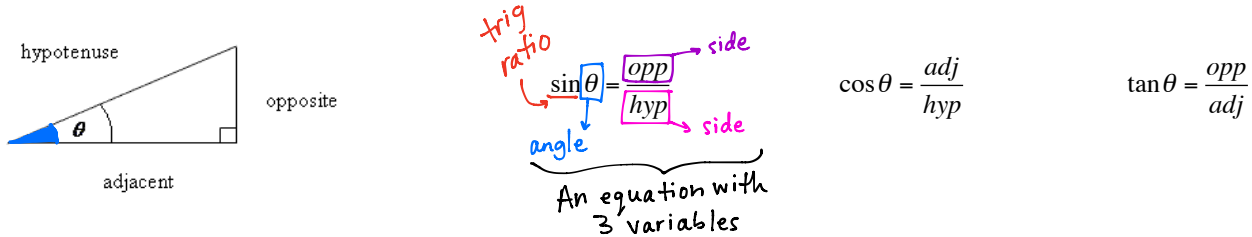


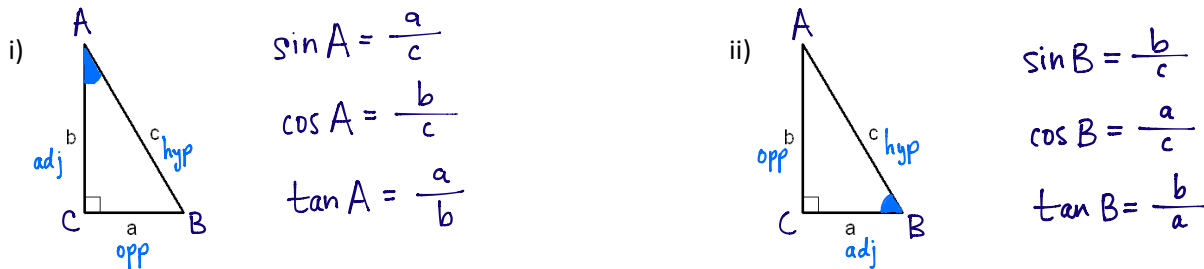
Reviewing the Trigonometry of Right Triangles

A. Reviewing the Primary Trigonometric Ratios: SOH CAH TOA

Recall that for any *right triangle*, we can write the three primary trigonometric ratios as equations in three unknowns, where the trig ratio of an angle is equal to one side divided by another side:



Ex. 1: Write the primary trig ratios for i) $\angle A$ and ii) $\angle B$ in the triangle shown below:



B. Solving for a Side Length

To solve for a **side length**, *clear the fraction* by multiplying both sides of the equation by the denominator or by cross-multiplying. Then, isolate the variable by *dividing out*, if necessary.

$$\begin{aligned} \cancel{\tan 72^\circ} &= \frac{y}{\cancel{6.1}} \\ y &= 6.1 (\tan 72^\circ) \\ \therefore y &\doteq 18.8 \text{ units} \end{aligned}$$

$$\begin{aligned} \cancel{\sin 58^\circ} &= \frac{18.8}{\cancel{x}} \\ x (\cancel{\sin 58^\circ}) &= \frac{18.8}{(\cancel{\sin 58^\circ})} \\ \therefore x &\doteq 22.2 \text{ units} \end{aligned}$$

C. Solving for an Angle

To solve for an **angle** when the ratio is known, take the *inverse trig operation* of both sides:

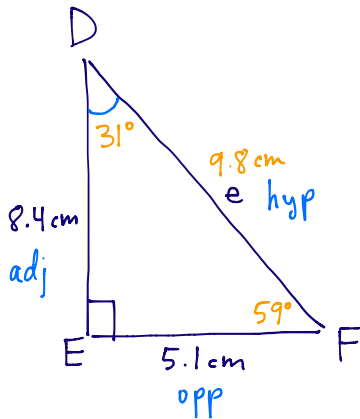
$$\begin{aligned} \cos C &= \frac{7.9}{13.5} \\ \cos^{-1}(\cos C) &= \cos^{-1}\left(\frac{7.9}{13.5}\right) \\ \angle C &= \cos^{-1}\left(\frac{7.9}{13.5}\right) \\ \therefore \angle C &\doteq 54^\circ \end{aligned}$$

D. Solving a Right Triangle

To **solve** a triangle means to find the value of *every side* and *every angle*. (Recall: the *Pythagorean Theorem* is $c^2 = a^2 + b^2$, where c is the hypotenuse of a right triangle).

Ex. 2: Solve $\triangle DEF$ where $\angle E = 90^\circ$, $d = 5.1$ cm and $f = 8.4$ cm. Round angles to the nearest degree and side lengths to one decimal place.

$$\begin{aligned}\angle D &= 31^\circ & d &= 5.1 \text{ cm} \\ \angle E &= 90^\circ & e &= 9.8 \text{ cm} \\ \angle F &= 59^\circ & f &= 8.4 \text{ cm}\end{aligned}$$



$$\begin{aligned}\textcircled{1} \text{ Find } \angle D: \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} \\ \tan D &= \frac{5.1}{8.4} \\ \angle D &= \tan^{-1}\left(\frac{5.1}{8.4}\right) \\ \therefore \angle D &\doteq 31^\circ\end{aligned}$$

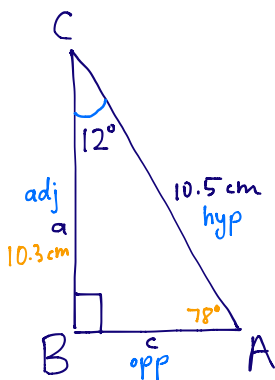
$$\begin{aligned}\textcircled{3} \text{ Find } e: \\ e^2 &= d^2 + f^2 \\ e &= \sqrt{(5.1)^2 + (8.4)^2} \\ \therefore e &\doteq 9.8 \text{ cm}\end{aligned}$$

$$\begin{aligned}\textcircled{2} \text{ Find } \angle F: \\ \angle F &= 180^\circ - 90^\circ - 31^\circ \\ \therefore \angle F &\doteq 59^\circ\end{aligned}$$

$$\begin{aligned}\therefore \angle D &\doteq 31^\circ, \\ \angle F &\doteq 59^\circ, \\ e &\doteq 9.8 \text{ cm}\end{aligned}$$

Ex. 3: Solve $\triangle ABC$ where $\angle B = 90^\circ$, $\angle C = 12^\circ$ and $b = 10.5$ cm. Round angles to the nearest degree and side lengths to one decimal place.

$$\begin{aligned}\angle A &= 78^\circ & a &= 10.3 \text{ cm} \\ \angle B &= 90^\circ & b &= 10.5 \text{ cm} \\ \angle C &= 12^\circ & c &= 2.2 \text{ cm}\end{aligned}$$



$$\begin{aligned}\textcircled{1} \text{ Find } \angle A: \\ \angle A &= 180^\circ - 90^\circ - 12^\circ \\ \angle A &= 78^\circ\end{aligned}$$

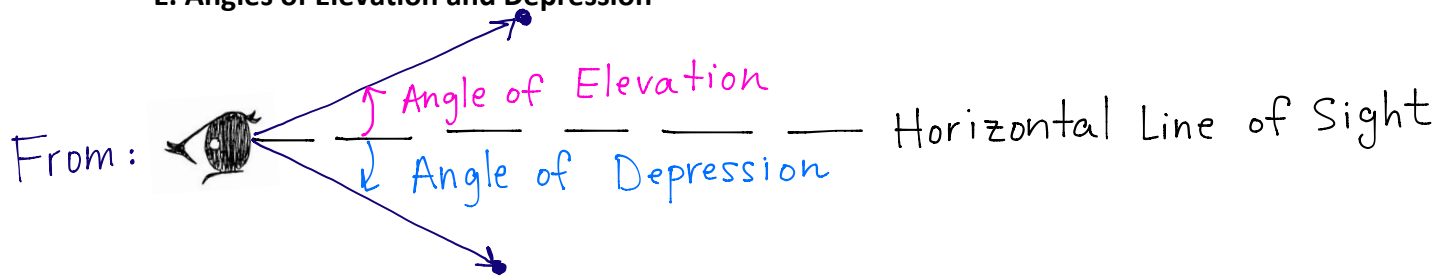
$$\begin{aligned}\textcircled{2} \text{ Find } a: \\ \cos 12^\circ &= \frac{a}{10.5} \\ a &= 10.5 (\cos 12^\circ) \\ \therefore a &\doteq 10.3 \text{ cm}\end{aligned}$$

$$\begin{aligned}\textcircled{3} \text{ Find } c: \\ \sin 12^\circ &= \frac{c}{10.5} \\ c &= 10.5 (\sin 12^\circ) \\ \therefore c &\doteq 2.2 \text{ cm}\end{aligned}$$

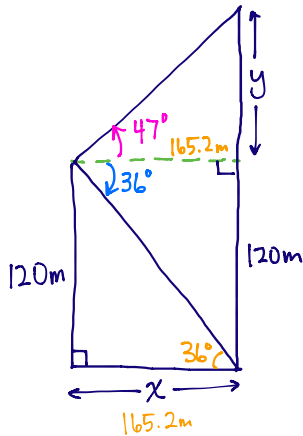
$$\begin{aligned}\therefore \angle A &= 78^\circ, \\ a &\doteq 10.3 \text{ cm}, \\ c &\doteq 2.2 \text{ cm}\end{aligned}$$

(or use Pythagorean theorem...)

E. Angles of Elevation and Depression



Ex. 4: From the top of a 120 m building, the angle of depression to the bottom of a second building is 36° , while the angle of elevation to the top of the second building is 47° . How far apart are the buildings? How tall is the second building?



Let x represent the distance between the buildings, in m.
Let $(y + 120)$ represent the height of the second building, in m.

① Find x :

$$\tan 36^\circ = \frac{120}{x}$$

$$\cancel{x} \tan 36^\circ = \frac{120}{\cancel{\tan 36^\circ}}$$

$$\therefore x \doteq 165.2 \text{ m}$$

② Find y :

$$\tan 47^\circ = \frac{y}{165.2}$$

$$165.2 \tan 47^\circ = y$$

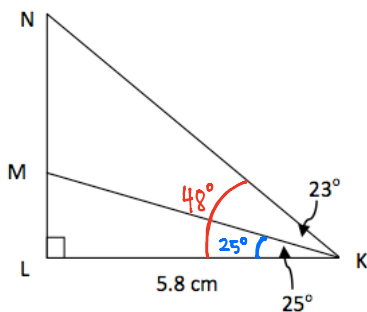
$$\therefore y \doteq 177.2 \text{ m}$$

\therefore the distance between the buildings is about 165.2 m and the height of the second building is about 297.2 m

③ Find $(y + 120)$:

$$y + 120 = 177.2 + 120 = 297.2 \text{ m}$$

Ex. 5: Find MN.



① Find LM:

$$\tan 25^\circ = \frac{LM}{5.8}$$

$$5.8 \tan 25^\circ = LM$$

$$\therefore LM \doteq 2.7 \text{ cm}$$

② Find LN:

$$\tan 48^\circ = \frac{LN}{5.8}$$

$$5.8 \tan 48^\circ = LN$$

$$\therefore LN \doteq 6.4 \text{ cm}$$

Strategy:

① Find LM in $\triangle KLM$

② Find LN in $\triangle KLN$

③ $MN = LN - LM$

③ Find MN:

$$MN = 6.4 - 2.7$$

$$\therefore MN \doteq 3.7 \text{ cm}$$

Angles in Standard Position and Radian Measure

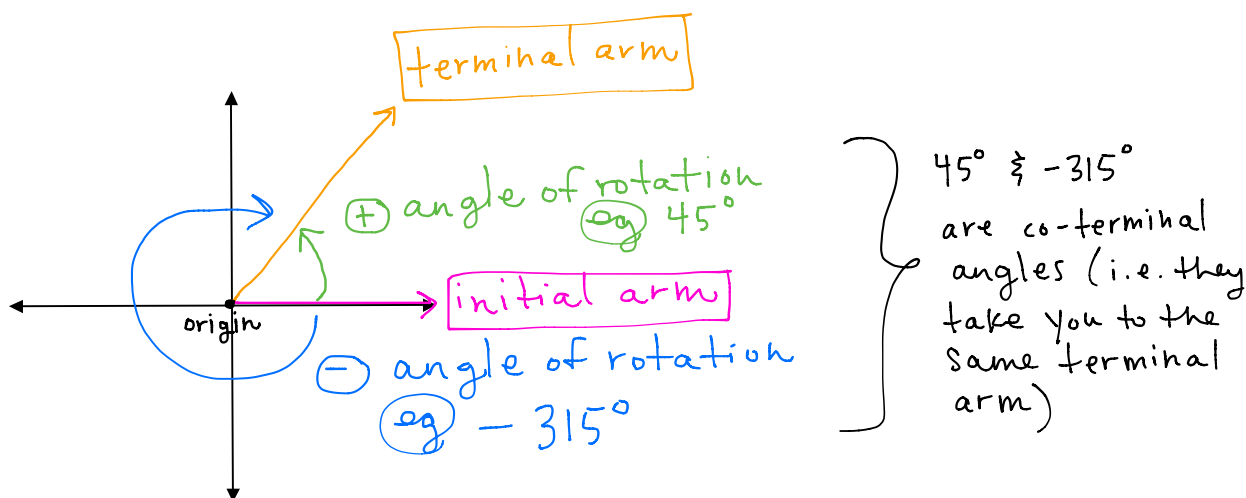
Up until this point, we have studied triangle trigonometry – where any angle, θ , was defined as $\{0^\circ < \theta < 180^\circ\}$. We will now explore the trigonometric ratios for any size of angle, such that $\{\theta \in \mathbb{R}\}$!

A. Angles in Standard Position

When drawing angles in on a coordinate plane in **standard position**:

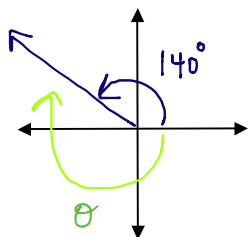
- The **vertex** is at the *origin*.
- The **initial arm** is on the *positive x-axis*.
- The **terminal arm** rotates through an angle about the origin:
 - if the direction of the rotation is counterclockwise, the measure of the angle is *positive*.
 - if the direction of the rotation is clockwise, the angle is *negative*.
- Co-terminal Angles** are angles in standard position that have the *same terminal arm*.

Note: To find co-terminal angles to θ , add or subtract 360° any number of times!



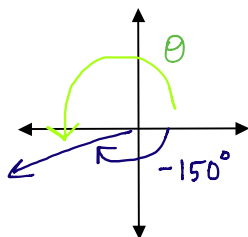
Ex. 1: Draw each angle in standard position. Name a **co-terminal** angle θ such that $-360^\circ < \theta < 360^\circ$.

a) 140°



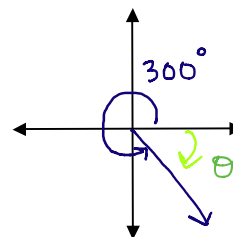
$$\theta = \frac{-220^\circ}{(140^\circ - 360^\circ)}$$

b) -150°



$$\theta = \frac{210^\circ}{(-150^\circ + 360^\circ)}$$

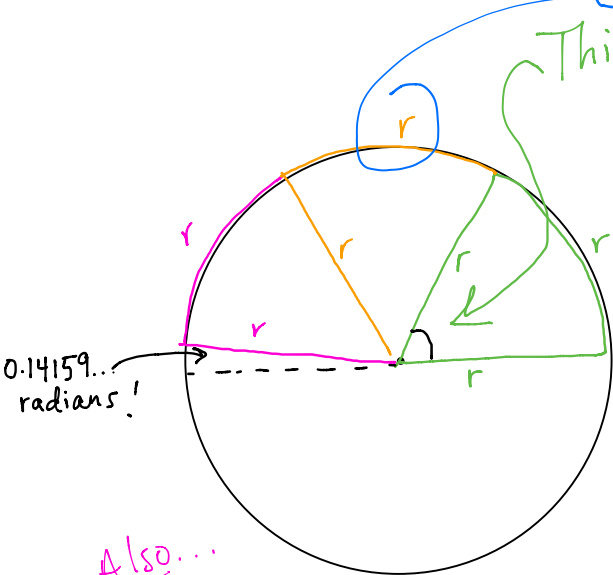
c) 300°



$$\theta = \frac{-60^\circ}{(300^\circ - 360^\circ)}$$

B. Radians

I) A **radian** is a unit, different from a degree, for measuring angles. One **radian** is "the measure of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle".



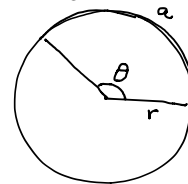
This angle is 1 radian
(also written as 1^r ,
1 rad, or just 1)

$$180^\circ = 3.14159... \text{ rad}$$

$$180^\circ = \pi \text{ rad}$$

* an angle θ in radians is found by dividing the arc length by the radius

$$\therefore \theta = \frac{a}{r}$$



Also...

$$360^\circ = 2\pi \text{ radians}$$

$$90^\circ = \frac{\pi}{2} \text{ radians}$$

and more!

II) Converting Between Degrees and Radians:

Since $180^\circ = \pi \text{ radians}$, $1^\circ = \frac{\pi}{180} \text{ rad}$ and $1 \text{ rad} = \frac{180^\circ}{\pi}$.

Therefore, we can use the conversion factors $\frac{180^\circ}{\pi \text{ rad}}$ and $\frac{\pi \text{ rad}}{180^\circ}$ to convert between units.

Ex. 2: Convert from degrees to radians. Give **exact** and **approximate** (to 2 decimal places) answers.

$$\text{a) } 45^\circ \times \frac{\pi}{180}$$

$$= \frac{45\pi}{180}$$

$$= \frac{\pi}{4} \quad \leftarrow \text{exact (keep } \pi \text{)}$$

$$\doteq 0.79 \quad \leftarrow \text{approximate (use 3.14159...)}$$

$$\text{b) } 72^\circ \times \frac{\pi}{180}$$

$$= \frac{72\pi}{180}$$

$$= \frac{2\pi}{5}$$

$$\doteq 1.26$$

$$\text{c) } 315^\circ \times \frac{\pi}{180}$$

$$= \frac{315\pi}{180}$$

$$= \frac{7\pi}{4}$$

$$\doteq 5.50$$

Ex. 3: Convert from radians to degrees. Give answers to one decimal place.

$$\text{a) } \frac{\pi}{4} \text{ radians} \times \frac{180^\circ}{\pi \text{ rad}}$$

$$= \frac{180^\circ}{4}$$

$$= 45^\circ$$

$$\text{b) } 4.3 \text{ radians} \times \frac{180^\circ}{\pi \text{ rad}}$$

$$= \frac{4.3(180^\circ)}{\pi}$$

$$\doteq 246.4^\circ$$

$$\text{c) } \frac{7\pi}{6} \times \frac{180^\circ}{\pi}$$

$$= \frac{7(180^\circ)}{6}$$

$$= 210^\circ$$

C. Problems Involving Arc Length

1) **Recall:** $\theta = \frac{a}{r}$, where θ is the measure of the angle in radians; a is the arc length; and r is the radius of the circle.

Ex. 4: Given a circle with the following measurements, find the unknown value.

a) $\theta = 2 \text{ rad}$
 $r = 6 \text{ cm}$
 $a = ?$

$$\theta = \frac{a}{r}$$

$$2 = \frac{a}{6}$$

$$a = 2(6)$$

$$\therefore a = 12 \text{ cm}$$

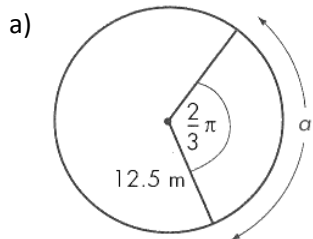
b) $\theta = ?$
 $r = 10 \text{ cm}$
 $a = 45 \text{ cm}$

$$\theta = \frac{a}{r}$$

$$\theta = \frac{45 \text{ cm}}{10 \text{ cm}}$$

$$\therefore \theta = 4.5 \text{ rad}$$

Ex. 5: Find the indicated quantity in each of the following diagrams.



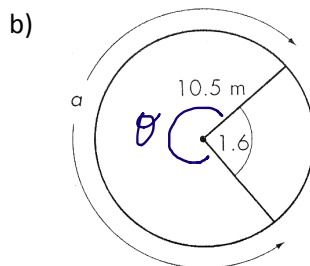
Find a :

$$\theta = \frac{a}{r}$$

$$\frac{2\pi}{3} = \frac{a}{12.5}$$

$$\frac{3a}{3} = \frac{2\pi(12.5)}{3}$$

$$\therefore a \approx 26.2 \text{ m}$$



Find θ :

$$\theta = 2\pi - 1.6$$

$$\theta = 4.7 \text{ rad}$$

Find a :

$$\theta = \frac{a}{r}$$

$$4.7 = \frac{a}{10.5}$$

$$a = 4.7(10.5)$$

$$\therefore a = 49.4 \text{ m}$$

II) Angular Velocity:

Ex. 6: An electric motor turns at 2000 revolutions/minute. Find the angular velocity in radians/second. Give an exact answer and an approximate answer.

$$2000 \frac{\cancel{\text{rev}}}{\cancel{\text{min}}} \times \frac{2\pi \text{ rad}}{1 \cancel{\text{rev}}} \times \frac{1 \cancel{\text{min}}}{60 \text{ sec}} = \frac{2000(2\pi) \text{ rad}}{60 \text{ sec}}$$

$$= \frac{200\pi}{3} \text{ rad/sec}$$

$$\approx 209.4 \text{ rad/sec}$$

Angles on a Coordinate Plane

A. Primary Trigonometric Ratios for Any Angle in Standard Position

<p>For an angle, θ, in standard position, with point $P(x, y)$ on the terminal arm: $r = \sqrt{x^2 + y^2}$</p> $\sin \theta = \frac{y}{r}$ $\cos \theta = \frac{x}{r}$ $\tan \theta = \frac{y}{x}$		<p>The CAST rule gives the initials of the trigonometric ratios that are positive in each quadrant:</p> <div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: center; margin-right: 20px;"> <p>II S</p> <hr/> <p>III T</p> </div> <div style="text-align: center; margin-right: 20px;"> <p>I A</p> <hr/> <p>IV C</p> </div> </div> <table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <thead> <tr> <th rowspan="2">Function</th> <th colspan="4">Quadrant</th> </tr> <tr> <th>I</th> <th>II</th> <th>III</th> <th>IV</th> </tr> </thead> <tbody> <tr> <td>sine</td> <td style="text-align: center;">+</td> <td style="text-align: center;">+</td> <td style="text-align: center;">-</td> <td style="text-align: center;">-</td> </tr> <tr> <td>cosine</td> <td style="text-align: center;">+</td> <td style="text-align: center;">-</td> <td style="text-align: center;">-</td> <td style="text-align: center;">+</td> </tr> <tr> <td>tangent</td> <td style="text-align: center;">+</td> <td style="text-align: center;">-</td> <td style="text-align: center;">+</td> <td style="text-align: center;">-</td> </tr> </tbody> </table>	Function	Quadrant				I	II	III	IV	sine	+	+	-	-	cosine	+	-	-	+	tangent	+	-	+	-
Function	Quadrant																									
	I	II	III	IV																						
sine	+	+	-	-																						
cosine	+	-	-	+																						
tangent	+	-	+	-																						

Note: The value of the “hypotenuse”, r , is always positive, as it is the **radius** of a circle that would have $P(x, y)$ on its **circumference**.

B. The CAST Rule

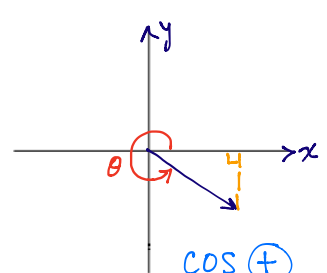
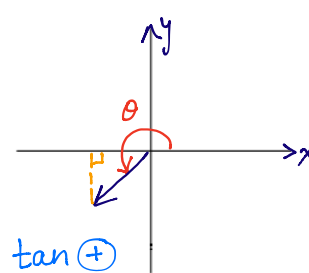
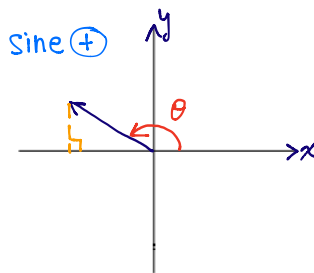
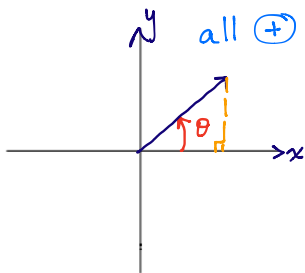
The coordinate plane can be divided into four **quadrants** using the x - and y -axes. The three primary trig ratios take on *different signs* depending on which quadrant the terminal arm is found, as summarized by the **CAST Rule**.

$$QI = \{0^\circ < \theta < 90^\circ\}$$

$$QII = \{90^\circ < \theta < 180^\circ\}$$

$$QIII = \{180^\circ < \theta < 270^\circ\}$$

$$QIV = \{270^\circ < \theta < 360^\circ\}$$



Ex. 1: Use the CAST rule to determine which quadrant(s) the terminal arm could be found for each trigonometric ratio.

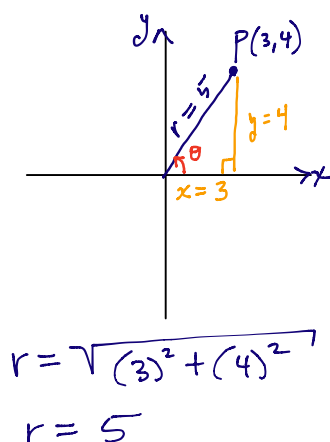
a) $\sin \theta = 0.5$ Q: I, II
⊕

b) $\cos \theta = -0.6432$ Q: II, III
⊖

c) $\tan \theta = -1.2$ Q: II, IV
⊖

Ex. 2: Find the exact primary trig ratios for each point on the terminal arm of θ , an angle in standard position.

a) (3, 4)

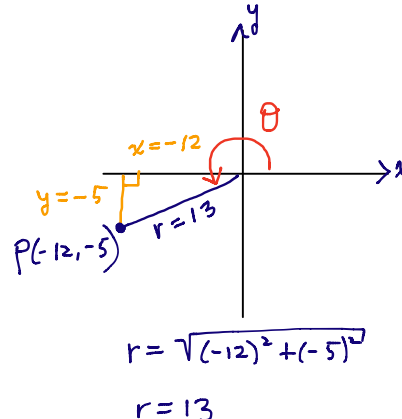


$$\sin \theta = \frac{4}{5}$$

$$\cos \theta = \frac{3}{5}$$

$$\tan \theta = \frac{4}{3}$$

b) (-12, -5)



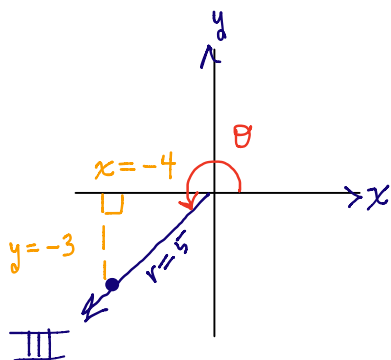
$$\sin \theta = -\frac{5}{13}$$

$$\cos \theta = -\frac{12}{13}$$

$$\tan \theta = \frac{-5}{-12} = \frac{5}{12}$$

Ex. 3: For each example below, $\angle \theta$ is given in standard position, and $0 \leq \theta \leq 2\pi$. A trigonometric ratio is given. Find all possible *exact values* of the other two trigonometric ratios.

a) $\sin \theta = -\frac{3}{5}$ and the terminal arm of $\angle \theta$ is in **quadrant III**



$$\sin \theta = \frac{y}{r} \therefore y = -3$$

$$r = 5$$

$$x = ?$$

$$x^2 + y^2 = r^2$$

$$x = \pm \sqrt{(5)^2 - (-3)^2}$$

$$x = \pm 4$$

$$\therefore \theta \text{ is in QIII}$$

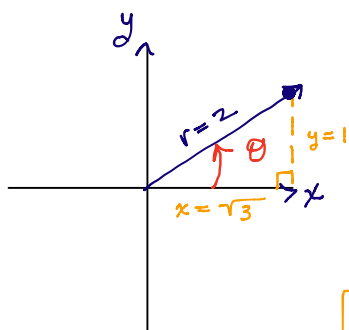
$$\therefore x = -4$$

$$\therefore \cos \theta = -\frac{4}{5}$$

$$\tan \theta = \frac{-3}{-4}$$

$$= \frac{3}{4}$$

b) $\tan \theta = \frac{1}{\sqrt{3}}$ $\tan \theta = + \therefore \text{QI, III}$ and $\tan \theta = \frac{y}{x}$



② In QI:

$$y = 1$$

$$x = \sqrt{3}$$

$$r = 2$$

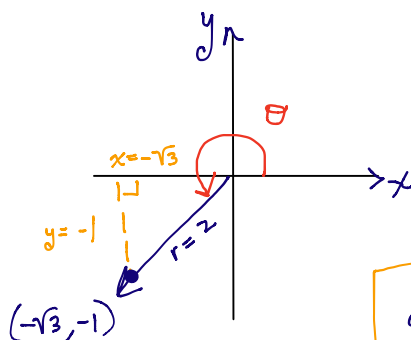
$$\therefore \sin \theta = \frac{1}{2}$$

$$\therefore \cos \theta = \frac{\sqrt{3}}{2}$$

① Find r:

$$r = \sqrt{(\sqrt{3})^2 + (1)^2}$$

$$\therefore r = 2$$



③ In QIII:

$$y = -1$$

$$x = -\sqrt{3}$$

$$r = 2$$

$$\therefore \sin \theta = -\frac{1}{2}$$

$$\therefore \cos \theta = -\frac{\sqrt{3}}{2}$$