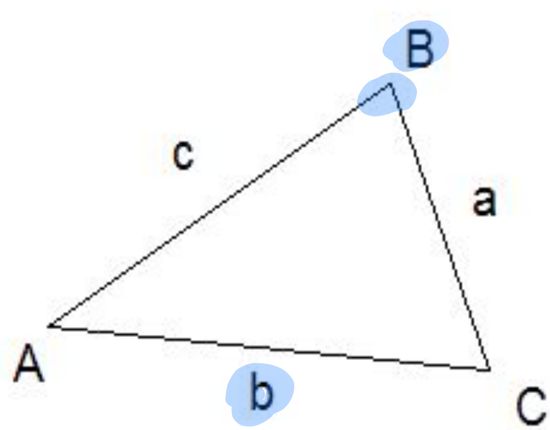
The Sine Law can be written in two forms, as the relationship represents a proportion. It is easiest to solve a proportion when **the unknown is in the numerator**, so when the unknown is an <u>angle</u>, use the <u>first form</u>, and when the unknown is a *side length*, use the *second form*.

When can we use the *Sine Law*?

1. When we are given **two angles and a side** for an acute triangle. (We use the angle sum of 180^o to find the third angle, and then set up a proportion to find any side we want!)

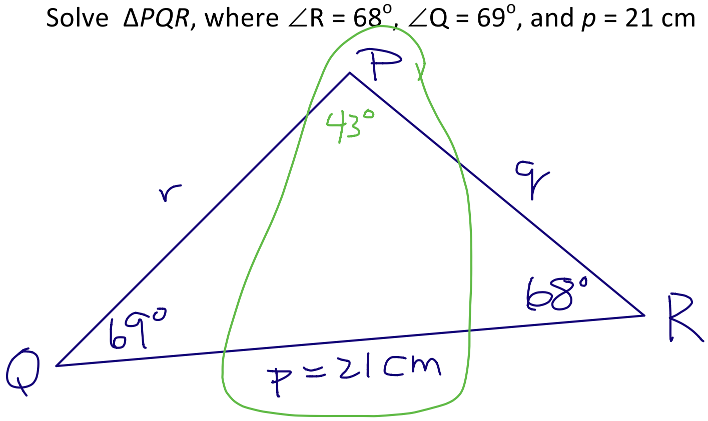
2. When we are given **two sides and an angle opposite one of them**. (We set up a proportion to find the second angle.)

Basically, we can use the ***<u>Sine Law</u>*** *when we have a* **corresponding side-angle pair plus one other piece of information** *about the triangle.*



1. Two Angles and a Side

Solve ΔPQR, where $\angle R = 68^{\circ}$, $\angle Q = 69^{\circ}$, and p = 21 cm



① Find $\angle P$:

$\angle P = 180^{\circ} - 69^{\circ} - 68^{\circ}$

$\boxed{\angle P = 43^{\circ}}$

② Find q:

$\frac{p}{\sin P} = \frac{q}{\sin Q}$

$\frac{21}{\sin 43^{\circ}} = \frac{q}{\sin 69^{\circ}}$

$\frac{21 \sin 69^{\circ}}{\sin 43^{\circ}} = q$

$\boxed{q \doteq 28.7 \text{ cm}}$

③ Find r:

$\frac{p}{\sin P} = \frac{r}{\sin R}$

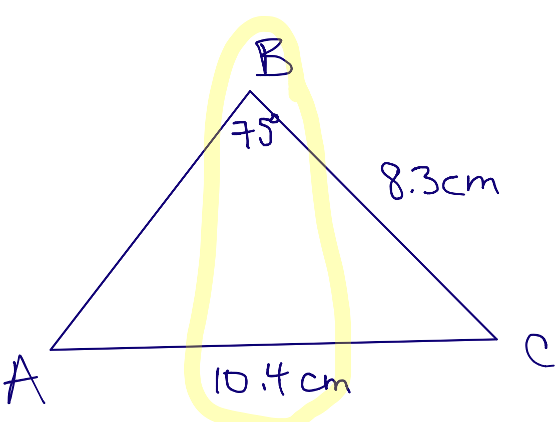
$\frac{21}{\sin 43^{\circ}} = \frac{r}{\sin 68^{\circ}}$

$\frac{21 \sin 68^{\circ}}{\sin 43^{\circ}} = r$

$\boxed{r \doteq 28.5 \text{ cm}}$

2. Two Sides and an Angle Opposite One of Them

Solve ΔABC, where $\angle B = 75^{\circ}$, a = 8.3 cm, and b = 10.4 cm



① Find $\angle A$

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin A}{8.3} = \frac{\sin 75^{\circ}}{10.4}$

$\sin A = \frac{8.3 \sin 75^{\circ}}{10.4}$

$\angle A = \sin^{-1}\left(\frac{8.3 \sin 75^{\circ}}{10.4}\right)$

$\angle A \doteq 50^{\circ}$

② Find $\angle C$:

$\angle C = 180^{\circ} - 75^{\circ} - 50^{\circ}$

$\angle C = 55^{\circ}$

③ Find c:

$\frac{c}{\sin C} = \frac{b}{\sin B}$

$\frac{c}{\sin 55^{\circ}} = \frac{10.4}{\sin 75^{\circ}}$

$c = \frac{10.4 \sin 55^{\circ}}{\sin 75^{\circ}}$

$c \doteq 8.8$ cm.

HW: p. 290-293 #1acde, 2abf, 3abd, 4a[*answer*: 41.2 m], 4ce, 5a, 6, 9

The Ambiguous Case of the Sine Law!

A. Trig Ratios for Angles and Their Supplements (*Recall*: supplementary angles add to 180°)

$\cos 50^\circ = 0.6428$ $\cos^{-1}(0.6428) = 50^\circ$

$\cos 130^\circ = -0.6428$ $\cos^{-1}(-0.6428) = 130^\circ$

$\sin 50^\circ = 0.7660$ $\sin^{-1}(0.7660) = 50^\circ$

$\sin 130^\circ = 0.7660$ $\sin^{-1}(0.7660) = 50^\circ$

$\triangle ABC$

$\angle A = 35^\circ$

$a = 21\text{cm}$

$c = 29\text{cm}$

Conclusion:

The **sine ratios** for an acute angle and its obtuse supplement are the same value and the **same sign**. However, when taking the **inverse sine** of the ratio, the angle returned is always the **acute** supplement!

Cosine ratios for an acute angle and its obtuse supplement are the same value but have **different signs**, so the correct angle is returned when you take the inverse cosine of the ratio. (That's why the *Cosine Law* is not ambiguous. ☺)

B. Recognizing When the Sine Law May Be Ambiguous

In some instances, the sine law is ambiguous (this means it will give you more than one answer). Imagine you are given a triangle with values for $\angle A$, side a, and side b. When side b is larger and you solve for the first unknown angle, $\angle B$, there are three possible cases, as shown in the figures below:

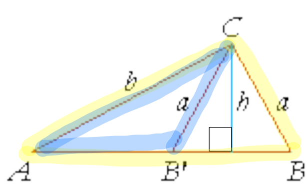


Fig. 1

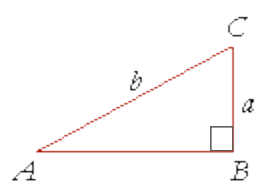


Fig. 2

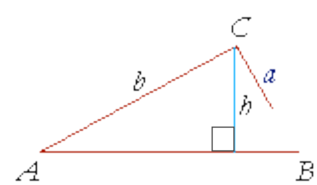


Fig. 3

1. $\angle B$ is an acute angle, which means that you will have to solve the triangle twice (*two possible triangles*): one triangle using $\angle B$, and one triangle using its *supplement* $\angle B'$ ($180^\circ - \angle B$)

2. $\angle B$ is a 90° angle (*one triangle*)

3. $\angle B$ cannot be found – "calculator error"! (*no triangles*)

- In general, the sine law may be ambiguous when you have ***a side-angle pair*** and a ***larger side***.
- It is NOT ambiguous when you have a side-angle pair and a smaller side or a side-angle pair and an equal length side.

Ex. 1: Determine the number of possible triangles that could be drawn with the given measures for each of the following: ① Check for possible ambiguous (side-angle pair + larger side) → ② Find angle → ③ determine # of Δ's

a) Given ΔDEF, where $\angle D = 44.3^{\circ}$, $d = 11.5$ cm, and $e = 7.7$ cm — Number of Triangles: one

side-angle + smaller side

① Not ambiguous.

b) Given ΔUVW, where $\angle W = 38.7^{\circ}$, $w = 10$ m, and $v = 25$ m — Number of Triangles: no

① possibly ambiguous — side-angle + larger side

② Find $\angle V$: $\frac{\sin V}{v} = \frac{\sin W}{w}$, $\frac{\sin V}{25} = \frac{\sin 38.7^{\circ}}{10}$ → $\angle V = \sin^{-1}\left(\frac{25\sin 38.7^{\circ}}{10}\right)$

$\angle V \rightarrow$ error

c) Given ΔPQR, where $\angle P = 30.0^{\circ}$, $p = 24.0$ cm, and $q = 48.0$ cm — Number of Triangles: one

① possibly ambiguous — side-angle + larger side

② Find $\angle Q$: $\frac{\sin Q}{q} = \frac{\sin P}{p}$, $\frac{\sin Q}{48} = \frac{\sin 30^{\circ}}{24}$ → $\angle Q = \sin^{-1}\left(\frac{48\sin 30^{\circ}}{24}\right)$

$\angle Q = 90^{\circ}$

d) Given ΔKLM, where $\angle K = 27^{\circ}$, $k = 25$ mm, and $m = 30$ mm — Number of Triangles: two

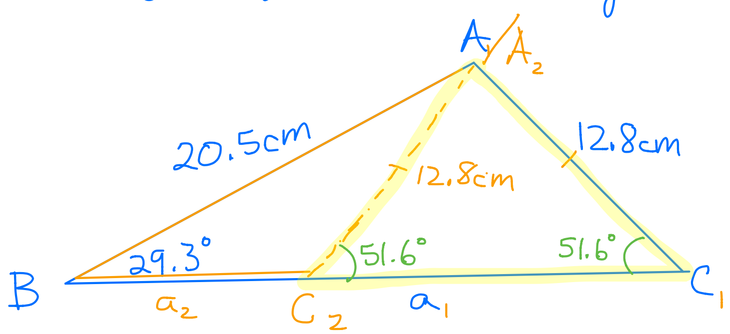
① possibly ambiguous — side-angle + larger side

② Find $\angle M$: $\frac{\sin M}{m} = \frac{\sin K}{k}$, $\frac{\sin M}{30} = \frac{\sin 27^{\circ}}{25}$ → $\angle M = \sin^{-1}\left(\frac{30\sin 27^{\circ}}{25}\right)$

$\angle M \doteq 33^{\circ}$

Ex. 2: Solve ΔABC, where $\angle B = 29.3^{\circ}$, $c = 20.5$ cm, and $b = 12.8$ cm

① Possibly Ambiguous — side-angle, larger



① Find $\angle C_1$:

$\frac{\sin C_1}{c} = \frac{\sin B}{b}$

$\frac{\sin C_1}{20.5} = \frac{\sin 29.3^{\circ}}{12.8}$

$\angle C_1 = \sin^{-1}\left(\frac{20.5\sin 29.3^{\circ}}{12.8}\right)$

$\boxed{\angle C_1 \doteq 51.6^{\circ}}$ ($\therefore$ 2 Δ's)

② Find $\angle A_1$:

$\angle A_1 = 180^{\circ} - 29.3^{\circ} - 51.6^{\circ}$

$\boxed{\angle A_1 = 99.1^{\circ}}$

③ Find a_1

$\frac{a_1}{\sin A} = \frac{b}{\sin B}$

$\frac{a_1}{\sin 99.1^{\circ}} = \frac{12.8}{\sin 29.3^{\circ}}$

$a_1 = \frac{12.8\sin 99.1^{\circ}}{\sin 29.3^{\circ}}$

$\boxed{a_1 \doteq 25.8\text{cm}}$

① Find $\angle C_2$:

$\angle C_2 = 180^{\circ} - 51.6^{\circ}$ (supp. ∠'s)

$\boxed{\angle C_2 = 128.4^{\circ}}$

② $\angle A_2 = 180^{\circ} - 29.3^{\circ} - 128.4^{\circ}$

$\boxed{\angle A_2 = 22.3^{\circ}}$

③ Find a_2:

$\frac{a_2}{\sin A_2} = \frac{b}{\sin B}$

$\frac{a_2}{\sin 22.3^{\circ}} = \frac{12.8}{\sin 29.3^{\circ}}$

$a_2 = \frac{12.8\sin 22.3^{\circ}}{\sin 29.3^{\circ}}$

$\boxed{a_2 \doteq 9.9\text{cm}}$

HW: p. 308-309 #3, 4, 5