Trig Identities – Part I

An IDENTITY is an equation that is true for all values of the variable for which the expressions on both sides of the equation are defined.

Recall: $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$

I. Simplify: $\frac{\sin\theta}{\cos\theta}$

$$= \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)}$$

$$= \frac{y}{r} \div \frac{x}{r}$$

$$= \frac{y}{\cancel{r}_1} \times \frac{\cancel{r}^1}{x}$$

$$= \frac{y}{x}$$

*Note:

$\tan^2\theta = [\tan\theta]^2$

The Quotient Identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

II. From the Pythagorean Theorem:

$x^2 + y^2 = r^2$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

The Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$
$$= (1-\sin\theta)(1+\sin\theta)$$

$$\sin^2\theta = 1 - \cos^2\theta$$
$$= (1-\cos\theta)(1+\cos\theta)$$

Recall:

III. Prove each of the following identities. Start with the more "complex" side and transform it algebraically into the exact form of the "simpler" side. * always replace $\tan\theta$ with $\frac{\sin\theta}{\cos\theta}$

a. Prove: $1-\sin^2 x = \dfrac{\sin^2 x}{\tan^2 x}$

$L.S. = 1-\sin^2 x$

$= \cos^2 x$

$R.S. = \dfrac{\sin^2 x}{\tan^2 x}$

$= \dfrac{\sin^2 x}{\left(\dfrac{\sin^2 x}{\cos^2 x}\right)}$

$= \dfrac{\cancel{\sin^2 x}^{\,1}}{1} \cdot \dfrac{\cos^2 x}{\cancel{\sin^2 x}_{\,1}}$

$= \cos^2 x$

$= 1-\sin^2 x$

$\because LS = RS$

$\therefore 1-\sin^2 x = \dfrac{\sin^2 x}{\tan^2 x}$

b. Prove: $\dfrac{\sin^2 x}{1-\cos x} = 1+\cos x$

$L.S. = \dfrac{\sin^2 x}{1-\cos x}$

$= \dfrac{1-\cos^2 x}{1-\cos x}$

$= \dfrac{\cancel{(1-\cos x)}^{\,1}(1+\cos x)}{\cancel{1-\cos x}_{\,1}}$

$= 1+\cos x$

$RS = 1+\cos x$

$\because LS = RS$

$\therefore$ Q.E.D.

c. Prove: $\cos x = \dfrac{1}{\cos x} - \sin x \tan x$

$LS = \cos x$

$$RS = \frac{1}{\cos x} - \sin x \tan x$$

$$= \frac{1}{\cos x} - \frac{\sin x}{1}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$$

$$= \frac{1 - \sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x}{\cos x}$$

$$= \frac{\cancel{\cos x}^{1} \cdot \cos x}{\cancel{\cos x}_{1}}$$

$$= \cos x$$

$\because LS = RS$

$\therefore$ Q.E.D.

d. Prove: $\dfrac{\sin x}{\sin x + \cos x} = \dfrac{\tan x}{1 + \tan x}$

$$LS = \frac{\sin x}{\sin x + \cos x}$$

$$RS = \frac{\tan x}{1 + \tan x}$$

$$= \frac{\left(\frac{\sin x}{\cos x}\right)}{\frac{\cos x}{\cos x}\cdot\frac{1}{1} + \left(\frac{\sin x}{\cos x}\right)}$$

$$= \frac{\left(\frac{\sin x}{\cancel{\cos x}_{1}}\right)}{\left(\frac{\cos x + \sin x}{\cancel{\cos x}_{1}}\right)} \qquad \begin{array}{l} \cdot\,\frac{\cancel{\cos x}^{1}}{1} \\ \cdot\,\frac{\cancel{\cos x}^{1}}{1} \end{array}$$

$$= \frac{\sin x}{\cos x + \sin x}$$

$\because LS = RS$

$\therefore$ QED.

Recall: $\tan x = \dfrac{\sin x}{\cos x}$, $\sin^2 x + \cos^2 x = 1$.

e. Prove: $\cos^2 x = \sin^2 x + 2\cos^2 x - 1$

$LS = \cos^2 x$

$$RS = \sin^2 x + 2\cos^2 x - 1$$
$$= (1 - \cos^2 x) + 2\cos^2 x - 1$$
$$= 1 - \cos^2 x + 2\cos^2 x - 1$$
$$= \cos^2 x$$

(collect like terms)

f. Prove: $\dfrac{1+\tan^2 x}{1-\tan^2 x} = \dfrac{1}{\cos^2 x - \sin^2 x}$

$$LS = \frac{1+\tan^2 x}{1-\tan^2 x}$$
$$= \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}}$$
$$= \frac{\left(\frac{\cos^2 x + \sin^2 x}{\cancel{\cos^2 x}}\right)}{\left(\frac{\cos^2 x - \sin^2 x}{\cancel{\cos^2 x}}\right)} \cdot \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos^2 x}}{\cos^2 x - \sin^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x}$$
$$= \frac{1}{\cos^2 x - \sin^2 x}$$

$$RS = \frac{1}{\cos^2 x - \sin^2 x}$$

$\therefore LS = RS$

$\therefore$ QED

g. Prove: $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

$$LS = \frac{\sin x - 1}{\sin x + 1}$$

$$RS = \frac{-\cos^2 x}{(\sin x + 1)^2}$$
$$= \frac{-(1 - \sin^2 x)}{(\sin x + 1)^2}$$
$$= \frac{\sin^2 x - 1}{(\sin x + 1)(\sin x + 1)}$$
$$= \frac{(\sin x - 1)\cancel{(\sin x + 1)}}{(\sin x + 1)\cancel{(\sin x + 1)}} = \frac{\sin x - 1}{\sin x + 1}$$

$\therefore LS = RS$

$\therefore$ Q.E.D.

HW: p. 398-399 # 2acegik, 3bdf, 4acehi

Trig Identities – Part II

I. Reciprocal Trigonometric Ratios and Identities

$cosecant\,\theta = \frac{r}{y}$ $secant\,\theta = \frac{r}{x}$ $cotangent\,\theta = \frac{x}{y}$

$csc\,\theta = \frac{r}{y}$ $sec\,\theta = \frac{r}{x}$ $cot\,\theta = \frac{x}{y}$

$csc\,\theta = \frac{1}{sin\theta}$ $sec\,\theta = \frac{1}{cos\theta}$ $cot\,\theta = \frac{1}{tan\theta}$

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$csc\theta = \frac{1}{sin\theta}$	$tan\theta = \frac{sin\theta}{cos\theta}$	$sin^2\theta + cos^2\theta = 1$
$sec\theta = \frac{1}{cos\theta}$		$sin^2\theta = 1 - cos^2\theta$
$cot\theta = \frac{1}{tan\theta}$	$\boxed{cot\theta = \frac{cos\theta}{sin\theta}}$	$cos^2\theta = 1 - sin^2\theta$

II. Prove each identity:

a. $csc\theta - \frac{cot\theta}{sec\theta} = sin\theta$

$LS = csc\theta - \frac{cot\theta}{sec\theta}$

$RS = sin\theta$

$= \frac{1}{sin\theta} - \frac{\left(\frac{cos\theta}{sin\theta}\right)}{\left(\frac{1}{cos\theta}\right)}$

$= \frac{1}{sin\theta} - \left(\frac{cos\theta}{sin\theta}\right)\cdot\left(\frac{cos\theta}{1}\right)$

$= \frac{1 - cos^2\theta}{sin\theta}$

$= \frac{\overset{sin\theta}{\cancel{sin^2\theta}}}{\cancel{sin\theta}\,1}$

$= \frac{sin\theta \cdot \overset{1}{\cancel{sin\theta}}}{\cancel{sin\theta}\,1}$

$= sin\theta$

$\therefore LS = RS$

$\therefore$ Q.E.D.

Recall: $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

b. $tan\theta + cot\theta = sec\theta csc\theta$

$LS = \tan\theta + \cot\theta$

$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$= \frac{\sin\theta \cdot \sin\theta}{\cos\theta \cdot \sin\theta} + \frac{\cos\theta \cdot \cos\theta}{\cos\theta \cdot \sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\cos\theta\sin\theta}$

$RS = \sec\theta\csc\theta$

$= \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}$

$= \frac{1}{\cos\theta\sin\theta}$

$\therefore LS = RS$

$\therefore$ QED

III. Prove the following identities for homework: #1 – 4 (bdf for all)

(A)

1. Prove each identity.
   a) $\tan\theta\cos\theta = \sin\theta$
   b) $\cot\theta\sec\theta = \csc\theta$
   c) $\sin\theta\cot\theta = \cos\theta$
   d) $\tan\theta\csc\theta = \sec\theta$
   e) $\sin\theta = \dfrac{\tan\theta}{\sec\theta}$
   f) $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$

2. Prove each identity.
   a) $\csc\theta(1 + \sin\theta) = 1 + \csc\theta$
   b) $\sin\theta(1 + \csc\theta) = 1 + \sin\theta$
   c) $\cos\theta(\sec\theta - 1) = 1 - \cos\theta$
   d) $\sin\theta\sec\theta\cot\theta = 1$
   e) $\dfrac{1 - \tan\theta}{1 - \cot\theta} = -\tan\theta$
   f) $\cot\theta = \dfrac{1 + \cot\theta}{1 + \tan\theta}$

(B)

3. Prove each identity.
   a) $\sin\theta\tan\theta + \sec\theta = \dfrac{\sin^2\theta + 1}{\cos\theta}$
   b) $\dfrac{1 + \cos\theta}{1 - \cos\theta} = \dfrac{1 + \sec\theta}{\sec\theta - 1}$
   c) $\dfrac{1 + \sin\theta}{1 - \sin\theta} = \dfrac{\csc\theta + 1}{\csc\theta - 1}$
   d) $\dfrac{1 + \tan\theta}{1 + \cot\theta} = \dfrac{1 - \tan\theta}{\cot\theta - 1}$
   e) $\dfrac{1 + \sin\theta}{1 + \csc\theta} = \sin\theta$
   f) $\dfrac{\sin\theta + \tan\theta}{\cos\theta + 1} = \tan\theta$

4. Prove each identity.
   a) $\sin^2\theta\cot^2\theta = 1 - \sin^2\theta$
   b) $\csc^2\theta - 1 = \csc^2\theta\cos^2\theta$
   c) $\sin^2\theta = \dfrac{\tan^2\theta}{1 + \tan^2\theta}$
   d) $\dfrac{\sin\theta + \cos\theta\cot\theta}{\cot\theta} = \sec\theta$
   e) $\sin\theta\cos\theta\tan\theta = 1 - \cos^2\theta$
   f) $\dfrac{\cos\theta}{1 + \sin\theta} + \dfrac{\cos\theta}{1 - \sin\theta} = 2\sec\theta$