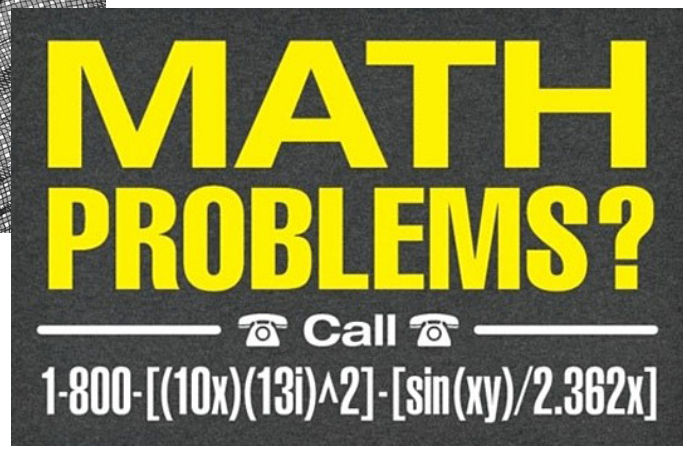
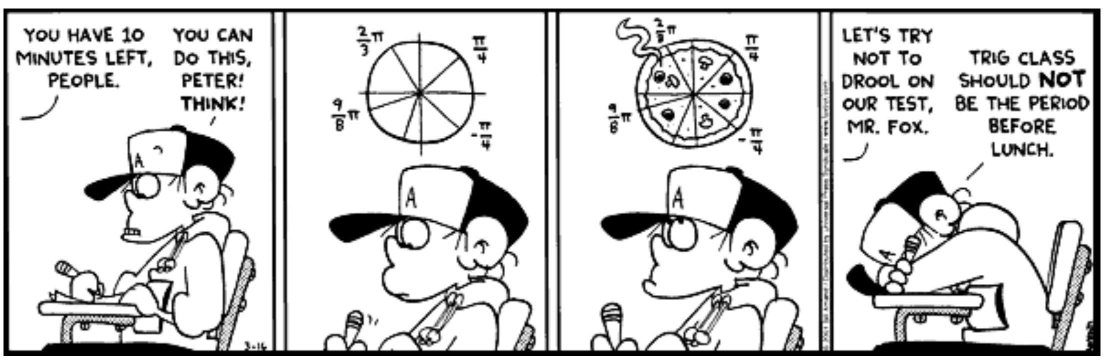




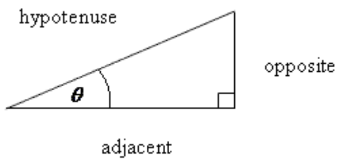
MCR3UI

Unit 4: Trigonometry

Reviewing the Trigonometry of Right Triangles

A. Reviewing the Primary Trigonometric Ratios: SOH CAH TOA

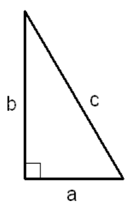
Recall that for any *right triangle*, we can write the three primary trigonometric ratios as equations in three unknowns, where the trig ratio of an <u>angle</u> is equal to one <u>side</u> divided by another <u>side</u>:



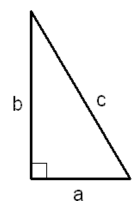
$\sin\theta = \dfrac{opp}{hyp}$ $\cos\theta = \dfrac{adj}{hyp}$ $\tan\theta = \dfrac{opp}{adj}$

Ex. 1: Write the primary trig ratios for i) ∠A and ii) ∠B in the triangle shown below:

i)



ii)



B. Solving for a Side Length

To solve for a **side length**, *clear the fraction* by multiplying both sides of the equation by the denominator or by cross-multiplying. Then, isolate the variable by *dividing out*, if necessary.	
$\tan 72° = \dfrac{y}{6.1}$	$\sin 58° = \dfrac{18.8}{x}$

C. Solving for an Angle

To solve for an **angle** when the ratio is known, take the *inverse trig operation* of both sides:

$$\cos C = \frac{7.9}{13.5}$$

D. Solving a Right Triangle

To ***solve*** a triangle means to find the value of *every side* and *every angle*. (Recall: the *Pythagorean Theorem* is $c^2 = a^2 + b^2$, where *c* is the hypoteneuse of a right triangle).

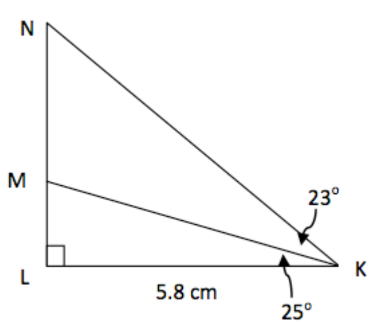
Ex. 2: Solve Δ*DEF* where ∠*E* = 90°, *d* = 5.1 cm and *f* = 8.4 cm. Round angles to the nearest degree and side lengths to one decimal place.

Ex. 3: Solve Δ*ABC* where ∠*B* = 90°, ∠*C* = 12° and *b* = 10.5 cm. Round angles to the nearest degree and side lengths to one decimal place.

E. Angles of Elevation and Depression

Ex. 4: From the top of a 120 m building, the angle of depression to the bottom of a second building is 36°, while the angle of elevation to the top of the second building is 47°. How far apart are the buildings? How tall is the second building?

Ex. 5: Find MN.



HW: p. 272-274 #1a, 2b, 3cd, 4a, 5- 8, 10, 11, 16, 18

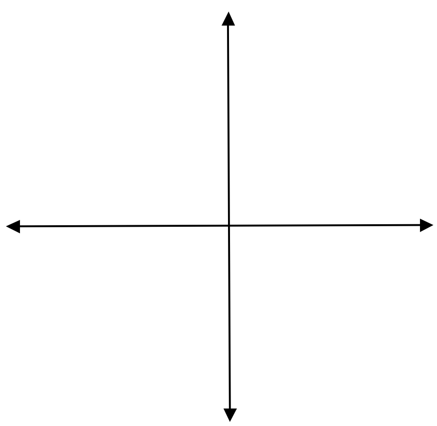
Angles in Standard Position and Radian Measure

Up until this point, we have studied triangle trigonometry – where any angle, θ, was defined as {0° < θ < 180°}. We will now explore the trigonometric ratios for <u>*any*</u> size of angle, such that $\{\theta \in R\}$!

A. Angles in Standard Position

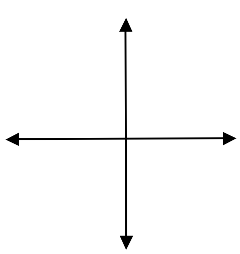
When drawing angles in on a coordinate plane in ***standard position***:

- The **vertex** is at the *origin.*
- The **initial arm** is on the *positive x-axis.*
- The **terminal arm** rotates through an angle about the origin:
  - if the direction of the rotation is counterclockwise, the measure of the angle is *positive*.
  - if the direction of the rotation is clockwise, the angle is *negative.*
- **Co-terminal Angles** are angles in standard position that have the *same terminal arm*.
  Note: To find co-terminal angles to θ, add *or* subtract 360° any number of times!



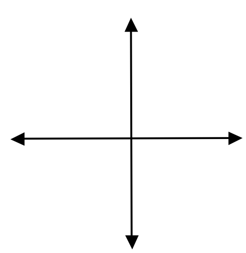
Ex. 1: Draw each angle in standard position. Name a **co-terminal** angle θ such that –360° < θ < 360°.

a) 140°



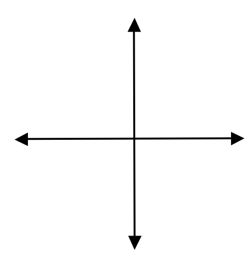
θ = _______________

b) –150°



θ = _______________

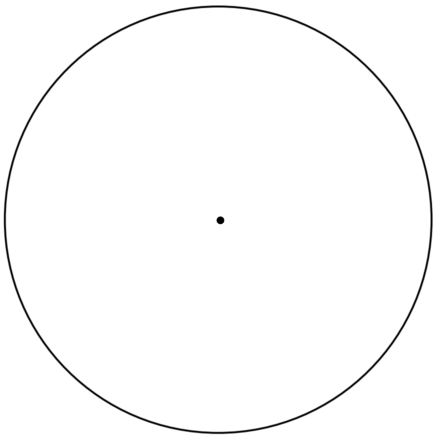
c) 300°



θ = _______________

B. Radians

I) A **radian** is a unit, different from a degree, for measuring angles. One **radian** is "the measure of the *angle* subtended at the centre of a circle by an *arc equal in length to the radius* of the circle".



II) ***Converting Between Degrees and Radians:***

Since **180° = π radians**, $1° = \frac{\pi}{180} rad$ and $1 rad = \frac{180°}{\pi}$.

Therefore, we can use the conversion factors $\frac{180°}{\pi \ rad}$ and $\frac{\pi \ rad}{180°}$ to convert between units.

Ex. 2: Convert from degrees to radians. Give *exact* and *approximate* (*to 2 decimal places*) answers.

a) 45°　　　　b) 72°　　　　c) 315°

Ex. 3: Convert from radians to degrees. Give answers to one decimal place.

a) $\frac{\pi}{4}$ radians　　　　b) 4.3 radians　　　　c) $\frac{7\pi}{6}$

C. Problems Involving Arc Length

I) Recall: $\theta = \frac{a}{r}$, where θ is the measure of the angle in radians; a is the arc length; and r is the radius of the circle.

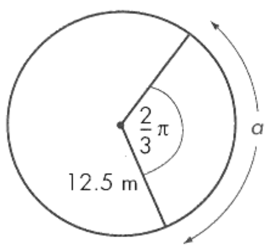
Ex. 4: Given a circle with the following measurements, find the unknown value.

a) θ = 2 rad
r = 6 cm
a = ?

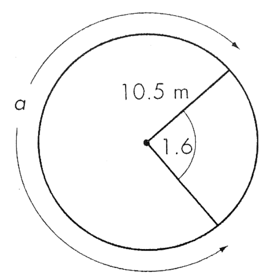
b) θ = ?
r = 10 cm
a = 45 cm

Ex. 5: Find the indicated quantity in each of the following diagrams.

a)



b)



II) *Angular Velocity:*

Ex. 6: An electric motor turns at 2000 revolutions/minute. Find the angular velocity in radians/second. Give an exact answer and an approximate answer.

HW: p. 334-339 #1acej, 2aceg, 3ac, 4cegi, 5a, 6a, 7a, 8b, 9, 11, 13, 18, 20a, 22

Angles on a Coordinate Plane

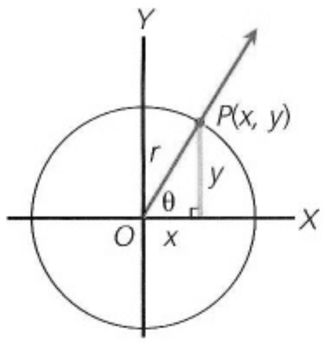
A. Primary Trigonometric Ratios for Any Angle in Standard Position

For an angle, θ, in standard position, with **point P(*x, y*)** on the terminal arm: $r = \sqrt{x^2 + y^2}$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



The **CAST rule** gives the initials of the trigonometric ratios that are positive in each quadrant:

Function	Quadrant			
	I	**II**	**III**	**IV**
sine	+	+	–	–
cosine	+	–	–	+
tangent	+	–	+	–

Note: The value of the "hypoteneuse", *r*, is always positive, as it is the **radius** of a circle that would have **P(*x, y*)** on its **circumference**.

B. The CAST Rule

The coordinate plane can be divided into four **quadrants** using the *x*- and *y*-axes. The three primary trig ratios take on *different signs* depending on which quadrant the terminal arm is found, as summarized by the **CAST Rule.**

QI = {0° < θ < 90°} QII = {90° < θ < 180°} QIII = {180° < θ < 270°} QIV = {270° < θ < 360°}

Ex. 1: Use the CAST rule to determine which quadrant(s) the terminal arm could be found for each trigonometric ratio.

a) sin θ = 0.5 Q:________ b) cos θ = – 0.6432 Q:________ c) tan θ = – 1.2 Q:________

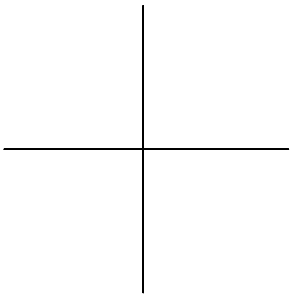
Ex. 2: Find the exact primary trig ratios for each point on the terminal arm of θ, an angle in standard position.

a) (3, 4)

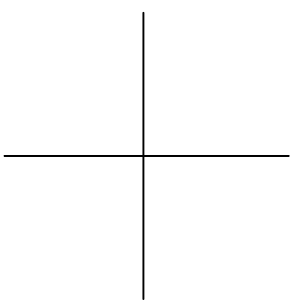
b) (–12, –5)

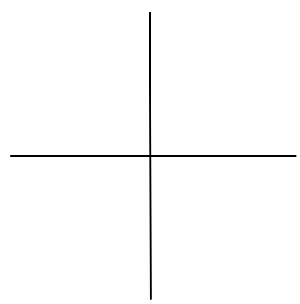
Ex. 3: For each example below, $\angle\theta$ is given in standard position, and $0 \le \theta \le 2\pi$. A trigonometric ratio is given. Find all possible *exact values* of the other two trigonometric ratios.

a) $\sin\theta = -\frac{3}{5}$ and the terminal arm of $\angle\theta$ is in quadrant III



b) $\tan\theta = \frac{1}{\sqrt{3}}$



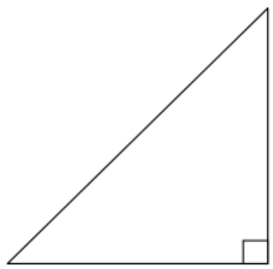


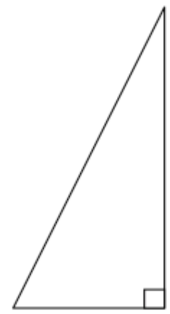
HW: p. 348-349 #1, 2, 6, 7 (Sketch the diagrams!)

Special Angles: Special Triangles and the Unit Circle

We have started to investigate the trigonometric ratios for any angle. We can further our discussion by examining the **special angles** using *special triangles* and the *unit circle*.

A. Special Triangles:

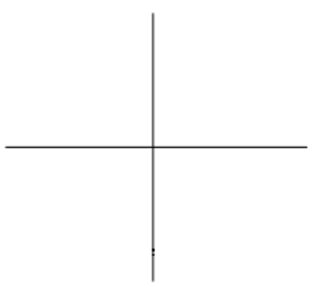




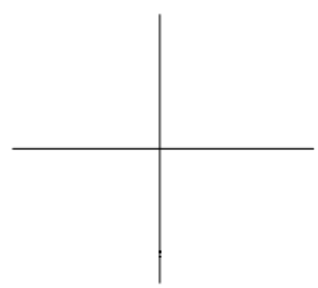
The **reference angle** (or *related acute angle **r.a.a.***) is the acute angle made **between** the <u>*x*-axis and the terminal arm</u> of an angle in standard position. When the reference angle is 30°, 45°, or 60°, we can use the special triangles to find the **exact values** (i.e. fractions – no decimals!) of the trig ratios. ☺

Ex. 1: Use special triangles and reference angles to find the **exact values** of the trig ratios for the following angles.

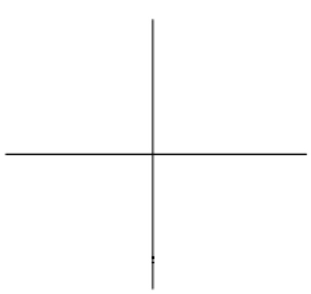
a) $\theta = 45°$



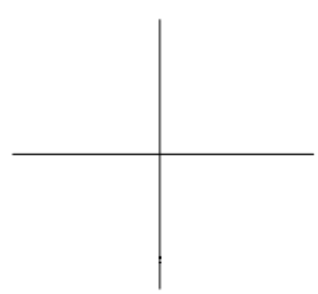
b) $\theta = 120°$



c) $\theta = \frac{7\pi}{6}$

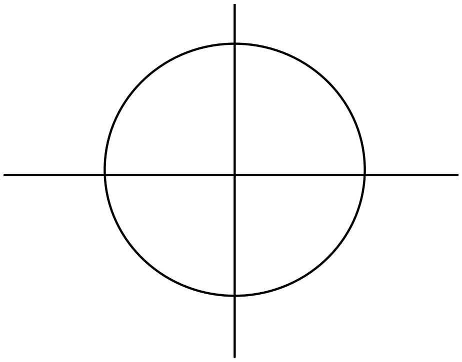


d) $\theta = \frac{11\pi}{6}$



B. The Unit Circle (a circle with radius = 1)

Previously, we have investigated angles on the coordinate plane *in the four quadrants*, but what about angles that lie *on the axes*? Enter the unit circle!



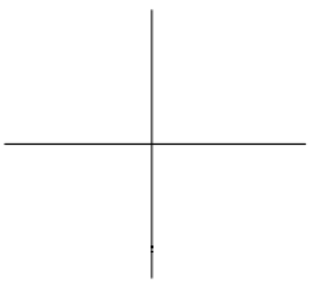
i) $\sin 90°$

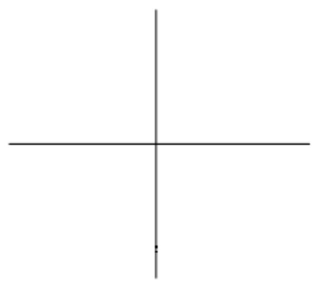
ii) $\tan\frac{3\pi}{2}$

iii) $\cos\pi$

iv) $\cos(-270°)$

Ex. 2: If $\sin\theta = \frac{1}{\sqrt{2}}$, $0° \le \theta \le 360°$, find the values of $\cos\theta$ and θ





Ex. 3: Find the *exact* value of 2sin30°cos30°.

HW: p. 348-349 #3abc, 4aef, 11-14, 18ace

Trigonometric Ratios for Positive and Negative Angles of Rotation

Find the exact values of the primary trig ratios for the following angles in standard position.

Steps: *1) Draw the angle in standard position*

2) If the terminal arm is on one of the axes, use the unit circle to find the exact values for the required ratios

3) If the terminal arm is in one of the four quadrants:

i) add or subtract 360° or 2π to determine the angle between the terminal arm and the x-axis

ii) Use the related acute angle and the CAST rule to determine the exact values for the required ratios

a) $\theta = 600°$

b) $\theta = -\frac{13\pi}{6}$

c) $\theta = -4\pi$

d) $\theta = -585°$

e) $\theta = \frac{13\pi}{2}$

f) $\theta = -810°$

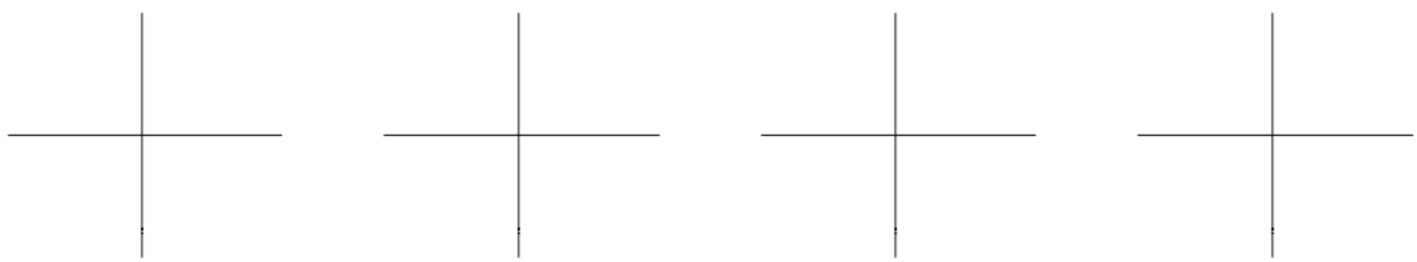
HW: p. 353 # 1, 2 and p. 354 # 3, 4

Combining the CAST Rule, Related Acute Angles, and Special Triangles

Steps for finding the possible value(s) of θ:

- determine the possible *quadrants* for the terminal arm using the **CAST rule**
- sketch the *terminal arm* in the appropriate quadrant(s)
- find the ***r.a.a.*** for θ using ***special angles*** or *inverse trig operations* on the value of the ratio (do not include the negative sign for this calculation – we are looking for the related *acute* angle!)
- add or subtract the *r.a.a.* from 180° (π) or 360° (2π) to get θ, the positive angle between the initial arm and the terminal arm for each quadrant
- provide a therefore statement for each quadrant

Given any trig ratio, we can find all angles that will satisfy the equation for $0° \le \theta \le 360°$ or $0 \le \theta \le 2\pi$.



QI: θ = _______________ **QII: θ = ________________** **QIII: θ = _______________** **QIV: θ = _______________**

A. Find θ, an angle in standard position, for $0° \le \theta \le 360°$. Round to the nearest whole degree, where necessary.

1. $\cos\theta = \frac{1}{2}$

2. $\tan\theta = -2.3546$

3. $\tan\theta = -1$

4. $\sin\theta = 0.1357$

5. $\cos\theta = -0.8436$

6. $\sin\theta = \frac{\sqrt{3}}{2}$

B. Find θ, an angle in standard position, for $0 \le \theta \le 2\pi$. Give the exact solution, where possible. Otherwise, round to the nearest hundredth of a radian.

1. $\sin\theta = -\frac{1}{2}$
2. $\tan\theta = 3.4$
3. $\cos\theta = \frac{1}{\sqrt{2}}$
4. $\tan\theta = -\sqrt{3}$
5. $\cos\theta = -0.9205$
6. $\sin\theta = \frac{1}{\sqrt{3}}$

C. Combine the CAST Rule, related acute angles, and special triangles ***or*** use the unit circle to find exact values of the following. **Rationalize the denominator when necessary.*

1. $(\sin 135°)(\cos 240°)$
2. $\tan\frac{11\pi}{6} - \cos\left(-\frac{3\pi}{4}\right)$
3. $\left(\cos\frac{11\pi}{6}\right)^2 + \left(\sin\frac{3\pi}{2}\right)^2$
4. $\tan(-300°)\cdot\cos(-210°)$
5. $2(\cos 2\pi) + \frac{1}{\left(\cos\frac{7\pi}{3}\right)}$
6. $\left(\sin\frac{5\pi}{4}\right)^2 + \left(\cos\frac{5\pi}{4}\right)^2$
7. $\tan(-510°) + (\sin 600° + \tan 540°)$
8. $\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) + \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$
9. $\dfrac{\tan\frac{\pi}{3} + \tan\frac{7\pi}{4}}{1 - \left(\tan\frac{\pi}{3}\right)\left(\tan\frac{7\pi}{4}\right)}$

Answers:

Part A:

1. $\cos\theta = \frac{1}{2}$

Q: I and IV, r.a.a. = 60°

$\therefore \theta = 60°$ *or* $300°$

2. $\tan\theta = -2.3546$

Q: III and IV, r.a.a. ≐ 67°

$\therefore \theta = 113°$ *or* $293°$

3. $\tan\theta = -1$

Q: II and IV, r.a.a. = 45°

$\therefore \theta = 135°$ *or* $315°$

4. $\sin\theta = 0.1357$

Q: I and II, r.a.a. ≐ 8°

$\therefore \theta = 8°$ *or* $172°$

5. $\cos\theta = -0.8436$

Q: II and III, r.a.a. ≐ 32°

$\therefore \theta = 148°$ *or* $212°$

6. $\sin\theta = \frac{\sqrt{3}}{2}$

Q: I and II, r.a.a. = 60°

$\therefore \theta = 60°$ *or* $120°$

Part B:

1. $\sin\theta = -\frac{1}{2}$

Q: III and IV, r.a.a. $= \frac{\pi}{6}$

$\therefore \theta = \frac{7\pi}{6}$ *or* $\frac{11\pi}{6}$

2. $\tan\theta = 3.4$

Q: I and III, r.a.a. ≐ 1.28

$\therefore \theta \doteq 1.28$ *or* 4.42

3. $\cos\theta = \frac{1}{\sqrt{2}}$

Q: I and IV, r.a.a. $= \frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$ *or* $\frac{7\pi}{4}$

4. $\tan\theta = -\sqrt{3}$

Q: II and IV, r.a.a. $= \frac{\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$ *or* $\frac{5\pi}{3}$

5. $\cos\theta = -0.9205$

Q: II and III, r.a.a. ≐ 0.40

$\therefore \theta \doteq 2.74$ *or* 3.54

6. $\sin\theta = \frac{1}{\sqrt{3}}$

Q: I and II, r.a.a. ≐ 0.62

$\therefore \theta \doteq 0.62$ *or* 2.52

Part C:

1. $(\sin 135°)(\cos 240°)$

$= -\frac{\sqrt{2}}{4}$

2. $\tan\frac{11\pi}{6} - \cos\left(-\frac{3\pi}{4}\right)$

$= \frac{3\sqrt{2} - 2\sqrt{3}}{6}$

3. $\left(\cos\frac{11\pi}{6}\right)^2 + \left(\sin\frac{3\pi}{2}\right)^2$

$= \frac{7}{4}$

4. $\tan(-300°)\cdot\cos(-210°)$

$= -\frac{3}{2}$

5. $2(\cos 2\pi) + \frac{1}{\left(\cos\frac{7\pi}{3}\right)}$

$= 4$

6. $\left(\sin\frac{5\pi}{4}\right)^2 + \left(\cos\frac{5\pi}{4}\right)^2$

$= 1$

7. $\tan(-510°) + (\sin 600° + \tan 540°)$

$= -\frac{\sqrt{3}}{6}$

8. $\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) + \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

9. $\frac{\tan\frac{\pi}{3} + \tan\frac{7\pi}{4}}{1 - \left(\tan\frac{\pi}{3}\right)\left(\tan\frac{7\pi}{4}\right)}$

$= 2 - \sqrt{3}$

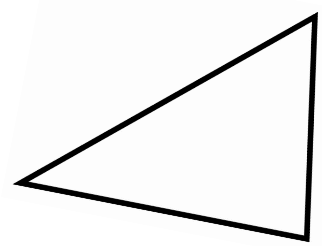
The Cosine Law

The Cosine Law is another expansion of a fundamental trig ratio into a law that can be used on any triangle (non-right triangles included)! It represents a general version of the Pythagorean Theorem, adapted to non-right triangles. It can also be rearranged to solve for the unknown angle.

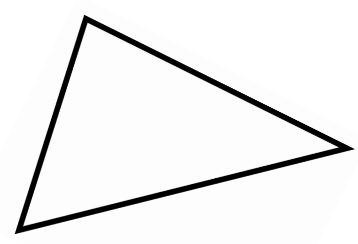
The Cosine Law	***Rearranged to Solve for the Angle***
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

When do we use the *Cosine Law*?

1. When we are given two sides and a contained angle, and we want to find the side opposite the angle.



2. When we are given three sides, and we want to find any other angle.
(When given a choice, find the largest angle first!)



1. Two Sides and a Contained Angle

Solve ΔPQR, where $\angle R = 121^{o}$, $p = 32$ cm, and $q = 27$ cm

2. Three Sides

Solve ΔDEF, where $d = 41$ cm, $e = 32$ cm, and $f = 21$ cm

HW: p. 290-293 #1bf, 2ce, 3cef, 4b, 5bd, 8, 10, 12

The Sine Law

The Sine Law is an expansion of a fundamental trig ratio into a law that can be used on any triangle (non-right triangles included)! It states that the sines of the angles of a triangle are proportional to the lengths of their opposite sides.

The Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

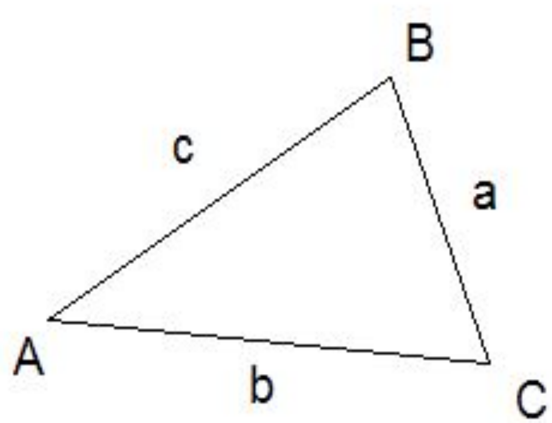
The Sine Law can be written in two forms, as the relationship represents a proportion. It is easiest to solve a proportion when **the unknown is in the numerator**, so when the unknown is an angle, use the first form, and when the unknown is a *side length*, use the *second form*.

When can we use the *Sine Law*?

1. When we are given **two angles and a side** for an acute triangle. (We use the angle sum of 180^{o} to find the third angle, and then set up a proportion to find any side we want!)

2. When we are given **two sides and an angle opposite one of them**. (We set up a proportion to find the second angle.)

Basically, we can use the ***Sine Law*** *when we have a* **corresponding side-angle pair plus one other piece of information** *about the triangle.*



1. Two Angles and a Side

Solve ΔPQR, where $\angle R = 68^o$, $\angle Q = 69^o$, and $p = 21$ cm

2. Two Sides and an Angle Opposite One of Them

Solve ΔABC, where $\angle B = 75^o$, $a = 8.3$ cm, and $b = 10.4$ cm

HW: p. 290-293 #1acde, 2abf, 3abd, 4a[*answer*: 41.2 m], 4ce, 5a, 6, 9

The Ambiguous Case of the Sine Law!

A. Trig Ratios for Angles and Their Supplements (*Recall*: supplementary angles add to __________)

$cos\ 50^{o}$ = ____________ cos^{-1}(_____________) = _________

$cos\ 130^{o}$ = ___________ cos^{-1}(_____________) = _________

$sin\ 50^{o}$ = ____________ sin^{-1}(_____________) = _________

$sin\ 130^{o}$ = ___________ sin^{-1}(_____________) = _________

Conclusion:

The **sine ratios** for an acute angle and its obtuse supplement are the same value and the **same sign**. However, when taking the **inverse sine** of the ratio, the angle returned is always the **acute** supplement!

Cosine ratios for an acute angle and its obtuse supplement are the same value but have **different signs**, so the correct angle is returned when you take the inverse cosine of the ratio. (That's why the *Cosine Law* is not ambiguous. ☺)

B. Recognizing When the Sine Law May Be Ambiguous

In some instances, the sine law is ambiguous (this means it will give you more than one answer). Imagine you are given a triangle with values for ∠ *A*, side *a*, and side *b*. When side *b* is larger and you solve for the first unknown angle, ∠*B*, there are three possible cases, as shown in the figures below:

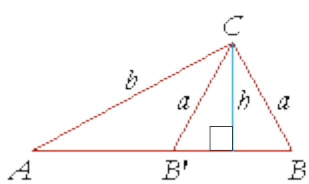


Fig. 1

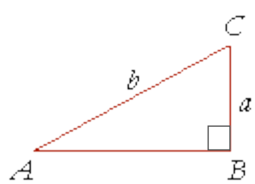


Fig. 2

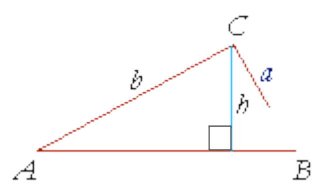


Fig. 3

1. ∠*B* is an acute angle, which means that you will have to solve the triangle twice (*two possible triangles*): one triangle using ∠*B,* and one triangle using its *supplement* ∠*B*′ (180° – ∠*B*)

2. ∠*B* is a 90° angle (*one triangle*)

3. ∠*B* cannot be found – "calculator error"! (*no triangles*)

- In general, the sine law may be ambiguous when you have ***a side-angle pair*** and a ***larger side***.
- It is NOT ambiguous when you have a side-angle pair and a smaller side or a side-angle pair and an equal length side.

Ex. 1: Determine the number of possible triangles that could be drawn with the given measures for each of the following:

a) Given ΔDEF, where $\angle D = 44.3^{\circ}$, $d = 11.5$ cm, and $e = 7.7$ cm Number of Triangles: _______

b) Given ΔUVW, where $\angle W = 38.7^{\circ}$, $w = 10$ m, and $v = 25$ m Number of Triangles: _______

c) Given ΔPQR, where $\angle P = 30.0^{\circ}$, $p = 24.0$ cm, and $q = 48.0$ cm Number of Triangles: _______

d) Given ΔKLM, where $\angle K = 27^{\circ}$, $k = 25$ mm, and $m = 30$ mm Number of Triangles: _______

Ex. 2: Solve ΔABC, where $\angle B = 29.3^{\circ}$, $c = 20.5$ cm, and $b = 12.8$ cm

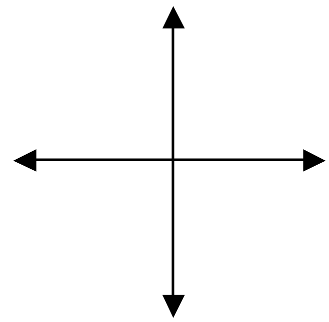
HW: p. 308-309 #3, 4, 5

Unit 4 Review

HW Part A: Complete the following:

See Lesson 4.2
& read p. 351

1. a) Draw an angle of 225° in standard position.
   b) Draw and label 2 co-terminal angles.



See Lesson 4.2
& p. 412 # 1-4

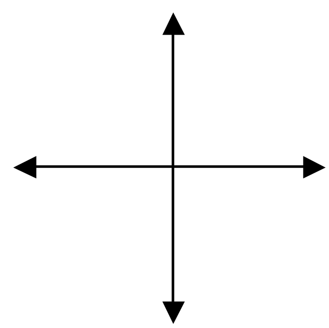
2. For the following conversions, give both exact and approximate answers, where possible.

   a) Convert 150° to radians.

   b) Convert $\frac{5\pi}{3}$ to degrees.

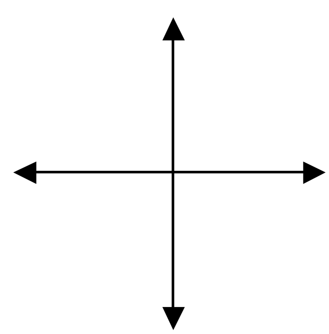
See Lesson 4.3
& p. 413 # 10

3. P(– 8, 15) is a point on the terminal arm of θ, an angle in standard position, where $0 \le \theta \le 2\pi$. Find the primary trig ratios for $\angle\theta$. Include a well-labelled diagram.



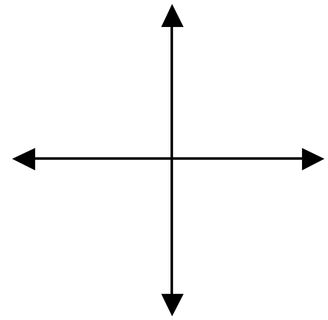
See Lesson 4.4
& p. 413 # 12

4. Find the exact value of $\sin\frac{5\pi}{4}$. Include a well-labelled diagram.



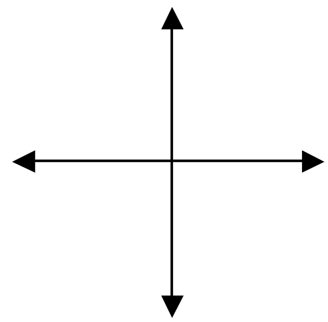
See Lesson 4.4
& p. 413 # 11

5. Find the exact value of tan 330°. Include a well-labelled diagram.



See Lesson 4.4
& p. 354 #3ceh,
4abc

6. State the primary trig ratios of $-\pi$, an angle in standard position.
Include a well-labelled diagram.



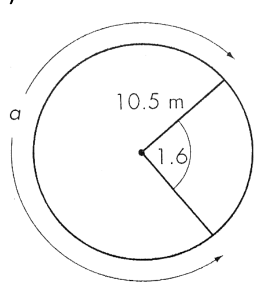
See Lesson 4.4
& p. 350 #18bd

7. Find the exact value for $3\sin 45^\circ \cos 60^\circ$

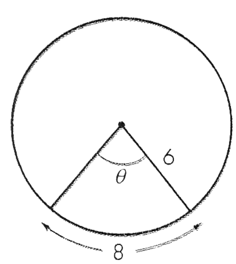
See Lesson 4.2
& p. 412 #5-7;
p. 336 #8bc

8. Find the indicated quantity in each diagram. Round decimal answers to the nearest tenth.

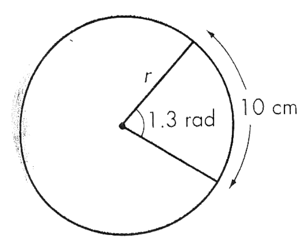
a)



b)



c)



See Lesson 4.4 & p. 413 #14

9. Find the value(s) of θ (in radians), an angle in standard position, where $0 \le \theta \le 2\pi$.

a) $\cos\theta = -\frac{1}{\sqrt{2}}$

b) $\sin\theta = -\frac{1}{2}$

See Lesson 4.6

10. Find the value(s) of θ (in degrees), an angle in standard position, where $0 \le \theta \le 360°$.

a) $\sin\theta = 0.6561$

b) $\tan\theta = -4.3315$

HW Part B: p.323 – "Chapter 4" #1, 4, 5, 6