
# MCR3UI

**Unit 7: Sequences and Series**

# Sequences

## Part I: Arithmetic Sequences

A sequence like **2, 5, 8, 11, . . .** where the difference between consecutive terms is a ________________, is called an ________________________ __________________________ . The first differences of an arithmetic sequence tell you the common difference between each term.

- In an arithmetic sequence, the **first term, $t_1$,** is the initial value and is denoted by the letter ________ ;
- The **common difference** (when the first differences are constant) is denoted by the letter _______ . ***The common difference of an arithmetic sequence is:*** $\boldsymbol{d} = t_2 - t_1 = t_3 - t_2 = \boldsymbol{t_n - t_{n-1}}$

**A. Writing the terms of a sequence:** Given $t_n = 3n - 1$ is the formula for the *n*th term of an arithmetic sequence, find the first 4 terms.

**B. Finding the general formula for an arithmetic sequence with first term *a* and common difference *d*:**

The **General Formula** for an **Arithmetic Sequence** is:

$$t_n = a + (n-1)d \quad or \quad t_n = a + d(n-1), \quad n \in \mathbb{N}$$

where ***a*** is the **first term**, ***d*** is the **common difference** between any 2 consecutive terms, and $\boldsymbol{d} = t_2 - t_1 = t_3 - t_2 = \boldsymbol{t_n - t_{n-1}}$

For the sequence 4, 9, 14, 19, . . . find $t_n$ and $t_{123}$.

*a)* $t_n =$

*b)* $t_{123} =$

**C. Finding the number of terms:** How many terms are in the sequence 12, 2, -8, -18, . . . , -298 ?

## Part II: Geometric Sequences

A sequence like **2, 6, 18, 54, . . .** where each term is found by multiplying the previous term by a ______________, is called a ____________________ ______________________. The **constant multiple** of a geometric sequence is called the **common ratio**.

- In a geometric sequence, the **first term, $t_1$,** is the initial value and is denoted by the letter _______ ;
- The **common ratio** (common multiple) is denoted by the letter ______ . ***The common ratio for a geometric sequence is:***

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_n}{t_{n-1}}$$

**A. Writing the terms of a sequence:** Given $t_n = 5(3)^{n-1}$ is the formula for the *n*th term of a geometric sequence, find the first 4 terms.

**B. Finding the general formula for a geometric sequence with first term *a* and common ratio *r*:**

The **General Formula** for a **Geometric Sequence** is:

$$t_n = a(r)^{n-1}, \quad n \in \mathbb{N}, \quad r \neq 0$$

where ***a*** is the **first term**, ***r*** is the **common ratio** between any 2 consecutive terms, and $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_n}{t_{n-1}}$

For 1, 3, 9, 27, . . . find $t_n$ and $t_7$

*a)* $t_n =$

*b)* $t_7 =$

**C. Finding the number of terms:** How many terms are in the geometric sequence 3, – 12, 48, – 192, . . . , – 3221225472?

**HW: p. 441-442 # 2ace, 3abe, 4acdeg, 5cdf, 6ace, 7ace; p. 452-453 #1ace, 2aceg, 3ace, 4ace, 5adef, 6ace**

# Series

## Part I: Arithmetic Series

**2, 4, 6, . . .** is called an **arithmetic sequence**; where $a$ = ______ and $d$ = ______ and $t_n = a + (n - 1)d$.

**2 + 4 + 6 + . . .** is called an **arithmetic series**. A series is the ______ of the terms in a sequence. The sum of the first $n$ terms of a series is denoted $S_n$. From p. 466 of the text:

The sum of the first $n$ terms in an **Arithmetic Series** is

$$S_n = \frac{n}{2}[a + t_n] \quad \text{or} \quad S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}[2a + d(n-1)]$$

**A. Sum of a series given the first terms.**

1) Given the arithmetic series 1 + 6 + 11 + 16 + . . . , find the **sum** of the first twenty terms.

2) Given the arithmetic series: –7 –11 –15 – . . . , find the **sum** of the first fifty-four terms.

**B. Sum of a series given the first and the last terms.**

1) Given the arithmetic series: 2 + 7 + 12 + . . . + 62 , find the **sum**. (Hint: to find $S_n$, we must first determine $n$).

## Part II: Geometric Series

**5, 10, 20, . . .** is called a **geometric sequence** ; where $a$ = ______ and $r$ = ______ and $t_n = a \cdot r^{n-1}$

**5 + 10 + 20 + . . .** is called a **geometric series** . The sum of the first $n$ terms of a series is denoted $S_n$. From p. 473 of the text:

The sum of the first $n$ terms in a **Geometric Series** is

$$S_n = \frac{a[(r)^n - 1]}{(r-1)} \quad , r \neq 1$$

**A. Sum of a series given the first terms.**

1) Given the geometric series 3 + 6 + 12 + 24 + . . . , find the **sum** of the first eight terms.

2) Given the geometric series 2 – 8 + 32 –128 + . . . , find the **sum** of the first thirteen terms.

**B. Sum of a series given the first and the last terms.**

1) Given the geometric series: 3 – 9 + 27 –. . . –6561 , find the **sum**. (Hint: to find $S_n$, we must first determine $n$).

**HW: p. 469 #1bcd, 2bcd, 3bcd, 4bd, 5bc; p. 476 #1ace, 2ace, 3ace**

# Recursion Formulas

***Recall:*** The formula $t_n = 3n - 1$ determines the **arithmetic sequence** __________________________________.

The formula $t_n = 2^{n-1}$ determines the **geometric sequence** __________________________________.

These formulas are called _______________________ ___________________________ because they can be used to calculate the $n^{th}$ term (any term) of a sequence without knowing the previous term(s).

**A. What is a recursion formula?**

$t_1 = 5$ ; $t_n = t_{n-1} + 2$ is an example of a ***recursion formula***, where you need to know the ___________________ ________________ to calculate the ______________ _____________.

**Recursion formulas** consist of at least **two parts:**

$$\begin{cases} t_1 = 5 \\ \\ t_n = t_{n-1} + 2 \end{cases}$$

Find the first four terms of this sequence:

**B. Use a recursion formula to write the first four terms of each sequence.**

a) $t_1 = 48$ ; $t_n = 0.5t_{n-1}$

b) $t_1 = 2$ ; $t_n = t_{n-1} - 2n + 1$

c) $t_1 = -2$ ; $t_2 = 3$ ; $t_n = 2t_{n-2} + t_{n-1}$

**C. Write an explicit formula for each sequence based on its recursion formula.**

a) Given $t_1 = 6$ ; $t_n = t_{n-1} + 4$ , write the first four terms and write a **simplified explicit formula** for the sequence.

b) Given $t_1 = 3$ ; $t_n = -2t_{n-1}$ , write the first four terms and write a **simplified explicit formula** for the sequence.

**D. Write a simplified explicit formula for the $n^{th}$ term of each sequence using only the recursion formula.**

a) $t_1 = 8$ ; $t_n = 2t_{n-1}$

b) $t_1 = 52$ ; $t_n = t_{n-1} - 3$

**HW: p. 461-463 #1ace, 2ace, 4ac, 5abc, 6, 7, 8, 9, 10, 17, 18**

# Pascal's Triangle and Binomial Expansions

*Blaise Pascal*, 1623-1662

**A. Binomial Expansions of the Form $(a + b)^n$ where $n \in W$.**

$(a + b)^0 =$

$(a + b)^1 =$

$(a + b)^2 =$

$(a + b)^3 =$

$(a + b)^4 =$

$(a + b)^5 =$

**B. Pascal's Triangle**

## Key Ideas

- The arrangement of numbers shown below is called Pascal's triangle. Each row is generated by calculating the sum of pairs of consecutive terms in the previous row.

| | |
|---|---|
| 1 | 0th row |
| 1 1 | 1st row |
| 1 2 1 | 2nd row |
| 1 3 3 1 | 3rd row |
| 1 4 6 4 1 | 4th row |
| 1 5 10 10 5 1 | 5th row |

- The numbers in Pascal's triangle correspond to the coefficients in the expansion of binomials raised to whole-number exponents.

## Need to Know

- Pascal's triangle has many interesting relationships among its numbers. Some of these relationships are recursive.
  - For example, down the sides are constant sequences: 1, 1, 1, ....
  - The diagonal beside that is the counting numbers, 1, 2, 3, ... , which form an arithmetic sequence.
  - The next diagonal is the triangular numbers, 1, 3, 6, 10, ... , which can be defined by the recursive formula

$$t_1 = 1, t_n = t_{n-1} + n, \text{ where } n \in \mathbf{N} \text{ and } n > 1.$$

- There are patterns in the expansions of a binomial $(a + b)^n$:
  - Each term in the expansion is the product of a number from Pascal's triangle, a power of $a$, and a power of $b$.
  - The coefficients in the expansion correspond to the numbers in the $n$th row in Pascal's triangle.
  - In the expansion, the exponents of $a$ start at $n$ and decrease by 1 down to zero, while the exponents of $b$ start at zero and increase by 1 up to $n$.
  - In each term, the sum of the exponents of $a$ and $b$ is always $n$.

**C. Expand and simplify each binomial power.**

a) $(x - 2)^5$

b) $(3x + 2y)^3$

**D. Expand and simplify *the first three terms* of each binomial power.**

a) $(2\ x + 3)^6$

b) $(5x^3 - 3y^2)^7$

**HW: WORKSHEET: Pascal's Triangle and Binomial Expansions**

# WORKSHEET: Pascal's Triangle and Binomial Expansions

1. Complete Pascal's triangle to the tenth level/row (for use in the following questions).
2. The first four entries of the 12th row of Pascal's triangle are 1, 12, 66, and 220. Determine the first four entries of the 13th row of the triangle.
3. Expand and simplify each binomial power.
   a. $(x+2)^5$
   b. $(x-1)^6$
   c. $(2x-3)^3$
4. Expand and simplify the ***first three terms*** of each binomial power.
   a. $(x+5)^{10}$
   b. $(x-2)^8$
   c. $(2x-7)^9$
5. Expand and simplify each binomial power.
   a. $(y-5)^6$
   b. $(2x+7y)^3$
   c. $(2z^3-3y^2)^5$
6. Expand and simplify the ***first three terms*** of each binomial power.
   a. $(x-2)^7$
   b. $\left(3b^2-\frac{2}{b}\right)^8$

*Answers:*

2. The first four entries of the 13th row are: 1, 13, 78, 286
3. a) $x^5+10x^4+40x^3+80x^2+80x+32$

   b) $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$

   c) $8x^3-36x^2+54x-27$
4. a) $x^{10}+50x^9+1125x^8$

   b) $x^8-16x^7+112x^6$

   c) $512x^9-16128x^8+225792x^7$
5. a) $y^6-30y^5+375y^4-2500y^3+9375y^2-18750y+15625$

   b) $8x^3+84x^2y+294xy^2+343y^3$

   c) $-243y^{10}+810y^8z^3-1080y^6z^6+720y^4z^9-240y^2z^{12}+32z^{15}$
6. a) $x^7-14x^6+84x^5$

   b) $6561b^{16}-34992b^{13}+81648b^{10}$

# Unit 7 Review

**1. i)** Determine the first five terms of each sequence, where $n \in \mathbf{N}$.
**ii)** Determine whether each sequence is arithmetic, geometric, or neither.

**a)** $t_n = 5 \times 3^{n+1}$

**b)** $t_n = \dfrac{3n + 2}{2n + 1}$

**c)** $t_n = 5n$

**d)** $t_1 = 5$, $t_n = 7t_{n-1}$, where $n > 1$

**e)** $t_1 = 19$, $t_n = 1 - t_{n-1}$, where $n > 1$

**f)** $t_1 = 7$, $t_2 = 13$, $t_n = 2t_{n-1} - t_{n-2}$, where $n > 2$

**2.** For each sequence, determine
**i)** the general term
**ii)** the recursive formula

**a)** a geometric sequence with $a = -9$ and $r = -11$

**b)** an arithmetic sequence with second term 123 and third term $-456$

**3.** Determine the number of terms in each sequence.

**a)** 18, 25, 32, ... , 193

**b)** 2, $-10$, 50, ... , $-156\,250$

**4.** Expand and simplify each binomial power.

**a)** $(x - 5)^4$

**b)** $(2x + 3y)^3$

**5.** Calculate the sum of each series.

**a)** 19 + 33 + 47 + ... + 439

**b)** the first 10 terms of the series 10 000 + 12 000 + 14 400 + ...

**6.** A sequence is defined by the recursive formula $t_1 = 4$, $t_2 = 5$, $t_n = \dfrac{t_{n-1} + 1}{t_{n-2}}$, where $n \in \mathbf{N}$ and $n > 2$. Determine $t_{123}$. Explain your reasoning.

**7.** Your grandparents put aside \$100 for you on your first birthday. Every following year, they put away \$75 more than they did the previous year. How much money will have been put aside by the time you are 21?

***Answers:***

1. a) i) the first five terms are 45, 135, 405, 1215, 3645 ii) it is geometric, r = 3
   b) i) the first five terms are $\frac{5}{3}, \frac{8}{5}, \frac{11}{7}, \frac{14}{9}, \frac{17}{11}$ ii) it is neither arithmetic nor geometric
   c) i) the first five terms are 5, 10, 15, 20, 25 ii) it is arithmetic, d = 5
   d) i) the first five terms are 5, 35, 245, 1715, 12 005 ii) it is geometric, r = 7
   e) i) the first five terms are 19, -18, 19, -18, 19 ii) it is neither arithmetic nor geometric
   f) i) the first five terms are 7, 13, 19, 25, 31 ii) it is arithmetic, d = 6

2. a) i) $t_n = (-9)(-11)^{n-1}$ ii) $t_1 = -9$ ; $t_n = -11t_{n-1}$
   b) i) $t_n = -579n + 1281$ ii) $t_1 = 702$ ; $t_n = t_{n-1} - 579$

3. a) n = 26
   b) n = 8

4. a) $(x - 5)^4 = x^4 - 20x^3 + 150x^2 - 500x + 625$
   b) $(2x + 3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3$

5. a) n = 31, $S_{31} = 7099$
   b) $S_{10} = 259\,586.8$

6. Find the first 7 terms and you will see a pattern repeating such that $t_{123} = 1.5$

7. a = 100, d= 75, n = 20, therefore $S_{20} = 16\,250$

**Textbook Review: p. 486-487 #1-10, 14, 15**