The Cosine Law

The Cosine Law is another expansion of a fundamental trig ratio into a law that can be used on any triangle (non-right triangles included)! It represents a general version of the Pythagorean Theorem, adapted to non-right triangles. It can also be rearranged to solve for the unknown angle.

SAS ✓

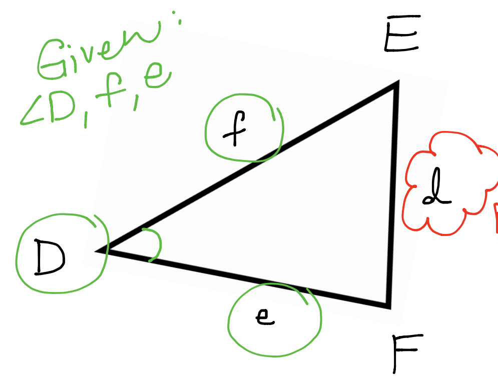
The Cosine Law	***Rearranged to Solve for the Angle***
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

SSS

When do we use the *Cosine Law*?

1. When we are given two sides and a contained angle, and we want to find the side opposite the angle.

Given: $\angle D$, f, e



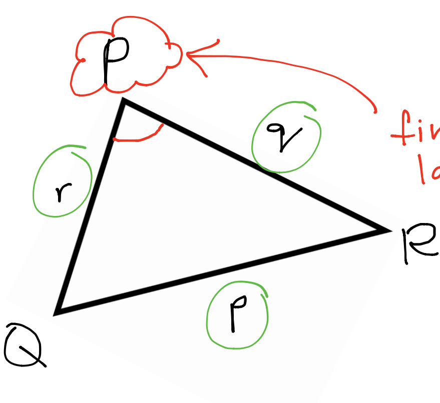
find the 3rd side

$$\underset{\text{side we want}}{d^2} = \underbrace{e^2}_{} + \underbrace{f^2}_{} - 2ef\cos \underset{\text{corresponding angle we're given}}{D}$$

other two sides

2. When we are given three sides, and we want to find any other angle. **(When given a choice, find the largest angle first!)**

Given: p, q, r



find the largest angle

$$\angle P = \cos^{-1}\left(\frac{r^2 + q^2 - p^2}{2rq}\right)$$

angle we want

other two sides

corresponding side we're given

1. Two Sides and a Contained Angle $\therefore$ SAS

Solve ΔPQR, where $\angle R = 121°$, $p = 32$ cm, and $q = 27$ cm

$\angle P \doteq \underline{33°}$ $\quad p = 32$ cm

$\angle Q \doteq \underline{26°}$ $\quad q = 27$ cm

$\angle R = 121°$ $\quad r \doteq \underline{51 \text{ cm}}$

(1) Find r:

$r^2 = p^2 + q^2 - 2pq\cos R$

$r^2 = (32)^2 + (27)^2 - 2(32)(27)\cos 121°$

$r^2 \doteq 1024 + 729 - (-889.9858)$

$r^2 \doteq 2642.9858$

$\boxed{r \doteq 51.4}$

(2) Find $\angle P$:

$\cos P = \dfrac{q^2 + r^2 - p^2}{2qr}$

$\cos P = \dfrac{(27)^2 + (51)^2 - (32)^2}{2(27)(51)}$

$\angle P \doteq \cos^{-1}(0.8373)$

$\boxed{\angle P \doteq 33°}$

(3) Find $\angle Q$:

$\angle Q \doteq 180° - 33° - 121°$

$\boxed{\angle Q \doteq 26°}$

$\boxed{\therefore \angle P \doteq 33°, \angle Q \doteq 26°, r \doteq 51 \text{ cm}}$

2. Three Sides $\therefore$ SSS $\rightarrow$ find largest angle first

Solve ΔDEF, where $d = 41$ cm, $e = 32$ cm, and $f = 21$ cm

$\angle D \doteq \underline{99°}$ $\quad d = 41$ cm

$\angle E \doteq \underline{50°}$ $\quad e = 32$ cm

$\angle F \doteq \underline{31°}$ $\quad f = 21$ cm

(1) Find $\angle D$ first:

$\angle D = \cos^{-1}\left(\dfrac{e^2 + f^2 - d^2}{2ef}\right)$

$\angle D = \cos^{-1}\left(\dfrac{(32)^2 + (21)^2 - (41)^2}{2(32)(21)}\right)$

$\boxed{\angle D \doteq 99°}$

(2) Find $\angle E$:

$\cos E = \dfrac{d^2 + f^2 - e^2}{2df}$

$\angle E = \cos^{-1}\left(\dfrac{41^2 + 21^2 - 32^2}{2(41)(21)}\right)$

$\boxed{\angle E \doteq 50°}$

(3) Find $\angle F$:

$\angle F \doteq 180° - 99° - 50°$

$\boxed{\angle F \doteq 31°}$

$\boxed{\therefore \angle D \doteq 99°, \angle E \doteq 50°, \angle F \doteq 31°}$

HW: p. 290-293 #1bf, 2ce, 3cef, 4b, 5bd, 8, 10, 12

The Sine Law

The Sine Law is an expansion of a fundamental trig ratio into a law that can be used on any triangle (non-right triangles included)! It states that the sines of the angles of a triangle are proportional to the lengths of their opposite sides.

The Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

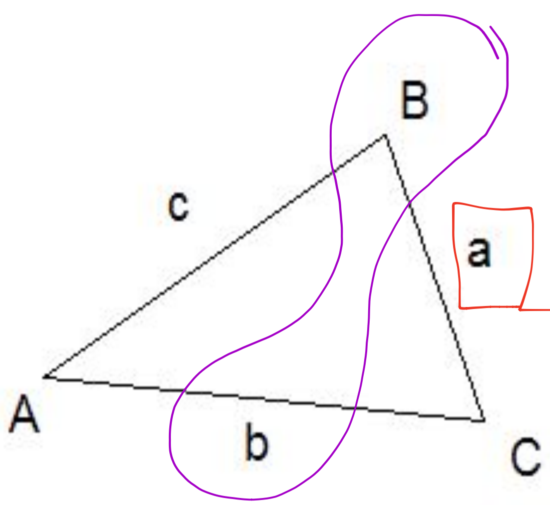
The Sine Law can be written in two forms, as the relationship represents a proportion. It is easiest to solve a proportion when **the unknown is in the numerator**, so when the unknown is an angle, use the first form, and when the unknown is a *side length*, use the *second form*.

When can we use the *Sine Law*?

1. When we are given **two angles and a side** for an acute triangle. (We use the angle sum of 180° to find the third angle, and then set up a proportion to find any side we want!)

2. When we are given **two sides and an angle opposite one of them**. (We set up a proportion to find the second angle.)

Basically, we can use the ***Sine Law*** *when we have a* **corresponding side-angle pair plus one other piece of information** *about the triangle.*



1. Two Angles and a Side

Solve ΔPQR, where $\angle R = 68^o$, $\angle Q = 69^o$, and $p = 21$ cm

$\angle P = \underline{43^o}$ $\quad p = 21$ cm
$\angle Q = 69^o$ $\quad q \doteq \underline{28.7 \text{ cm}}$
$\angle R = 68^o$ $\quad r \doteq \underline{28.5 \text{ cm}}$

① Find $\angle P$:

$\angle P = 180^o - 69^o - 68^o$

$\boxed{\angle P = 43^o}$

② Find q:

$$\frac{q}{\sin Q} = \frac{p}{\sin P}$$

$$\frac{q}{\sin 69^o} = \frac{21}{\sin 43^o}$$

$$q = \frac{21}{\sin 43^o}(\sin 69^o)$$

$\boxed{q \doteq 28.7 \text{ cm}}$

③ Find r:

$$\frac{r}{\sin R} = \frac{p}{\sin P}$$

$$\frac{r}{\sin 68^o} = \frac{21}{\sin 43^o}$$

$$r = \frac{21}{\sin 43^o}(\sin 68^o)$$

$\boxed{r \doteq 28.5 \text{ cm}}$

$\boxed{\therefore \angle P = 43^o, \; q \doteq 28.7 \text{ cm}, \; r \doteq 28.5 \text{ cm}}$

2. Two Sides and an Angle Opposite One of Them

Solve ΔABC, where $\angle B = 75^o$, $a = 8.3$ cm, and $b = 10.4$ cm

$\angle A \doteq \underline{50^o}$ $\quad a = 8.3$ cm
$\angle B = 75^o$ $\quad b = 10.4$ cm
$\angle C \doteq \underline{55^o}$ $\quad c \doteq \underline{8.8 \text{ cm}}$

① Find $\angle A$:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{8.3} = \frac{\sin 75^o}{10.4}$$

$$\sin A = \frac{\sin 75^o}{10.4} \times 8.3$$

$$\angle A = \sin^{-1}\left(\frac{\sin 75^o}{10.4} \times 8.3\right)$$

$\boxed{\angle A \doteq 50^o}$

② Find $\angle C$:

$\angle C \doteq 180^o - 50^o - 75^o$

$\boxed{\angle C \doteq 55^o}$

③ Find c:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 55^o} = \frac{10.4}{\sin 75^o}$$

$$c = \frac{10.4}{\sin 75^o}(\sin 55^o)$$

$\boxed{c \doteq 8.8 \text{ cm}}$

$\therefore \angle A \doteq 50^o$,
$\angle C \doteq 55^o$,
$c \doteq 8.8$ cm

HW: p. 290-293 #1acde, 2abf, 3abd, 4a[*answer*: 41.2 m], 4ce, 5a, 6, 9

The Ambiguous Case of the Sine Law!

A. Trig Ratios for Angles and Their Supplements (*Recall*: supplementary angles add to 180)

$\cos 50^\circ$ = 0.6428 $\cos^{-1}$(0.6428) = 50°

$\cos 130^\circ$ = −0.6428 $\cos^{-1}$(−0.6428) = 130°

$\sin 50^\circ$ = 0.7660 $\sin^{-1}$(0.7660) = 50°

$\sin 130^\circ$ = 0.7660 $\sin^{-1}$(0.7660) = 50°

always returns the acute angle (raa)

Conclusion:

The **sine ratios** for an acute angle and its obtuse supplement are the same value and the **same sign**. However, when taking the **inverse sine** of the ratio, the angle returned is always the **acute** supplement!

Cosine ratios for an acute angle and its obtuse supplement are the same value but have **different signs**, so the correct angle is returned when you take the inverse cosine of the ratio. (That's why the *Cosine Law* is not ambiguous. ☺)

B. Recognizing When the Sine Law May Be Ambiguous

In some instances, the sine law is ambiguous (this means it will give you more than one answer). Imagine you are given a triangle with values for $\angle A$, side a, and side b. When side b is larger and you solve for the first unknown angle, $\angle B$, there are three possible cases, as shown in the figures below:

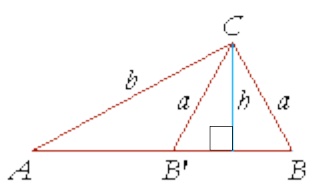


Fig. 1

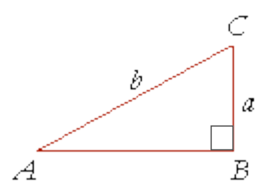


Fig. 2

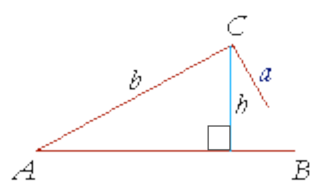


Fig. 3

3 possible cases given side-angle pair + larger side

1. $\angle B$ is an acute angle, which means that you will have to solve the triangle twice (*two possible triangles*): one triangle using $\angle B$, and one triangle using its *supplement* $\angle B'$ (180° – $\angle B$)

∴ ambiguous!

2. $\angle B$ is a 90° angle (*one triangle*)

3. $\angle B$ cannot be found – "calculator error"! (*no triangles*)

- In general, the sine law may be ambiguous when you have ***a side-angle pair*** and a ***larger side***.
- It is NOT ambiguous when you have a side-angle pair and a smaller side or a side-angle pair and an equal length side.

Ex. 1: Determine the number of possible triangles that could be drawn with the given measures for each of the following: ① Check for possible ambiguous → ② Find the angle → ③ Use 3 cases to determine # of Δs

a) Given Δ*DEF*, where $\angle D = 44.3^\circ$, $d = 11.5$ cm, and $e = 7.7$ cm → Number of Triangles: one

side-angle pair + smaller side

① Not ambiguous

b) Given Δ*UVW*, where $\angle W = 38.7^\circ$, $w = 10$ m, and $v = 25$ m → Number of Triangles: none

side-angle pair + larger side

① Possible ambiguous...

② Find angle V

$$\frac{\sin V}{25} = \frac{\sin 38.7^\circ}{10} \rightarrow \angle V = \sin^{-1}\left(\frac{\sin 38.7^\circ}{10} \cdot 25\right) \rightarrow \angle V = \text{error}$$

③ ∴ no solutions

c) Given Δ*PQR*, where $\angle P = 30.0^\circ$, $p = 24.0$ cm, and $q = 48.0$ cm → Number of Triangles: one

side-angle pair + larger side

① Possible ambiguous...

② Find angle Q

$$\frac{\sin Q}{48.0} = \frac{\sin 30.0^\circ}{24.0} \rightarrow \angle Q = \sin^{-1}\left(\frac{\sin 30.0^\circ}{24.0} \cdot 48.0\right) \rightarrow \angle Q = 90^\circ$$

③ ∴ one solution

d) Given Δ*KLM*, where $\angle K = 27^\circ$, $k = 25$ mm, and $m = 30$ mm → Number of Triangles: two

side-angle pair + larger side

① Possible ambiguous...

② Find angle M

$$\frac{\sin M}{30} = \frac{\sin 27^\circ}{25} \rightarrow \angle M = \sin^{-1}\left(\frac{\sin 27^\circ}{25} \cdot 30\right) \rightarrow \angle M \doteq 33^\circ$$

③ ∴ two solutions

QI: i) $\angle M = 33^\circ$

QII: ii) $\angle M' = 180^\circ - 33^\circ = 147^\circ$

side-angle pair

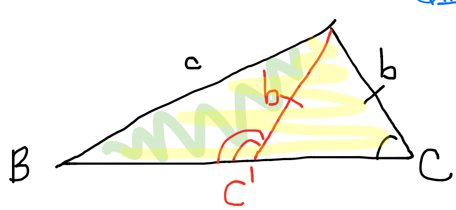
Ex. 2: Solve ΔABC, where $\angle B = 29.3^\circ$, $c = 20.5$ cm, and $b = 12.8$ cm

+ larger side

$\angle A \doteq 99.1^\circ$ $a \doteq 25.8$ cm

$\angle B = 29.3^\circ$ $b = 12.8$ cm

$\angle C \doteq 51.6^\circ$ $c = 20.5$ cm



① Find ∠C:

$$\frac{\sin C}{20.5} = \frac{\sin 29.3}{12.8}$$

$$\angle C = \sin^{-1}\left(\frac{\sin 29.3}{12.8} \times 20.5\right)$$

$\boxed{\angle C \doteq 51.6^\circ}$ → acute!!

∴ two solutions

② Find ∠A:

$\angle A \doteq 180^\circ - 29.3^\circ - 51.6^\circ$

$\boxed{\angle A \doteq 99.1^\circ}$

③ Find a:

$$\frac{a}{\sin 99.1^\circ} = \frac{12.8}{\sin 29.3^\circ}$$

$\boxed{a \doteq 25.8 \text{ cm}}$

$\angle A' \doteq 22.3^\circ$ $a' \doteq 9.9$ cm

$\angle B = 29.3^\circ$ $b = 12.8$ cm

$\angle C' \doteq 128.4^\circ$ $c = 20.5$ cm

∴ find the supplement

① $\angle C' = 180^\circ - \text{raa}$

$\boxed{\angle C' \doteq 128.4^\circ}$

② $\angle A' \doteq 180^\circ - 128.4^\circ - 29.3^\circ$

$\boxed{\angle A' \doteq 22.3^\circ}$

③ Find a':

$$\frac{a'}{\sin 22.3^\circ} = \frac{12.8}{\sin 29.3^\circ}$$

$\boxed{a' \doteq 9.9 \text{ cm}}$

∴ $\angle A \doteq 99.1^\circ$, $\angle C \doteq 51.6^\circ$, $a \doteq 25.8$ cm OR $\angle A \doteq 22.3^\circ$, $\angle C \doteq 128.4^\circ$, $a \doteq 9.9$ cm

HW: p. 308-309 #3, 4, 5

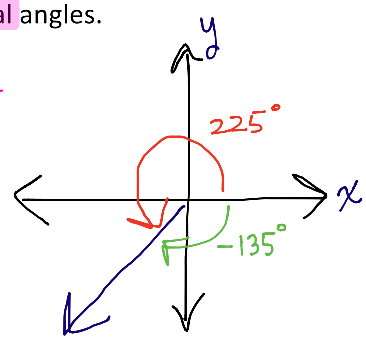
Unit 4 Review

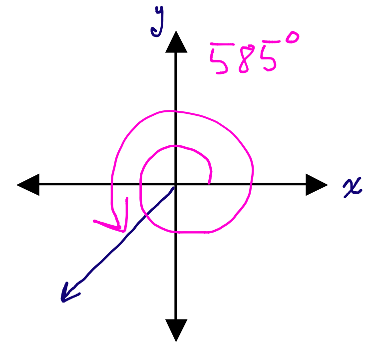
HW Part A: Complete the following:

See Lesson 4.2
& read p. 351

1. a) Draw an angle of 225° in standard position.
   b) Draw and label 2 co-terminal angles.

[add or subtract 360° (or 2π) any number of times...]





See Lesson 4.2
& p. 412 # 1-4

2. For the following conversions, give both exact and approximate answers, where possible.

Keep π → exact; approximate → use 3.14159...

a) Convert 150° to radians.

$$150^\circ \times \frac{\pi}{180^\circ} = \boxed{\frac{5\pi}{6}}$$

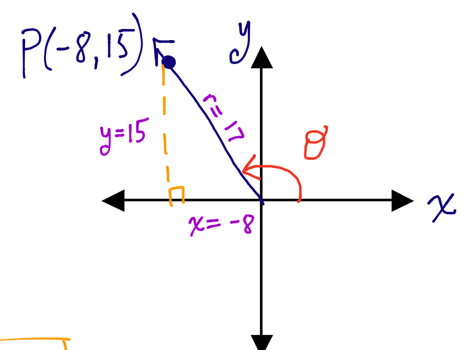
$$\boxed{\doteq 2.62}$$

b) Convert $\frac{5\pi}{3}$ to degrees.

$$\frac{5\cancel{\pi}}{\cancel{3}_1} \times \frac{\cancel{180}^{60\,\circ}}{\cancel{\pi}_1} = 300^\circ$$

See Lesson 4.3
& p. 413 # 10

3. P(– 8, 15) is a point on the terminal arm of θ, an angle in standard position, where $0 \le \theta \le 2\pi$. Find the primary trig ratios for $\angle\theta$. Include a well-labelled diagram.



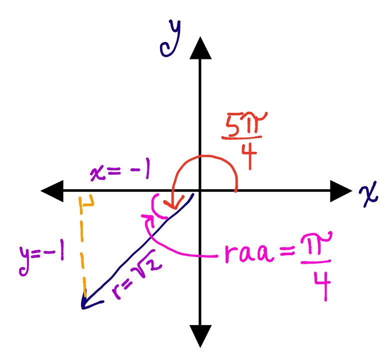
$$r = \sqrt{(-8)^2 + (15)^2}$$

$$r = 17$$

$$\boxed{\therefore \sin\theta = \frac{15}{17} \qquad \cos\theta = -\frac{8}{17} \qquad \tan\theta = -\frac{15}{8}}$$

See Lesson 4.4
& p. 413 # 12

4. Find the exact value of $\sin\frac{5\pi}{4}$. Include a well-labelled diagram.



$$\sin\theta = \frac{y}{r}$$

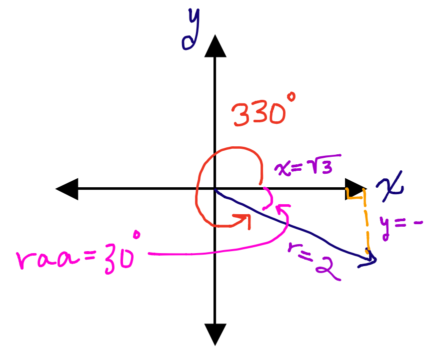
$$\boxed{\therefore \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}}$$

See Lesson 4.4
& p. 413 # 11

5. Find the exact value of tan 330°. Include a well-labelled diagram.

$\tan\theta = \dfrac{y}{x}$

$\boxed{\therefore \tan 330° = -\dfrac{1}{\sqrt{3}}}$



See Lesson 4.4
& p. 354 #3ceh,
4abc

6. State the primary trig ratios of $-\pi$, an angle in standard position. Include a well-labelled diagram.

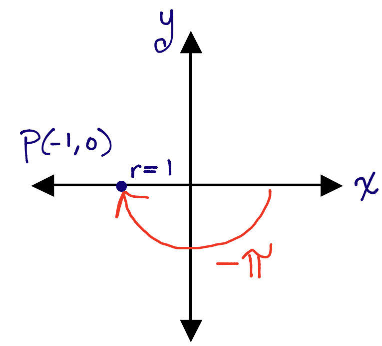
Use a unit circle for angles on the axes, like $(-\pi)$:

$r = 1$, $x = -1$, $y = 0$

$\sin(-\pi) = 0$

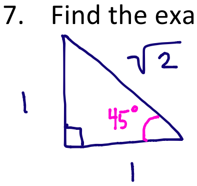
$\cos(-\pi) = -1$

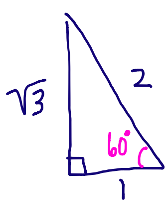
$\tan(-\pi) = 0$



See Lesson 4.4
& p. 350 #18bd

7. Find the exact value for $3\sin 45° \cos 60°$





$\sin 45° = \dfrac{1}{\sqrt{2}}$ $\qquad \cos 60° = \dfrac{1}{2}$

$3\sin 45° \cos 60° = 3\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{2}\right)$

$= \dfrac{3}{2\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{3\sqrt{2}}{2\sqrt{4}}$

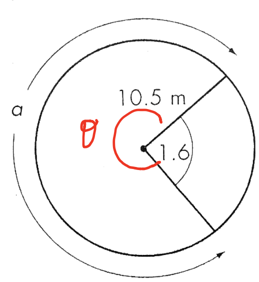
$= \dfrac{3\sqrt{2}}{4}$

$\boxed{\therefore 3\sin 45° \cos 60° = \dfrac{3\sqrt{2}}{4}}$

See Lesson 4.2
& p. 412 #5-7;
p. 336 #8bc

8. Find the indicated quantity in each diagram. Round decimal answers to the nearest tenth.

a)



$a =$ ____

$r = 10.5$ m

$\theta = 2\pi - 1.6$

$= 4.7$

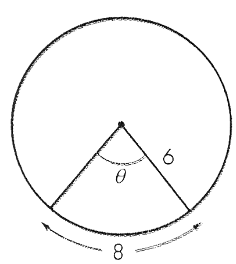
$\boxed{\theta = \dfrac{a}{r}}$

$4.7 = \dfrac{a}{10.5}$

$a = (4.7)(10.5)$

$\boxed{\therefore a = 49.4 \text{ m}}$

b)



$a = 8$ units

$r = 6$ units

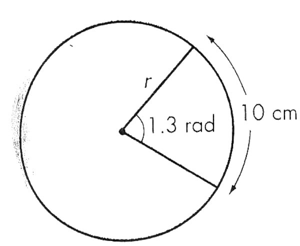
$\theta =$ ____

$\boxed{\theta = \dfrac{a}{r}}$

$\theta = \dfrac{8}{6}$

$\boxed{\therefore \theta \doteq 1.3}$

c)



$a = 10$ cm

$r =$ ____

$\theta = 1.3$

$\boxed{\theta = \dfrac{a}{r}}$

$1.3 = \dfrac{10}{r}$

$r = \dfrac{10}{1.3}$

$\boxed{\therefore r = 7.7 \text{ cm}}$

See Lesson 4.4
& p. 413 #14

9. Find the value(s) of θ (in radians), an angle in standard position, where $0 \le \theta \le 2\pi$.

a) $\cos\theta = -\frac{1}{\sqrt{2}}$

$\therefore$ QII & QIII

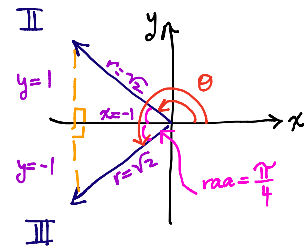
$x = -1$
$r = \sqrt{2}$

In QII:

$\theta = \pi - raa$
$= \frac{4\pi}{4} - \frac{\pi}{4}$
$= \frac{3\pi}{4}$

In QIII:

$\theta = \pi + raa$
$= \frac{4\pi}{4} + \frac{\pi}{4}$
$= \frac{5\pi}{4}$



$\boxed{\therefore \theta = \frac{3\pi}{4}, \frac{5\pi}{4}}$

b) $\sin\theta = -\frac{1}{2}$

$\therefore$ QIII & QIV

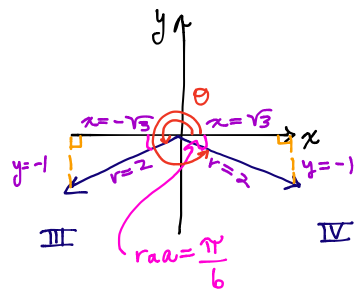
$y = -1$
$r = 2$

In QIII:

$\theta = \pi + raa$
$= \frac{6\pi}{6} - \frac{\pi}{6}$
$= \frac{7\pi}{6}$

In QIV:

$\theta = 2\pi - raa$
$= \frac{12\pi}{6} - \frac{\pi}{6}$
$= \frac{11\pi}{6}$



$\boxed{\therefore \theta = \frac{7\pi}{6}, \frac{11\pi}{6}}$

See Lesson 4.6

10. Find the value(s) of θ (in degrees), an angle in standard position, where $0 \le \theta \le 360°$.

a) $\sin\theta = 0.6561$

$raa = \sin^{-1}(0.6561)$
$raa = 41°$

QI & QII

In QI:

$\theta = raa$
$= 41°$

In QII:

$\theta = 180° - raa$
$= 180° - 41°$
$= 139°$

$\boxed{\therefore \theta = 41°, 139°}$

b) $\tan\theta = -4.3315$

$raa = \tan^{-1}(4.3315)$
$raa = 77°$

QII & QIV

In QII:

$\theta = 180° - raa$
$= 180° - 77°$
$= 103°$

In QIV:

$\theta = 360° - raa$
$= 360° - 77°$
$= 283°$

$\boxed{\therefore \theta = 103°, 283°}$

HW Part B: p.323 – "Chapter 4" #1, 4, 5, 6