Properties of Exponential Functions

Exponential functions are curves that ***increase*** or ***decrease*** throughout their domains. They have the basic form $y = b^x$, where $b > 1$ *or* $0 < b < 1$. They model many different phenomena, including population growth and radioactive decay.

I. Comparing the graphs of $y = b^x$ when $b > 1$

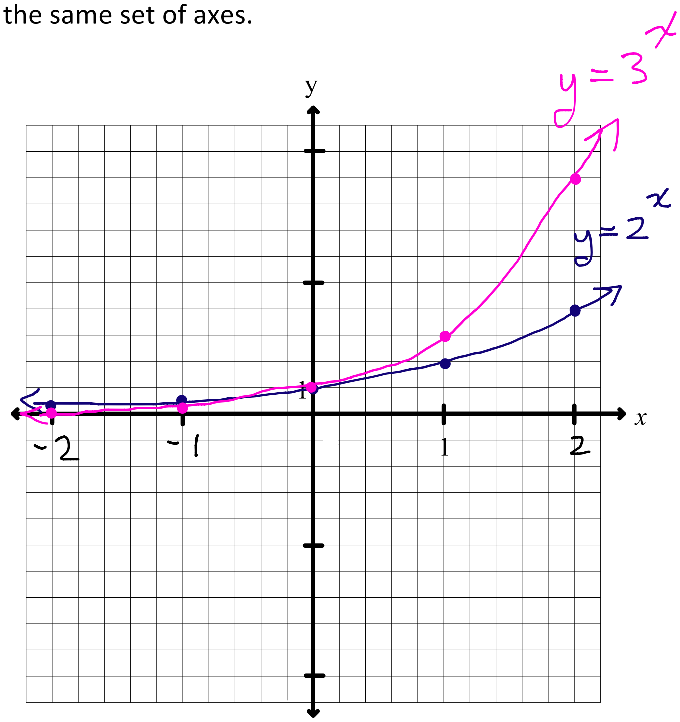
1. Create a table of values for each function, and graph them both on the same set of axes.

i) $y = 2^x$

x	y
-2	$\frac{1}{4}$ → 2^{-2}
-1	$\frac{1}{2}$ → 2^{-1}
0	1 → 2^0
1	2 → 2^1
2	4 → 2^2

ii) $y = 3^x$

x	y
-2	$1/9$ → 3^{-2}
-1	$1/3$ → 3^{-1}
0	1 → 3^0
1	3 → 3^1
2	9 → 3^2



2. Sketch the functions $y = 2^x$, $y = 5^x$ and $y = 10^x$ on the same set of axes. Summarize your findings below.

For $y = b^x$ when $b > 1$:

Domain = $\{x \mid x \in \mathbb{R}\}$

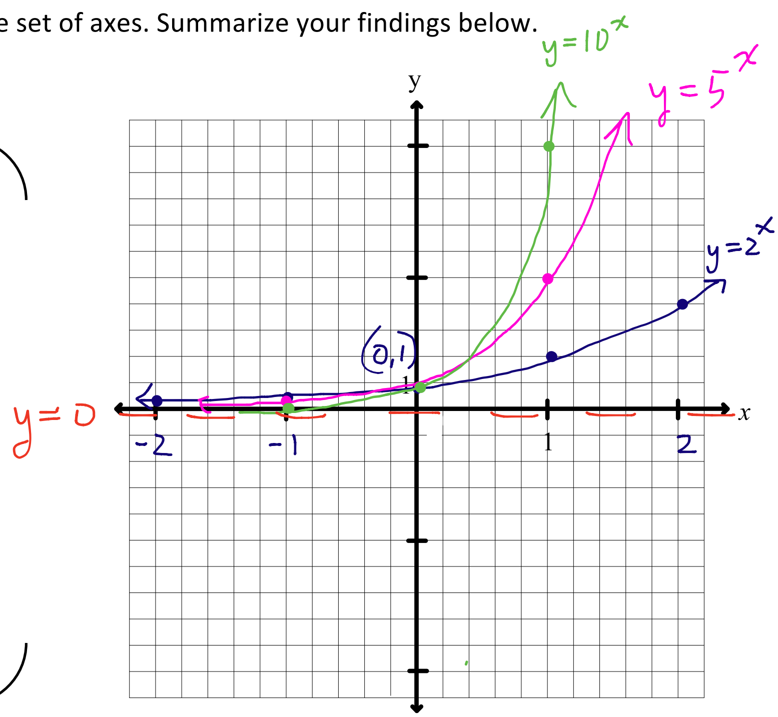
Range = $\{y \mid y \in \mathbb{R}, y > 0\}$

y-intercept = $(0, 1)$

Horizontal Asymptote: $y = 0$

Increasing or decreasing? increasing

As *b* ↑: the curve increases at an increasing rate (steeper curve)



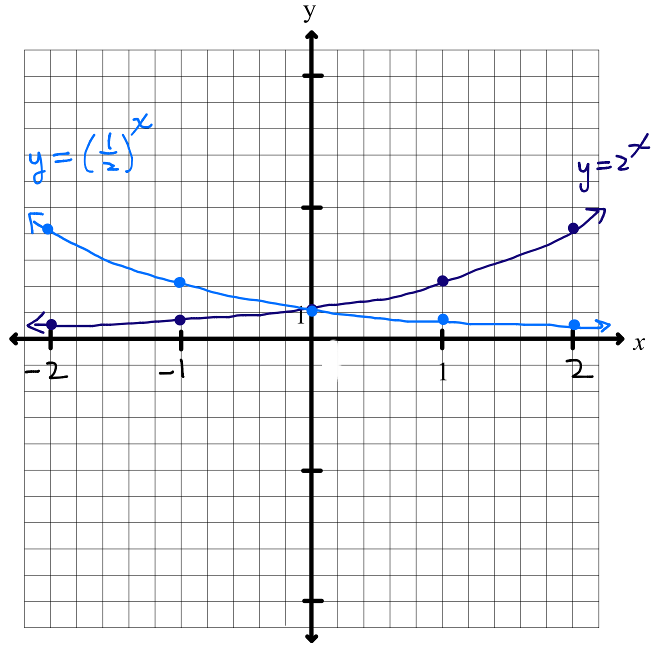
II. Comparing the graphs of $y = b^x$ when $0 < b < 1$

1. Create a table of values for each function, and graph them both on the same set of axes.

i) $y = \left(\frac{1}{2}\right)^x$ ii) $y = 2^x$

x	y
-2	4
-1	2
0	1
1	1/2
2	1/4

x	y
-2	1/4
-1	1/2
0	1
1	2
2	4



Note that the equation $y = \left(\frac{1}{2}\right)^x$ is equivalent to $y = 2^{-x}$! During transformations, when we make the x-value of a function negative, we cause a reflection of the parent function across the y-axis. Have we done that here, also? Yes! Now, check that the graph of $y = \left(\frac{1}{2}\right)^{-x}$ is the same as $y = 2^x$!

2. Sketch the functions $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{5}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$ on the same set of axes. Summarize your findings below.

For $y = b^x$ when $0 < b < 1$:

Domain = $\{x \mid x \in \mathbb{R}\}$

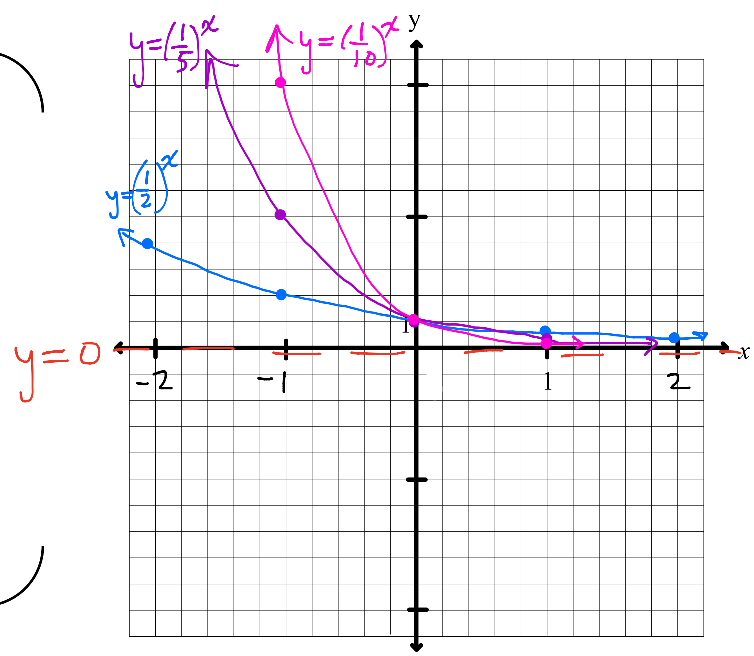
Range = $\{y \mid y \in \mathbb{R}, y > 0\}$

y-intercept = $(0, 1)$

Horizontal Asymptote: $y = 0$

Increasing or decreasing? decreasing

As *b* ↓: curve is steeper (closer to the y-axis)



3. Extension:

a) Investigate why an exponential function should not have a base that is *negative*.

The graph oscillates between (−) and (+), causing it to decrease & increase repeatedly. This violates the rule that exponential functions ↑ or ↓. (not both)

b) Investigate why an exponential function should not have a base that is *0 or 1.*

This creates a horizontal line → neither ↑ nor ↓.

III. Comparing Linear, Quadratic, and Exponential Functions

1. Create a table of values for each function, and calculate their first and second differences and their common ratio.

i) $f(x) = -2x^1 + 3$

x	y	Δy ($y_2 - y_1$)	$\Delta^2 y$ ($\Delta y_2 - \Delta y_1$)
-3	9		
-2	7	$7-9 = -2$	
-1	5	$5-7 = -2$	
0	3	$3-5 = -2$	
1	1	$1-3 = -2$	
2	−1	$-1-1 = -2$	
3	−3	$-3-(-1) = -2$	

ii) $g(x) = x^2 - 5$

x	y	Δy ($y_2 - y_1$)	$\Delta^2 y$ ($\Delta y_2 - \Delta y_1$)
-3	4		
-2	−1	$-1-4 = -5$	
-1	−4	$-4-(-1) = -3$	$-3-(-5) = 2$
0	−5	$-5-(-4) = -1$	$-1-(-3) = 2$
1	−4	$-4-(-5) = 1$	$1-(-1) = 2$
2	−1	$-1-(-4) = 3$	$3-1 = 2$
3	4	$4-(-1) = 5$	$5-3 = 2$

iii) $h(x) = 3(2)^x$

x	y	Δy ($y_2 - y_1$)	$\Delta^2 y$ ($\Delta y_2 - \Delta y_1$)	Ratio ($y_2 \div y_1$)
-3	$\frac{3}{8}$			
-2	$\frac{3}{4}$ ×2 ×2	$\frac{3}{4} - \frac{3}{8} = \frac{3}{8}$		2
-1	$\frac{3}{2}$ ×2 ×2	$\frac{3}{2} - \frac{3}{4} = \frac{3}{4}$	$\frac{3}{8}$	2
0	$\frac{3}{1}$ ×2 ×2	$3 - \frac{3}{2} = \frac{3}{2}$	$\frac{3}{4}$	2
1	6	$6-3 = 3$	$\frac{3}{2}$	2
2	12	$12-6 = 6$	3	2
3	24	$24-12 = 12$	6	2

2. a) How do the differences for exponential functions differ from those for linear or quadratic functions?

No constant differences can be found for exponential f'ns. (In a polynomial, the "degree" tells you the constant difference.)

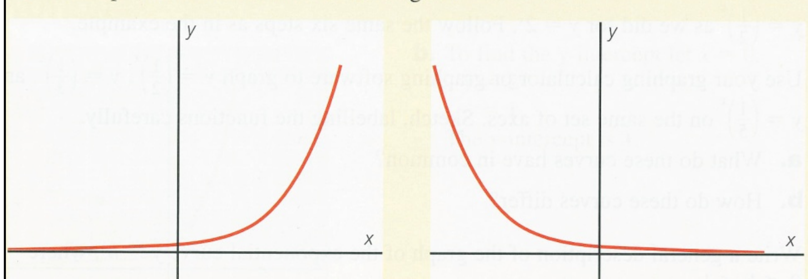
b) How can you tell whether a function is exponential given a table of values?

Find the common ratio by dividing the y-values. If the common ratio is constant → it's exponential!

Summary:

Properties of the Exponential Function $f(x) = b^x$, $b > 0$

- The base b is positive.
- $f(0) = 1$. (The y-intercept of the graph is 1.)
- The x-axis is a horizontal asymptote.
- The domain is the set of real numbers R.
- The range is the set of positive real numbers R.
- The exponential function is increasing if $b > 1$.
- The exponential function is decreasing if $0 < b < 1$.



$y = b^x$, $b > 1$
The function is increasing as x increases.

$y = b^x$, $0 < b < 1$
The function is decreasing as x increases.

Transformations of Exponential Functions of the Form $y = a \cdot b^{k(x-d)} + c$

Exponential functions of the form $y = b^x$ can be transformed using the same algorithm as for other functions we have studied this semester, where the point (x, y) on $y = b^x$ maps onto the point $(\frac{1}{k}x + d, ay + c)$ on $y = a \cdot b^{k(x-d)} + c$.

$$y = a \cdot b^{k(x-d)} + c$$

- Vertical reflection if $a < 0$
- Vertical expansion/compression by a factor of "a"

- Horizontal expansion/compression by a factor of "$\frac{1}{k}$"

- Horizontal translation "d" units left/right

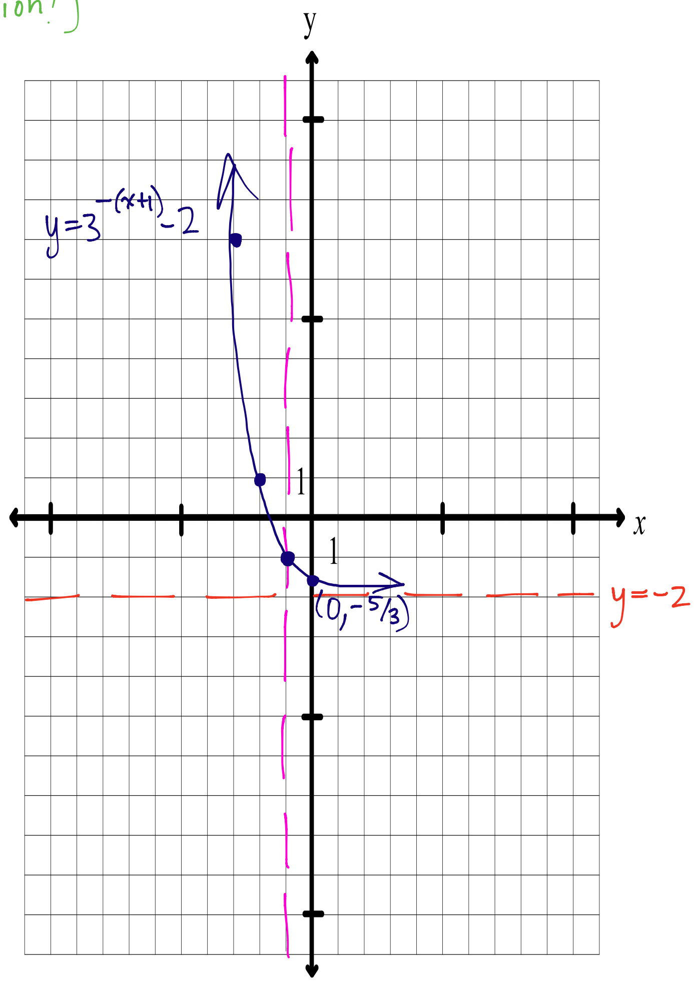
- Vertical translation "c" units up/down
- ha: $y = c$

1. Given the function $y = 3^{-(x+1)} - 2$, use the base function ($y = 3^x$) to graph the transformed function using the following steps:

base = 3 (not a transformation!)

a) List the transformations in order:

i) Horizontal reflection across y-axis
ii) Horizontal translation 1 unit left
iii) Vertical translation 2 units down



b) Describe the transformation that maps a point on the base function to a point on the transformed function:

(x, y) → ($-x-1$ **,** $y-2$ **)**

c) Map the changes onto a few key points from the base function:

$y = 3^x$ $\quad$ $y = 3^{-(x+1)} - 2$

$(x, y) \longrightarrow (-x-1, y-2)$

$(-1, \frac{1}{3}) \rightarrow (0, -\frac{5}{3})$

$(0, 1) \longrightarrow (-1, -1)$

$(1, 3) \rightarrow (-2, 1)$

$(2, 9) \rightarrow (-3, 7)$

d) Sketch the transformed function and label the horizontal asymptote, $y = -2$ **.**

e) State the following:

Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \in \mathbb{R}, y > -2\}$

2. Given the function $y = -2^{2(x-3)} + 5$**, use the base function (** $y = 2^x$ **) to graph the transformed function using the following steps:**

base = 2 (not a transformation!)

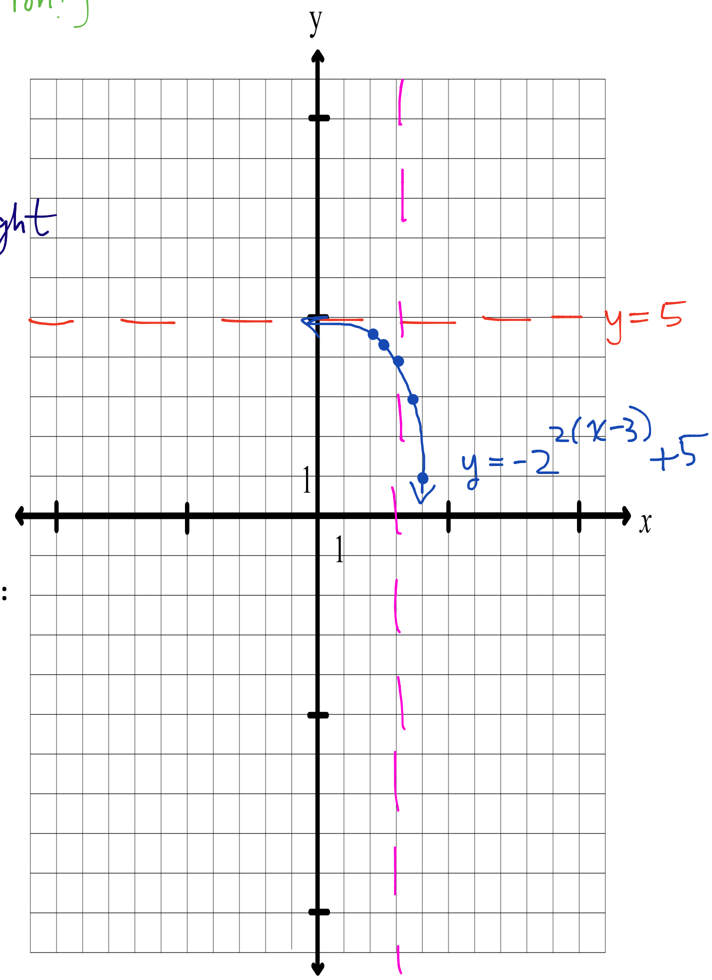
a) List the transformations in order:

i) vertical reflection across the x-axis

ii) horizontal compression by a factor of ½

iii) horizontal translation 3 units right

iv) vertical translation 5 units up



b) Describe the transformation that maps a point on the base function to a point on the transformed function:

(x, y) → $\left(\frac{1}{2}x+3,\ -y+5\right)$

c) Map the changes onto a few key points from the base function:

$y = 2^x$

x	y
−2	¼
−1	½
0	1
1	2
2	4

→

$y = -2^{2(x-3)} + 5$

$\frac{1}{2}x+3$	$-y+5$
2	4¾
2½	4½
3	4
3½	3
4	1

d) Sketch the transformed function and label the horizontal asymptote, $y = 5$.

e) State the following:

Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \in \mathbb{R},\ y < 5\}$

Summary:

- The horizontal asymptote changes when vertical translations (+/– c) are applied; h.a. equation is ***y = c***
- The domain is always D = $\{x \in R\}$.
- The range depends on the location of the horizontal asymptote and whether the function is above or below the asymptote. If it is above the asymptote, the range is R = $\{y \in R, y > c\}$ If it is below the asymptote, the range is R = $\{y \in R, y < c\}$.

HW: WORKSHEET – Transforming Exponential Functions

WORKSHEET: Transforming Exponential Functions

Name: Solutions

Part 1: List the transformations for each function in order, and write the equation of the horizontal asymptote.

1. $f(x) = (3)4^{x} - 5$

VE by a factor of 3
VT down 5 units
ha: $y = -5$

2. $f(x) = \left(\frac{1}{4}\right)2^{-2x} + 11$

VC by a factor of 1/4
HR across y-axis
HC by a factor of 1/2
VT up 11 units
ha: $y = 11$

3. $f(x) = -7 - 3^{x}$

$f(x) = -3^{x} - 7$
VR across x-axis
VT down 7 units
ha: $y = -7$

4. $f(x) + 3 = 2^{-x-5}$

$f(x) = 2^{-(x+5)} - 3$
HR across y-axis
HT 5 units left
VT 3 units down
ha: $y = -3$

Part 2: Write the equation of the horizontal asymptote for each exponential function.

5. $f(x) = 2^{x}$

ha: $y = 0$

6. $f(x) = 2^{x} - 0.5$

ha: $y = -0.5$

7. $f(x) = 24.25 + 2^{x}$

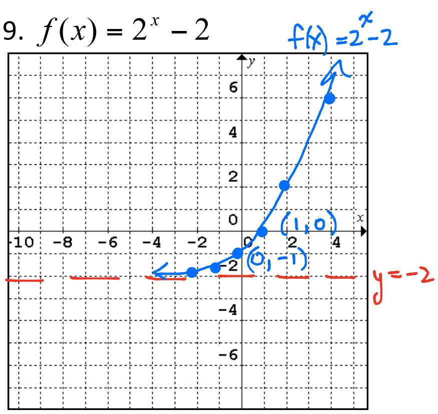
ha: $y = 24.25$

8. $f(x) - 5 = 2^{x} - 8$

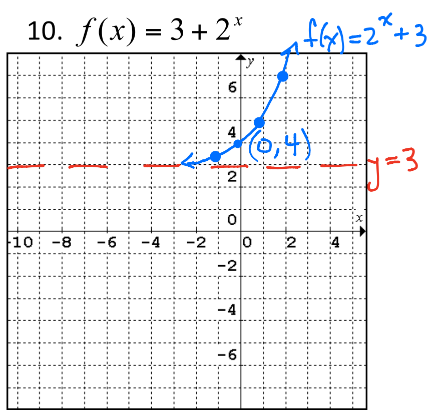
$f(x) = 2^{x} - 8 + 5$
ha: $y = -3$

Part 3: Make an accurate sketch of each function.

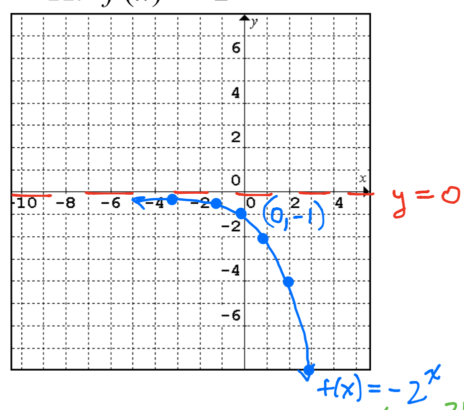
9. $f(x) = 2^{x} - 2$



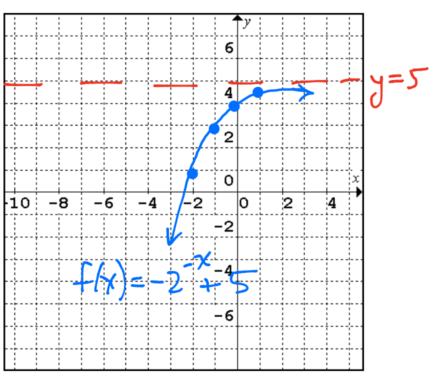
10. $f(x) = 3 + 2^{x}$



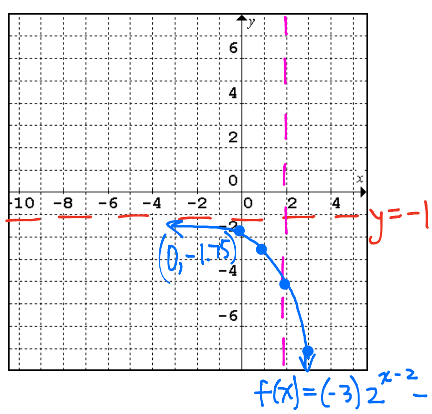
11. $f(x) = -2^{x}$



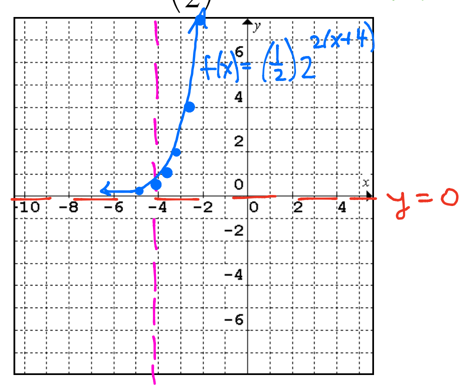
12. $f(x) = -2^{-x} + 5$



13. $f(x) = (-3)2^{x-2} - 1$



14. $f(x) = \left(\frac{1}{2}\right)2^{2x+8}$ $\quad f(x) = \left(\frac{1}{2}\right)2^{2(x+4)}$



Part 3: Write the equation resulting from the following transformations on $y = 6^{x}$:

15. a vertical translation up 1.71 units: $y = 6^{x} + 1.71$

16. a vertical translation down 1.6 units and a reflection across the x-axis: $y = -6^{x} - 1.6$

17. a vertical expansion by a factor of 5, a reflection across the y-axis, a horizontal expansion by a factor of 2, and a vertical translation up 4 units: $y = (5)6^{-\frac{1}{2}x} + 4$

Applications Involving Exponential Functions

A. Exponential growth or ***decay*** occurs when quantities increase or decrease at a rate proportional to the initial quantity present. This ***growth*** or ***decay*** occurs in savings accounts, the size of populations, appreciation, depreciation, and with radioactive chemicals.

All situations of this type can be modeled using the **exponential function** $y = a \cdot b^x$ or $f(x) = a \cdot b^x$ where:

- y or $f(x)$ is the *final* value
- b is the *growth factor* if $b > 1$
- *the growth or decay rate is* $|b - 1|$
- a is the *initial* or *original* value
- b is the *decay factor* if $0 < b < 1$
- x is the *number of growth* or *decay periods*

Ex. 1: If the growth in population of the regional municipality of Wood Buffalo, Alberta since 1996 is given by the function $P(n) = 35\,000(1.08)^n$, determine the:

a) initial population of the municipality 35 000

b) growth factor 1.08

c) growth rate 0.08 = 8%

d) population in the year 2010 ~102 802

c) $|b-1| = |1.08-1|$
$= 0.08$

d) $n = 14$
$P(14) = 35\,000(1.08)^{14}$
$\doteq 102\,801.8$

Ex. 2: If the value of a car after it is purchased depreciates according to the formula $V = 30\,000(0.85)^n$, where V is the car's value in the n^{th} year since it was purchased, determine the:

a) purchase price of the car \$30 000

b) decay factor 0.85

c) annual rate of depreciation 0.15 = 15%

d) car's value at the end of 3 years \$18 423.75

e) car's value at the end of 30 months \$19 983.36

c) $|b-1| = |0.85-1|$
$= 0.15$

e) $V\left(\frac{30}{12}\right) = 30\,000(0.85)^{\frac{30}{12}}$
$\doteq 19\,983.36$

d) $V(3) = 30\,000(0.85)^3$
$\doteq 18\,423.75$

Ex. 3: An antique vase was purchased in 2000 for \$8 000 and appreciates in value by 6% per year.

a) Write an equation that represents the value of the vase, V, in dollars, n years after 2000 using function notation.

$a = 8000$
$b = 1.06$

$\boxed{\therefore V(n) = 8000(1.06)^n}$

b) In what year did the owner double his investment?

$\dfrac{16\,000}{8000} = \dfrac{\cancel{8000}(1.06)^n}{\cancel{8000}}$

$2 = (1.06)^n$

Use systematic trial & error:

$2 \doteq 1.06^{11.9}$

$\boxed{\therefore \text{the investment was doubled at the end of 2011}}$

Ex. 4: A colony of insects has a current population of 80 insects and doubles every 10 days.

a) Write an equation that represents the population, *P*, after *t* days using function notation.
b) Determine the number of insects in the colony 3 days ago.
c) Exactly how long would it take for the population to reach 40 960?

a) $a = 80$
$b = 2$
doubling time $= 10$ days

$\boxed{\therefore P(t) = 80(2)^{t/10}}$

b) $P(-3) = 80(2)^{-3/10}$

$\boxed{\therefore P(-3) \doteq 65 \text{ insects}}$

c) $\frac{40\,960}{80} = \frac{\cancel{80}(2)^{t/10}}{\cancel{80}}$

$512 = 2^{t/10}$

$2^9 = 2^{t/10}$

$\because$ bases are equal

$9 = \frac{t}{10}$

$\boxed{\therefore t = 90 \text{ days}}$

Ex. 5: A sample of radioactive polonium-210 has a *half-life* of 138 days. This means that every 138 days, the amount of polonium left in a sample is half of the original amount.

a) If the original sample was 200 g, write an equation that represents the mass, *M*, of polonium, in grams, that remains after *t* days. Use this equation to determine exactly how long it takes for this sample to decay to 12.5 grams.
b) If there are 80 grams remaining after 365 days, what was the mass of the original sample (to the nearest gram)?

a) $\underbrace{M(t)}_{\text{amount remaining}} = \underbrace{200}_{\text{initial value}}\underbrace{\left(\frac{1}{2}\right)}_{\text{decay factor}}^{\frac{t}{138}}$ } half-life

$\frac{12.5}{200} = \frac{\cancel{200}\left(\frac{1}{2}\right)^{\frac{t}{138}}}{\cancel{200}}$

$\frac{1}{16} = \left(\frac{1}{2}\right)^{t/138}$

$\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{t/138}$

$\because$ bases are equal

$4 = \frac{t}{138}$

$\boxed{\therefore t = 552 \text{ days}}$

b) $80 = a\left(\frac{1}{2}\right)^{\frac{365}{138}}$

$80 \doteq a(0.16)$

$\frac{80}{0.16} \doteq a$

$\therefore a \doteq 500$

$\boxed{\therefore \text{the original sample was approximately 500 grams}}$

B. Compound interest is a type of exponential growth and can be modeled by the function $A = P(1+i)^n$ where:

- *A* is the final amount that the investment is worth.
- *P* is the principal (the original amount of money invested).
- *i* is the interest rate *per compounding period*
- *n* is the number of times interest is calculated over the length of the investment.

Ex. 6: \$3 500 is invested at 6% per annum. What is the accumulated amount after 5 years if the interest was compounded:

a) *quarterly?*

$A = ?$
$P = 3\,500$
$i = 6\% \div 4$ (divide by the number of times the interest is paid out)
$= 0.06 \div 4$
$= 0.015$
$n = 5 \times 4$ (multiply by the number of times the interest is paid out)
$= 20$

$A = P(1+i)^n$
$A = 3500(1+0.015)^{20}$
$A = 3500(1.015)^{20}$
$A \doteq 4713.99$

$\boxed{\therefore \$4\,713.99 \text{ has accumulated after 5 years compounded quarterly}}$

b) *monthly?*

$A = ?$
$P = 3\,500$
$i = 6\% \div 12$
$= 0.06 \div 12$
$= 0.005$
$n = 5 \times 12$
$= 60$

$A = P(1+i)^n$
$A = 3500(1+0.005)^{60}$
$A = 3500(1.005)^{60}$
$A \doteq 4720.98$

$\boxed{\therefore \$4\,720.98 \text{ has accumulated after 5 years compounded monthly}}$

HW: "WORKSHEET: Applications Involving Exponential Functions" #1cd, 2-6, 8-12 [for #10: when using percents, *a* = 100; when using fractions, *a* = 1]; Textbook p. 508 #3; p. 573 #6