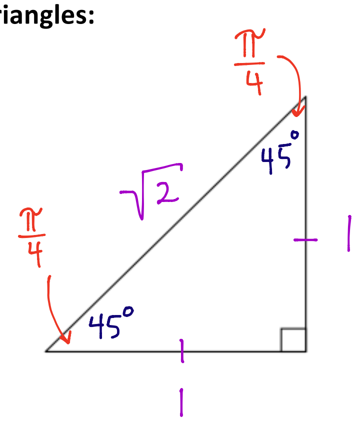
Special Angles: Special Triangles and the Unit Circle

We have started to investigate the trigonometric ratios for any angle. We can further our discussion by examining the **special angles** using *special triangles* and the *unit circle*.

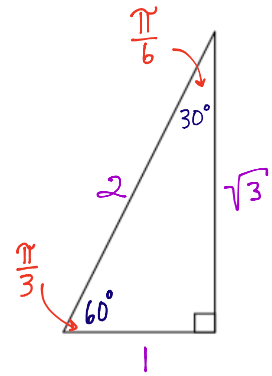
A. Special Triangles:



eg $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan\frac{\pi}{4} = 1$



eg $\sin\frac{\pi}{6} = \frac{1}{2}$

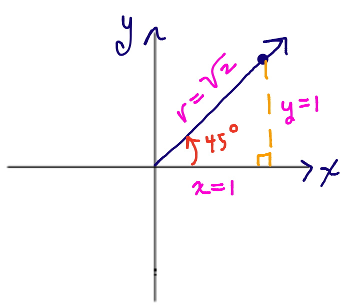
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

The **reference angle** (or *related acute angle* ***r.a.a.****)* is the acute angle made **between** the x-axis and the terminal arm of an angle in standard position. When the reference angle is 30°, 45°, or 60°, we can use the special triangles to find the **exact values** (i.e. fractions – no decimals!) of the trig ratios. ☺

Ex. 1: Use special triangles and reference angles to find the **exact values** of the trig ratios for the following angles.

a) $\theta = 45^\circ$

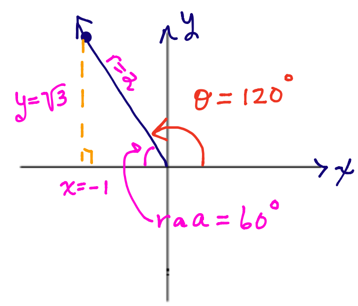


$\sin 45^\circ = \frac{1}{\sqrt{2}}$

$\cos 45^\circ = \frac{1}{\sqrt{2}}$

$\tan 45^\circ = 1$

b) $\theta = 120^\circ$



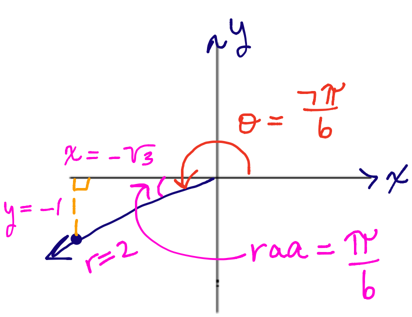
$\sin 120^\circ = \frac{\sqrt{3}}{2}$

$\cos 120^\circ = -\frac{1}{2}$

$\tan 120^\circ = \frac{\sqrt{3}}{-1}$

$= -\sqrt{3}$

c) $\theta = \frac{7\pi}{6}$



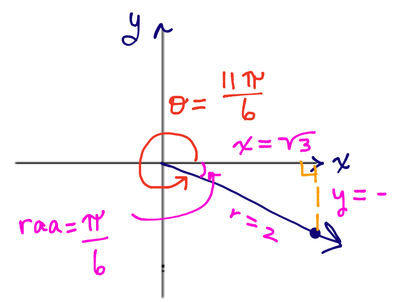
$\sin\frac{7\pi}{6} = -\frac{1}{2}$

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$\tan\frac{7\pi}{6} = \frac{-1}{-\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$

d) $\theta = \frac{11\pi}{6}$



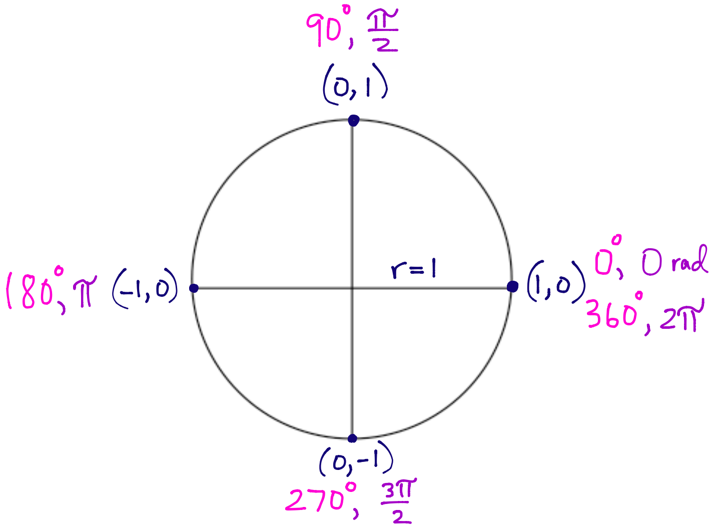
$\sin\frac{11\pi}{6} = -\frac{1}{2}$

$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

B. The Unit Circle (a circle with radius = 1)

Previously, we have investigated angles on the coordinate plane *in the four quadrants*, but what about angles that lie <u>*on the axes*</u>? Enter the unit circle!



i) $\sin 90^\circ = \frac{y}{r}$

@ 90°: $x = 0$, $y = 1$, $r = 1$

$= \frac{1}{1}$

$= 1$

ii) $\tan\frac{3\pi}{2} = \frac{y}{x}$

@ $\frac{3\pi}{2}$ rad: $x = 0$, $y = -1$, $r = 1$

$= \frac{-1}{0}$

= undefined

iii) $\cos\pi = \frac{x}{r}$

@ π rad: $x = -1$, $y = 0$, $r = 1$

$= \frac{-1}{1}$

$= -1$

iv) $\cos(-270^\circ) = \frac{x}{r}$

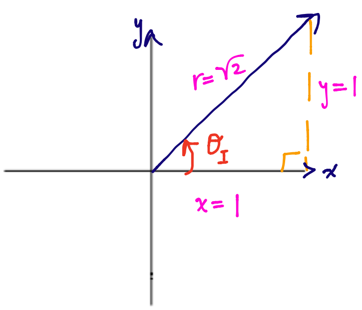
@ −270°: $x = 0$, $y = 1$, $r = 1$

$= \frac{0}{1}$

$= 0$

Ex. 2: If $\sin\theta = \frac{1}{\sqrt{2}}$ ($\frac{y}{r}$), $0^\circ \le \theta \le 360^\circ$, find the values of $\cos\theta$ and θ

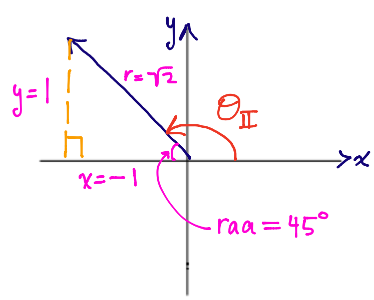
∴ special triangle; QI, QII



In QI:

$\cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta_I = 45^\circ$



In QII:

$\cos\theta = -\frac{1}{\sqrt{2}}$

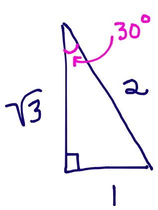
$\theta_{II} = 180^\circ - 45^\circ$

$\therefore \theta_{II} = 135^\circ$

Ex. 3: Find the *exact* value of 2sin30°cos30°.

$\sin 30^\circ = \frac{1}{2}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$



$$\boxed{\begin{aligned} 2\sin 30^\circ\cos 30^\circ &= \cancel{2}^{1}\left(\frac{1}{\cancel{2}_{1}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}}$$

Note: if the denominator is a radical, you will need to rationalize!

HW: p. 348-349 #3abc, 4aef, 11-14, 18ace

Trigonometric Ratios for Positive and Negative Angles of Rotation

Find the <u>exact</u> values of the primary trig ratios for the following angles in standard position.

Steps: *1) Draw the angle in standard position*

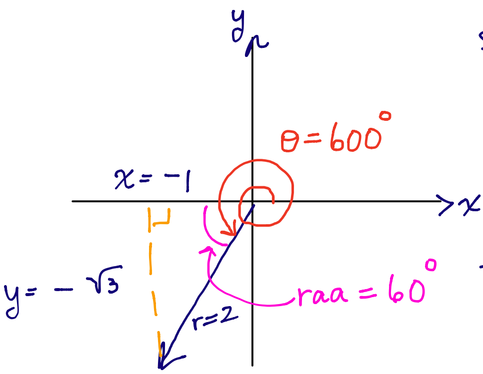
2) If the terminal arm is on one of the axes, use the unit circle to find the exact values for the required ratios

3) If the terminal arm is in one of the four quadrants:

i) add or subtract 360° or 2π to determine the angle between the terminal arm and the x-axis

ii) Use the related acute angle and the CAST rule to determine the exact values for the required ratios

a) $\theta = 600°$



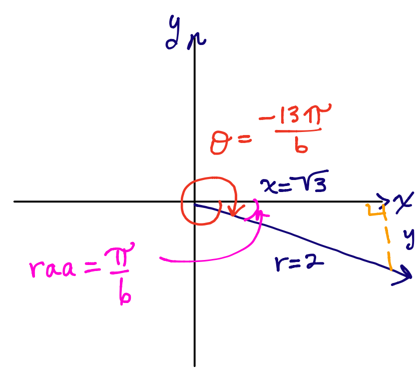
$\sin 600° = \frac{-\sqrt{3}}{2}$

$\cos 600° = \frac{-1}{2}$

$\tan 600° = \frac{-\sqrt{3}}{-1}$

$= \sqrt{3}$

b) $\theta = -\frac{13\pi}{6}$

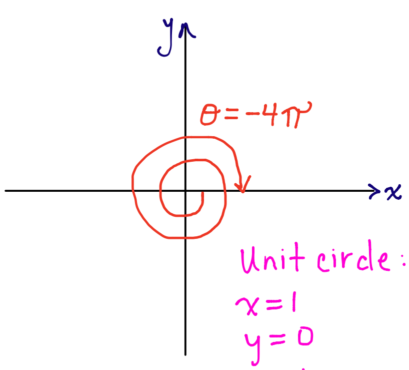


$\sin\left(\frac{-13\pi}{6}\right) = \frac{-1}{2}$

$\cos\left(\frac{-13\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\tan\left(\frac{-13\pi}{6}\right) = \frac{-1}{\sqrt{3}}$

c) $\theta = -4\pi$



Unit circle:
$x = 1$
$y = 0$
$r = 1$

$\sin(-4\pi) = \frac{0}{1}$

$= 0$

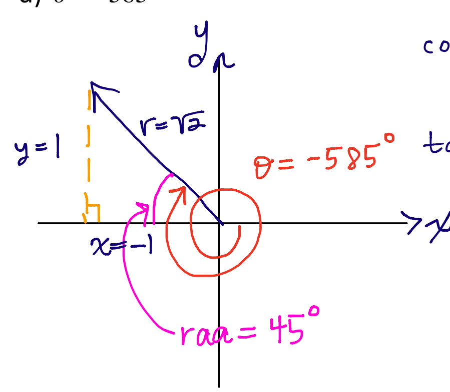
$\cos(-4\pi) = \frac{1}{1}$

$= 1$

$\tan(-4\pi) = \frac{0}{1}$

$= 0$

d) $\theta = -585°$



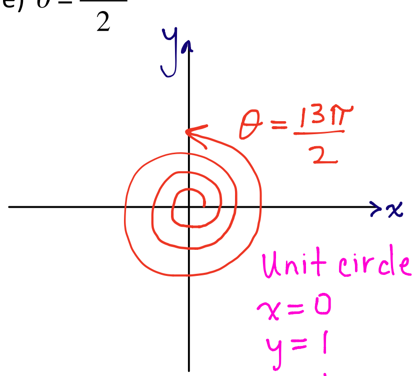
$\sin(-585°) = \frac{1}{\sqrt{2}}$

$\cos(-585°) = \frac{-1}{\sqrt{2}}$

$\tan(-585°) = \frac{1}{-1}$

$= -1$

e) $\theta = \frac{13\pi}{2}$



Unit circle:
$x = 0$
$y = 1$
$r = 1$

$\sin\left(\frac{13\pi}{2}\right) = \frac{1}{1}$

$= 1$

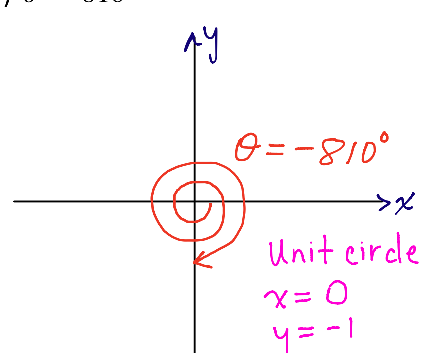
$\cos\left(\frac{13\pi}{2}\right) = \frac{0}{1}$

$= 0$

$\tan\left(\frac{13\pi}{2}\right) = \frac{1}{0}$

$=$ undefined

f) $\theta = -810°$



Unit circle:
$x = 0$
$y = -1$
$r = 1$

$\sin(-810°) = \frac{-1}{1}$

$= -1$

$\cos(-810°) = \frac{0}{1}$

$= 0$

$\tan(-810°) = \frac{-1}{0}$

$=$ undefined

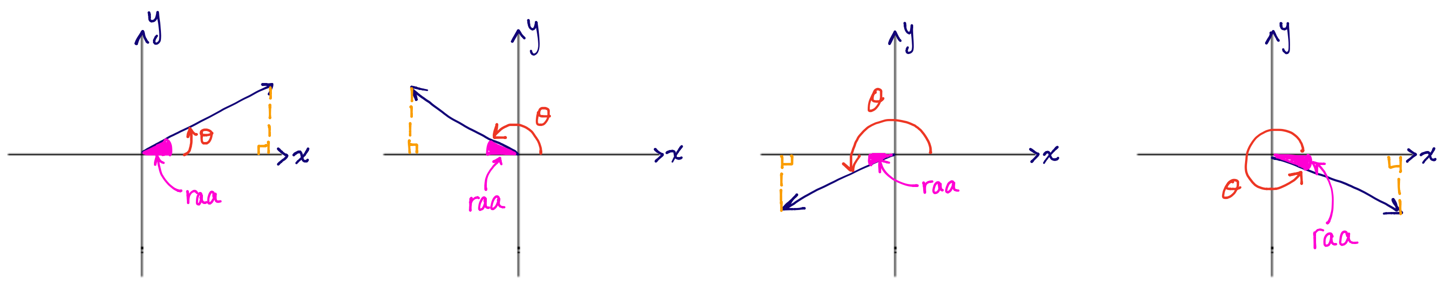
HW: p. 353 # 1, 2 and p. 354 # 3, 4

Combining the CAST Rule, Related Acute Angles, and Special Triangles

Steps for finding the possible value(s) of θ:

- determine the possible *quadrants* for the terminal arm using the **CAST rule**
- sketch the *terminal arm* in the appropriate quadrant(s)
- find the ***r.a.a.*** for θ using ***special angles*** or *inverse trig operations* on the value of the ratio (do not include the negative sign for this calculation – we are looking for the related *acute* angle!)
- add or subtract the *r.a.a.* from 180° (π) or 360° (2π) to get θ, the positive angle between the initial arm and the terminal arm for each quadrant
- provide a therefore statement for each quadrant

Given any trig ratio, we can find all angles that will satisfy the equation for $0° \le \theta \le 360°$ or $0 \le \theta \le 2\pi$.



QI: θ = raa

QII: θ = π − raa or $180°$ − raa

QIII: θ = π + raa or $180°$ + raa

QIV: θ = 2π − raa or $360°$ − raa

A. Find θ, an angle in standard position, for $0° \le \theta \le 360°$. Round to the nearest whole degree, where necessary.

1. $\cos\theta = \frac{1}{2}$ QI, IV; raa = 60°

QI:
$\theta = 60°$

QIV:
$\theta = 360° - 60°$
$= 300°$

$\boxed{\therefore \theta = 60° \text{ or } 300°}$

2. $\tan\theta = -2.3546$ QII, IV; raa ≐ 67°

$\boxed{raa = \tan^{-1}(2.3546)}$

QII:
$\theta = 180° - 67°$
$= 113°$

QIV:
$\theta = 360° - 67°$
$= 293°$

$\boxed{\therefore \theta = 113° \text{ or } 293°}$

3. $\tan\theta = -1$

4. $\sin\theta = 0.1357$

5. $\cos\theta = -0.8436$

6. $\sin\theta = \frac{\sqrt{3}}{2}$

B. Find θ, an angle in standard position, for $0 \le \theta \le 2\pi$. Give the exact solution, where possible. Otherwise, round to the nearest hundredth of a radian.

1. $\sin\theta = -\frac{1}{2}$ QIII, IV ; raa $= \frac{\pi}{6}$

QIII: $\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

QIV: $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$\boxed{\therefore \theta = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}}$

2. $\tan\theta = 3.4$ QI, III; raa $\doteq 1.28$ $\boxed{\text{raa} = \tan^{-1}(3.4)}$

QI: $\theta \doteq 1.28$

QIII: $\theta \doteq \pi + 1.28 \doteq 4.42$

$\boxed{\therefore \theta \doteq 1.28 \text{ or } 4.42}$

3. $\cos\theta = \frac{1}{\sqrt{2}}$

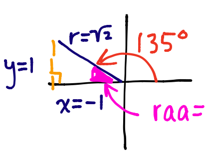
4. $\tan\theta = -\sqrt{3}$

5. $\cos\theta = -0.9205$

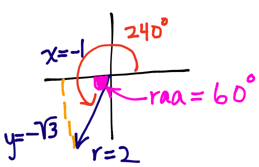
6. $\sin\theta = \frac{1}{\sqrt{3}}$

C. Combine the CAST Rule, related acute angles, and special triangles ***or*** use the unit circle to find exact values of the following. **Rationalize the denominator when necessary.*

1. $(\sin 135°)(\cos 240°)$



$\sin 135° = \frac{1}{\sqrt{2}}$



$\cos 240° = -\frac{1}{2}$

$\therefore (\sin 135°)(\cos 240°)$
$= \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right)$
$= -\frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$
$= -\frac{\sqrt{2}}{4}$

2. $\tan\frac{11\pi}{6} - \cos\left(-\frac{3\pi}{4}\right)$

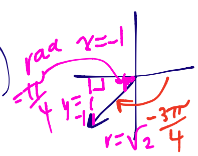
$\theta = \frac{11\pi}{6}$, QIV; raa $= \frac{\pi}{6}$; $x = \sqrt{3}$, $y = -1$, $r = 2$

$\therefore \tan\left(\frac{11\pi}{6}\right) = \frac{-1}{\sqrt{3}}$

$\theta = -\frac{3\pi}{4}$, QIII; raa $= \frac{\pi}{4}$; $x = -1$, $y = -1$, $r = \sqrt{2}$

$\therefore \cos\left(-\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

$\therefore \tan\left(\frac{11\pi}{6}\right) - \cos\left(-\frac{3\pi}{4}\right)$



$= -\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{2}}\right)$
$= -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{-\sqrt{2} + \sqrt{3}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}$
$= \frac{-\sqrt{12} + \sqrt{18}}{\sqrt{36}}$
$= \frac{-2\sqrt{3} + 3\sqrt{2}}{6}$

3. $\left(\cos\frac{11\pi}{6}\right)^2 + \left(\sin\frac{3\pi}{2}\right)^2$

4. $\tan(-300°) \cdot \cos(-210°)$

5. $2(\cos 2\pi) + \frac{1}{\left(\cos\frac{7\pi}{3}\right)}$

6. $\left(\sin\frac{5\pi}{4}\right)^2 + \left(\cos\frac{5\pi}{4}\right)^2$

7. $\tan(-510°) + (\sin 600° + \tan 540°)$

8. $\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) + \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$

9. $\frac{\tan\frac{\pi}{3} + \tan\frac{7\pi}{4}}{1 - \left(\tan\frac{\pi}{3}\right)\left(\tan\frac{7\pi}{4}\right)}$