Date:______________

UNIT 2: GRAPHING FUNCTIONS

2.1 Graphing Quadratic, Cubic, Square Root, Absolute Value and Reciprocal Functions Using Transformations

Given $y = a\,f[k(x-d)]+c$, the **transformations** on the graph of $y = f(x)$ are as follows:

i) ***vertical reflection*** in the x-axis if $a < 0$

ii) ***vertical stretch*** by a factor of $|a|$

*Note: A stretch is an **expansion** if the stretch factor is more than 1 or a **compression** if the stretch factor is between 0 and 1.*

iii) ***horizontal reflection*** in the y-axis if $k < 0$

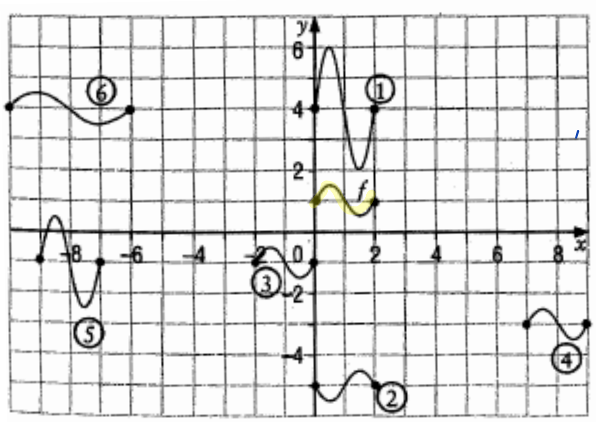
iv) ***horizontal stretch*** by a factor of $\dfrac{1}{|k|}$

v) ***horizontal translation*** **right** $|d|$ units if $d > 0$ or **left** $|d|$ units if $d < 0$

vi) ***vertical translation*** **up** $|c|$ units if $c > 0$ or **down** $|c|$ units if $c < 0$

$$(x, y) \to \left(\frac{1}{k}x + d, ay + c\right)$$

1. The graph of $y = f(x)$ is shown. Match each equation with its graph by describing the transformations applied to the graph of $y = f(x)$.



a) $y = f(x-7)-4$

i) H.T. 7units right
ii) V.T. 4units down
$(x,y) \to (x+7, y-4)$
4

b) $y = -f(x)-4$

i) V.R. across x-axis
ii) V.T. 4units down
$(x,y) \to (x, -y-4)$
2

c) $y = 4f(x)$

i) V.S. by a factor of 4
$(x,y) \to (x, 4y)$
1

d) $y = -f(-x)$

i) V.R. across x-axis
ii) H.R. across y-axis
$(x,y) \to (-x, -y)$
3

e) $y = 3f(x+9)-4$

i) VS by a factor of 3
ii) H.T. 9 units left
iii) V.T. 4 units down
$(x,y) \to (x-9, 3y-4)$
5

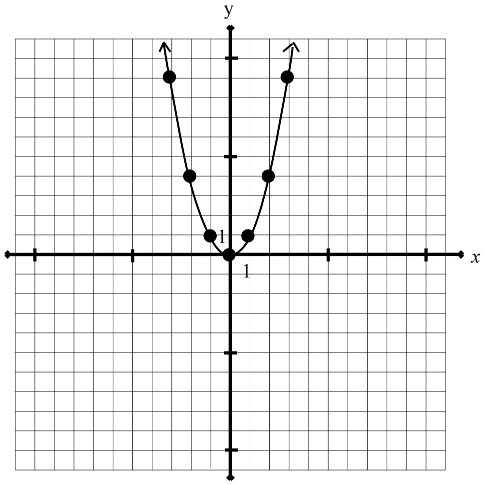
f) $y = f\left(\frac{1}{2}x+5\right)+3$

$y = f\left[\frac{1}{2}(x+10)\right]+3$
i) HS. by a factor of 2
ii) H.T. 10units left
iii) V.T. 3units up
6

2. Memorize each of the following base functions.

a) ***quadratic function*** $f(x) = x^2$

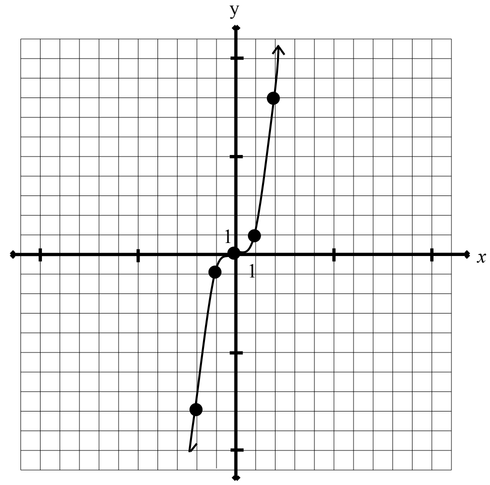
x	y
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16



Domain: $\{x \in \Re\}$, Range: $\{y \in \Re | y \geq 0\}$

b) ***cubic function*** $f(x) = x^3$

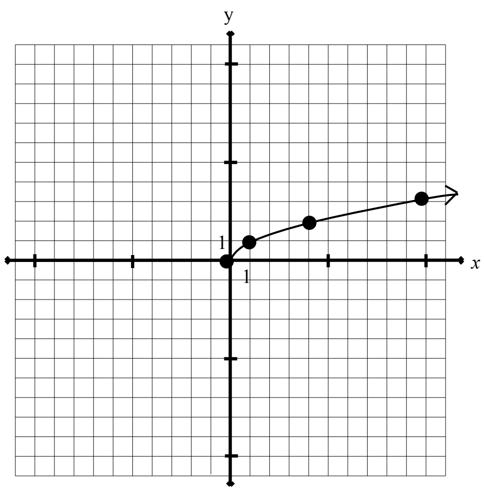
x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27



Domain: $\{x \in \Re\}$, Range: $\{y \in \Re\}$

c) ***square root function*** $f(x) = \sqrt{x}$

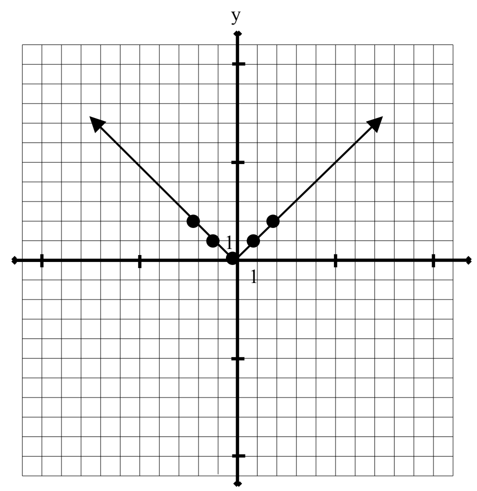
x	y
0	0
1	1
4	2
9	3
16	4
25	5



Domain: $\{x \in \Re | x \geq 0\}$, Range: $\{y \in \Re | y \geq 0\}$

d) ***absolute value function*** $f(x) = |x|$

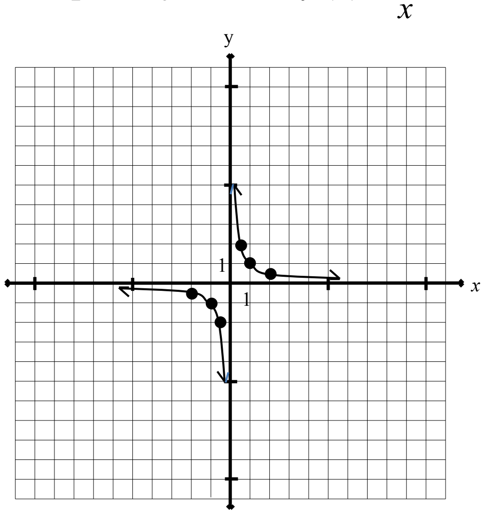
x	y
-4	4
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3
4	4



Domain: $\{x \in \Re\}$, Range: $\{y \in \Re | y \geq 0\}$

e) ***reciprocal function*** $f(x) = \dfrac{1}{x}$

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	*dne*
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



Domain: $\{x \in \Re | x \neq 0\}$, Range: $\{y \in \Re | y \neq 0\}$

The ***vertical asymptote*** of $f(x) = \dfrac{1}{x}$ is $x = 0$.

As $x \to 0^-$, $f(x) \to -\infty$ *and* as $x \to 0^+$, $f(x) \to +\infty$.

"As x approaches 0 from the left, $f(x)$ approaches positive infinity" "as x approaches 0 from the right, $f(x)$ approaches negative infinity."

The ***horizontal asymptote*** of $f(x) = \dfrac{1}{x}$ is $y = 0$.

As $x \to -\infty$, $f(x) \to 0$ *and* as $x \to +\infty$, $f(x) \to 0$.

Ex. 1. Graph each of the following by naming and applying transformations on an appropriate function.

a) $y=-2\sqrt{-3(x-5)}+4 \leftrightarrow y=a\sqrt{k(x-d)}+c$

Transformations on $y=\sqrt{x}$ *are:*

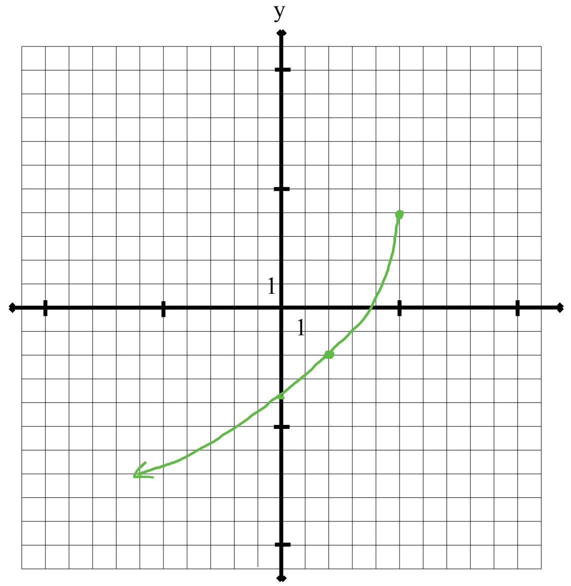
i) H.R. in the y-axis
ii) V.R. in the x-axis
iii) H.C. by a factor of $\frac{1}{3}$
iv) V.E. by a factor of 2
v) H.T. 5 units right
vi) V.T. 4 units up.

x	y
0	0
1	1
4	2
9	3
16	4

$\rightarrow$

x	y
5	4
$4\frac{2}{3}$	2
$3\frac{2}{3}$	0
2	-2
$-\frac{1}{3}$	-4

$(x,y)\rightarrow\left(-\frac{1}{3}x+5,\ -2y+4\right)$



i) Domain: $\{x\in\mathbb{R}\mid x\le 5\}$

ii) Range: $\{y\in\mathbb{R}\mid y\le 4\}$

b) $f(x)=-|2(x+1)|-3 \leftrightarrow f(x)=a|k(x-d)|+c$

Transformations on $y=|x|$ *are:*

i) V.R. in the x-axis
ii) H.C. by a factor of $\frac{1}{2}$
iii) H.T. 1 unit left
iv) V.T. 3 units down

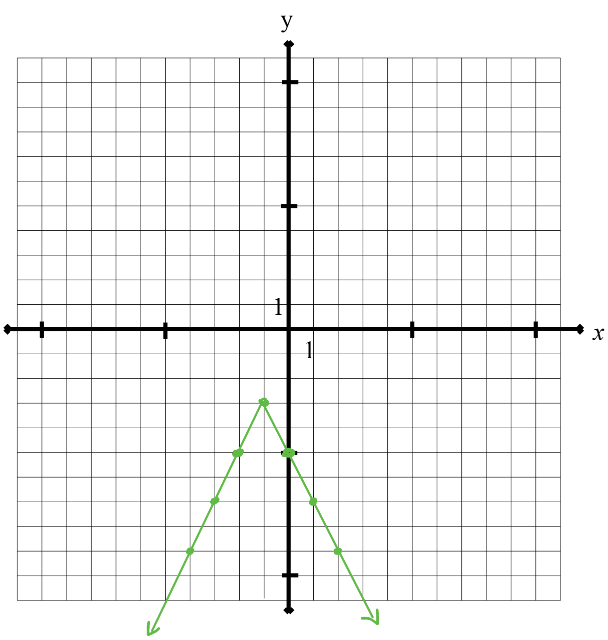
$(x,y)\rightarrow\left(\frac{1}{2}x-1,\ -y-3\right)$

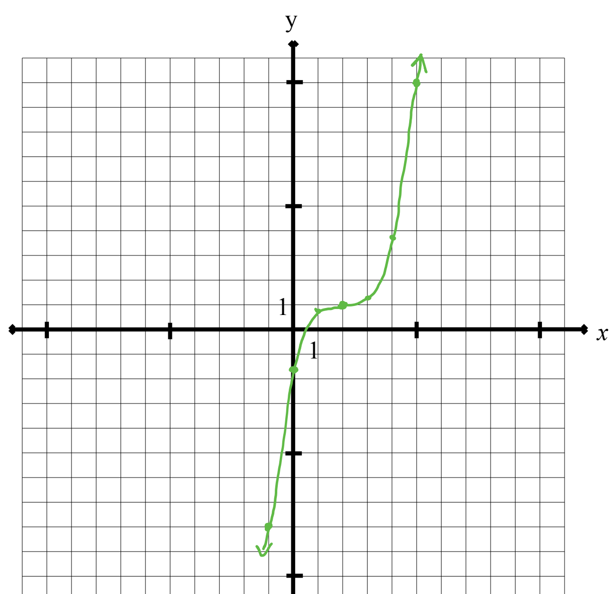
c) $g(x)=\frac{1}{3}(x-2)^3+1 \leftrightarrow g(x)=a[k(x-d)]^3+c$

Transformations on $y=x^3$ *are:*

i) V.C by a factor of $\frac{1}{3}$
ii) H.T. 2 units right
iii) V.T. 1 unit up.

$(x,y)\rightarrow\left(x+2,\ \frac{1}{3}y+1\right)$





Ex. 2. Graph each of the following by naming and applying transformations on an appropriate function. Then, state the domain and range.

a) $y = \left(0.4x + \frac{4}{5}\right)^2 \leftrightarrow y = a[k(x-d)]^2 + c$

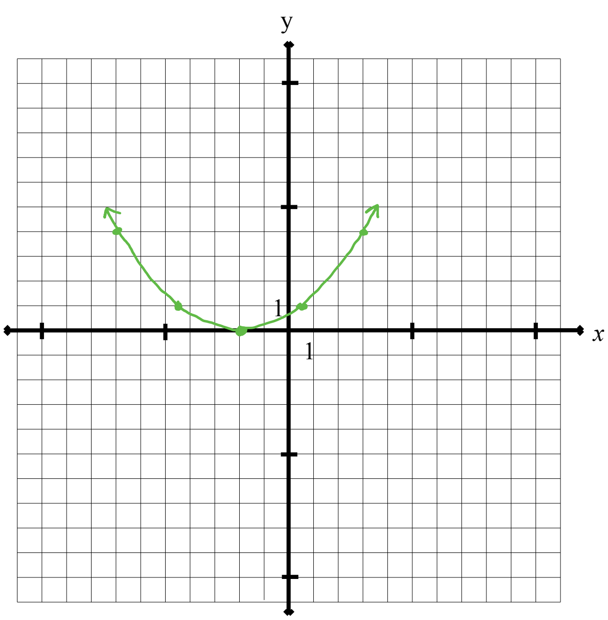
$y = \left(\frac{2}{5}x + \frac{4}{5}\right)^2$

$y = \left[\frac{2}{5}(x+2)\right]^2$

Transformations on $y = x^2$ *are:*

i) H.E. by a factor of $\frac{5}{2}$

ii) H.T. 2 units left



i) Domain: $\{x \in \mathbb{R}\}$

ii) Range: $\{y \in \mathbb{R} \mid y \geq 0\}$

b) $f(x) = \frac{1}{3-x} - 2 \leftrightarrow f(x) = a\left[\frac{1}{k(x-d)}\right] + c$

$f(x) = \frac{1}{-x+3} - 2$

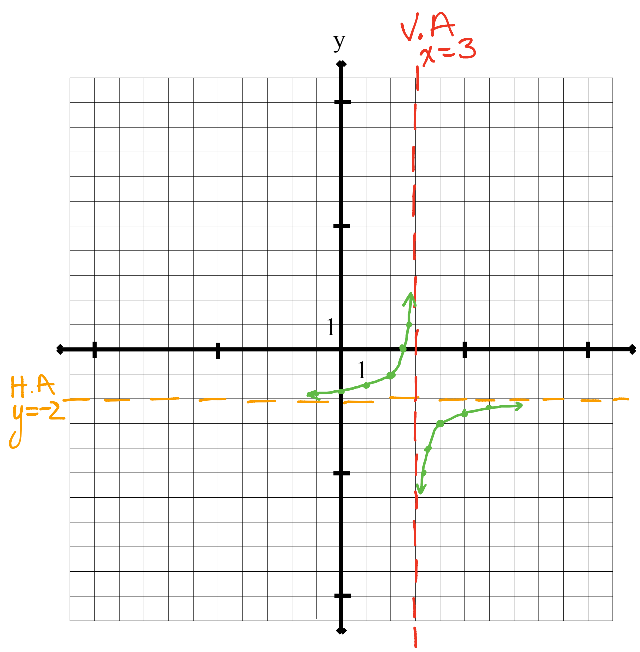
$f(x) = \frac{1}{-(x-3)} - 2$

Transformations on $y = \frac{1}{x}$ *are:*

i) H.R. in y-axis

ii) H.T. 3 units right

iii) V.T 2units down



i) Domain: $\{x \in \mathbb{R} \mid x \neq 3\}$

ii) Range: $\{y \in \mathbb{R} \mid y \neq -2\}$

HW. Exercise 2.1

Date:_______________ **2.2 Graphing Reciprocal and Absolute Value Functions of $y = f(x)$**

PART I: ***Using the graph of*** $y = f(x)$ ***to graph its reciprocal function*** $y = \dfrac{1}{f(x)}$

1. *If* $f(x) = 0$, $\dfrac{1}{f(x)}$ *dne. Draw vertical asymptotes at the zeros.*

2. *If* $f(x) = \pm 1$, $\dfrac{1}{f(x)} = \pm 1$. *Mark these invariant points.*

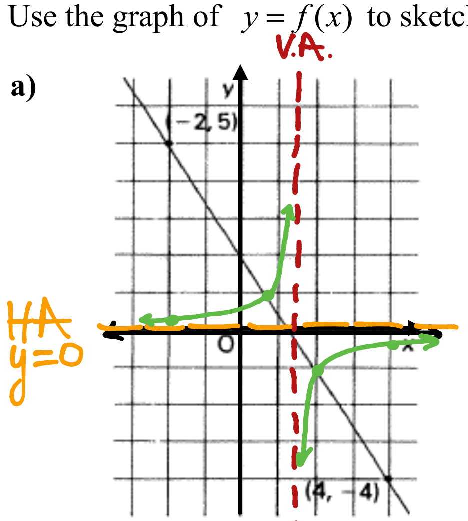
3. i) *If* $f(x)$ *increases,* $\dfrac{1}{f(x)}$ *decreases.* ***ii)*** *If* $f(x)$ *decreases,* $\dfrac{1}{f(x)}$ *increases.*

iii) *If* $f(x)$ *is constant,* $\dfrac{1}{f(x)}$ *is also constant. Graph accordingly.*

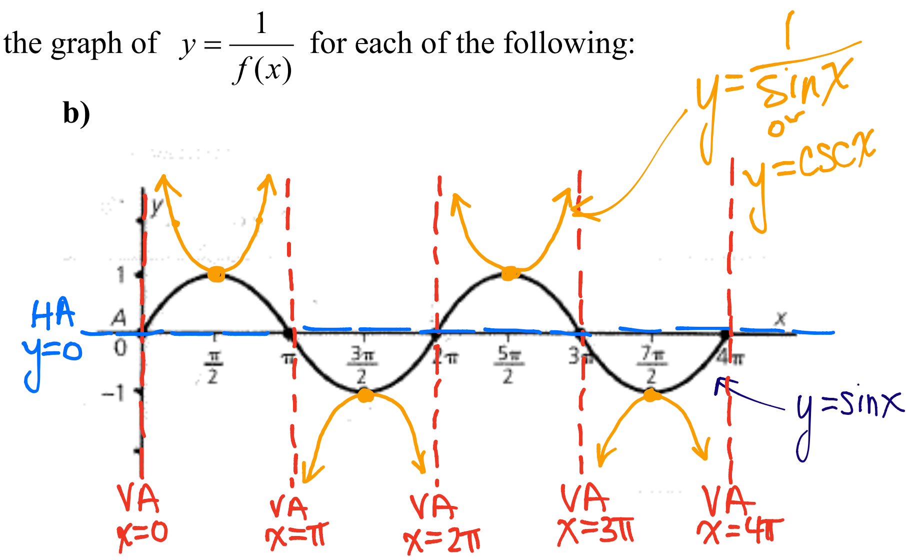
4. *If* $f(x) \to \pm\infty$, $\dfrac{1}{f(x)} \to 0$. *Draw a horizontal asymptote at* $y = 0$.

Ex. 1. Use the graph of $y = f(x)$ to sketch the graph of $y = \dfrac{1}{f(x)}$ for each of the following:

a)

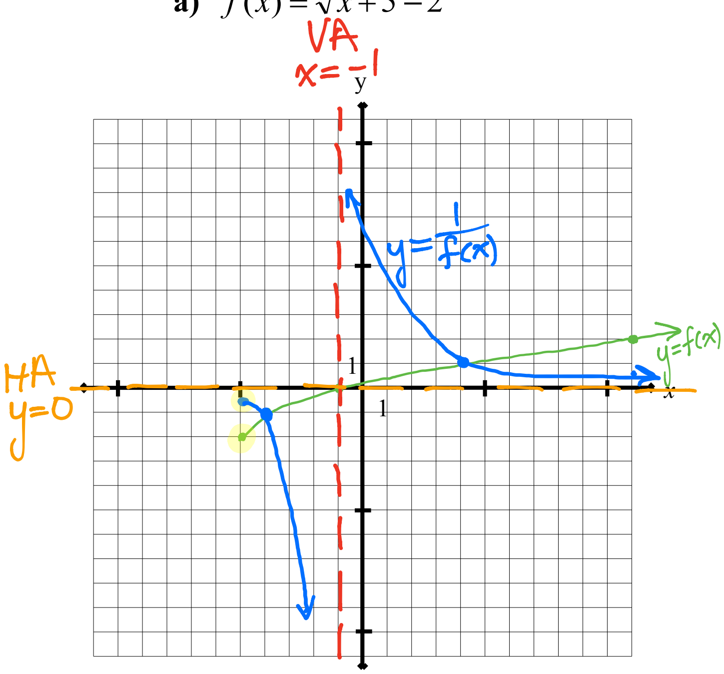


b)

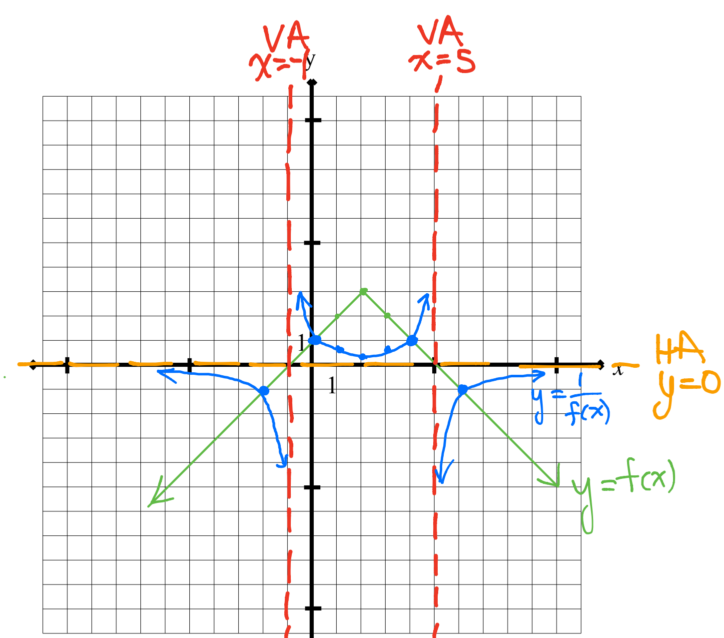


Ex. 2. Graph each function $y = f(x)$ and its reciprocal function $y = \dfrac{1}{f(x)}$ on the same grid.

a) $f(x) = \sqrt{x+5} - 2$



b) $f(x) = -|x-2| + 3$

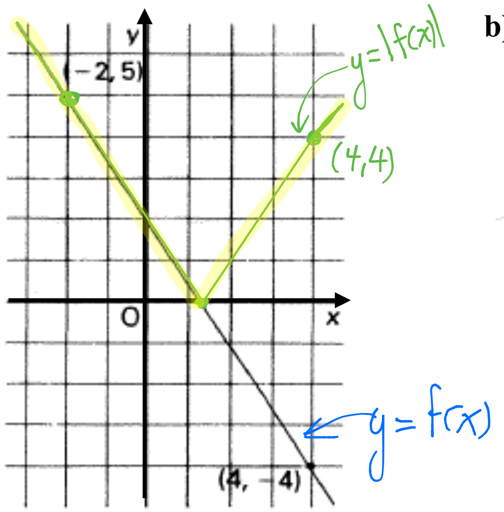


PART II: *Using the graph of* $y=f(x)$ *to graph its absolute value function* $y=|f(x)|$

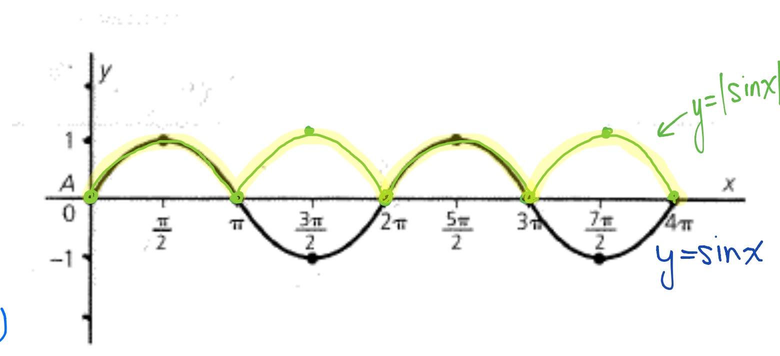
1. *All points on the graph of* $y=f(x)$ *where* $f(x)\geq 0$ *are also on the graph of* $y=|f(x)|$.
   Graph over these ***invariant*** *points with a different colour.*
2. *All points on the graph of* $y=f(x)$ *where* $f(x)<0$ *are vertically reflected in the x-axis.*
   Use the same colour to reflect these points in order to complete the graph.

Ex. 1. Use the graph of $y=f(x)$ to sketch the graph of $y=|f(x)|$ for each of the following:

a)



b)



Ex. 2. Graph each function $y=f(x)$ and its absolute value function $y=|f(x)|$ on the same grid.

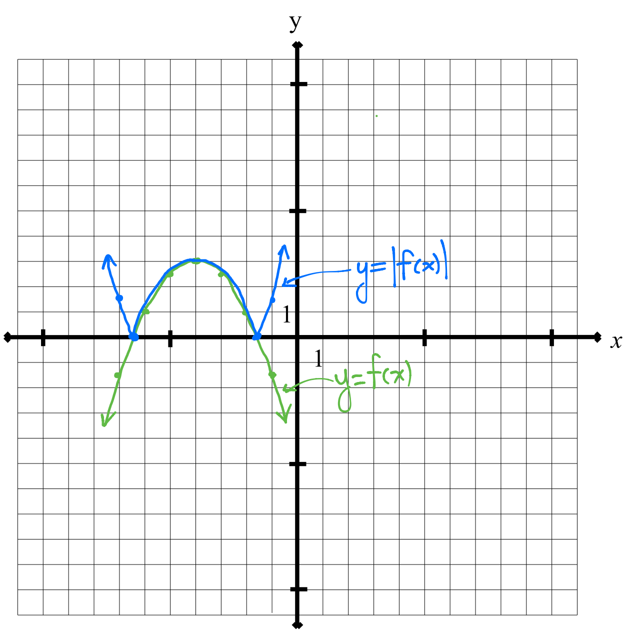
a) $f(x)=-0.5x^2-4x-5$ complete the square

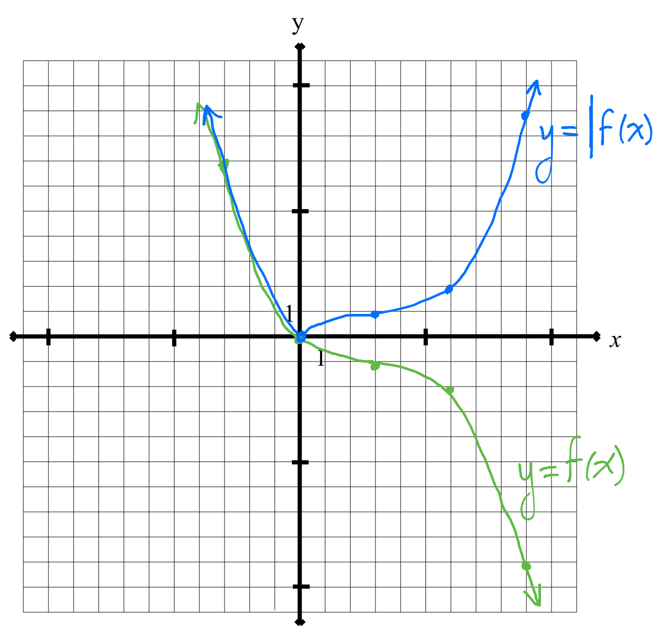
$f(x)=-\frac{1}{2}(x^2+8x+16-16)-5$

$f(x)=-\frac{1}{2}(x+4)^2+3$

b) $f(x)=\left(-\frac{1}{3}x+1\right)^3-1$

$f(x)=\left[-\frac{1}{3}(x-3)\right]^3-1$

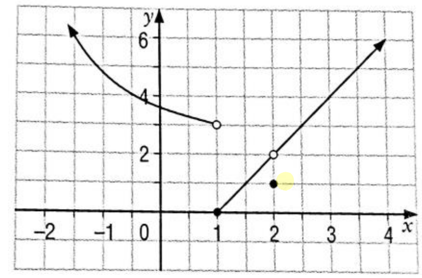




HW. Exercise 2.2

Date:_______________ **2.3 Graphing Piecewise Functions**

Ex. 1. The graph of a ***piecewise*** function ***f*** is shown. Use the graph to determine the following:



a) $f\left(-\frac{1}{2}\right)$ $=4$ **b)** $f(3)$ $=4$ **c)** $f(1)$ $=0$ **d)** $f(2)$ $=1$

e) the value(s) of x at which the function is ***discontinuous***

at $x=1$ & $x=2$

f) as $x \to 1^-$, $f(x) \to$ <u>3</u> and as $x \to 1^+$, $f(x) \to$ <u>0</u>

"as x approaches 1 from the left" "as x approaches 1 from the right"

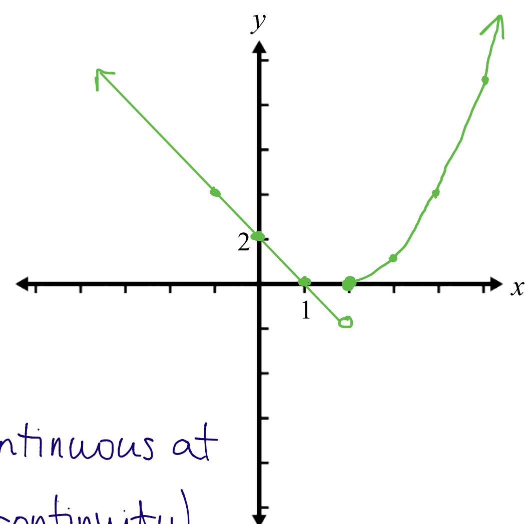
g) the end behaviour of the function ***f***

As $x \to -\infty$, $f(x) \to +\infty$

As $x \to +\infty$, $f(x) \to +\infty$

Ex. 2. For each piecewise function sketch the graph and determine the value(s) of x at which the function is discontinuous. Identify the discontinuities as *jump*, *removable* or *infinite*.

a) $f(x) = \begin{cases} 2-2x & if\ x<2 \quad ① \\ (x-2)^2 & if\ x \geq 2 \quad ② \end{cases}$



① if $x<2$,
$f(x) = -2x+2$

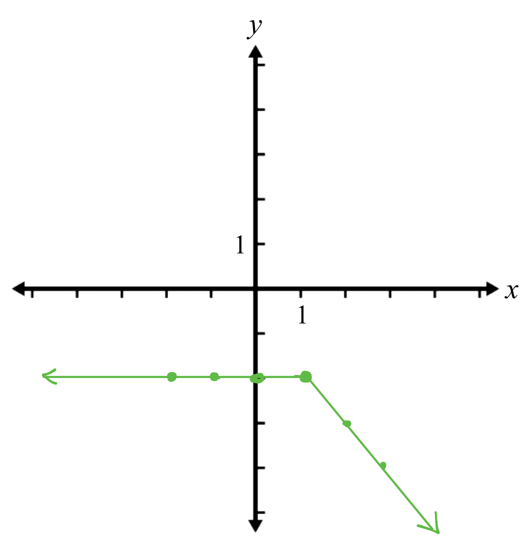
② if $x \geq 2$,
$f(x) = (x-2)^2$

	x	$f(x)$
(open)	2	−2
	1	0
	0	2
	−1	4

	x	$f(x)$
(closed)	2	0
	3	1
	4	4
	5	9

$\therefore f(x)$ is discontinuous at $x=2$.
(jump discontinuity)

b) $f(x) = \begin{cases} -2 & if\ x \in (-\infty, 1) \quad ① \\ -x-1 & if\ x \in [1, \infty) \quad ② \end{cases}$



① If $x<1$,
$f(x) = -2$

② If $x \geq 1$,
$f(x) = -x-1$

	x	$f(x)$
open	1	−2
	0	−2
	−1	−2
	−2	−2

	x	$f(x)$
closed	1	−2
	2	−3
	3	−4
	4	−5

$f(x)$ is continuous for all $x \in \mathbb{R}$

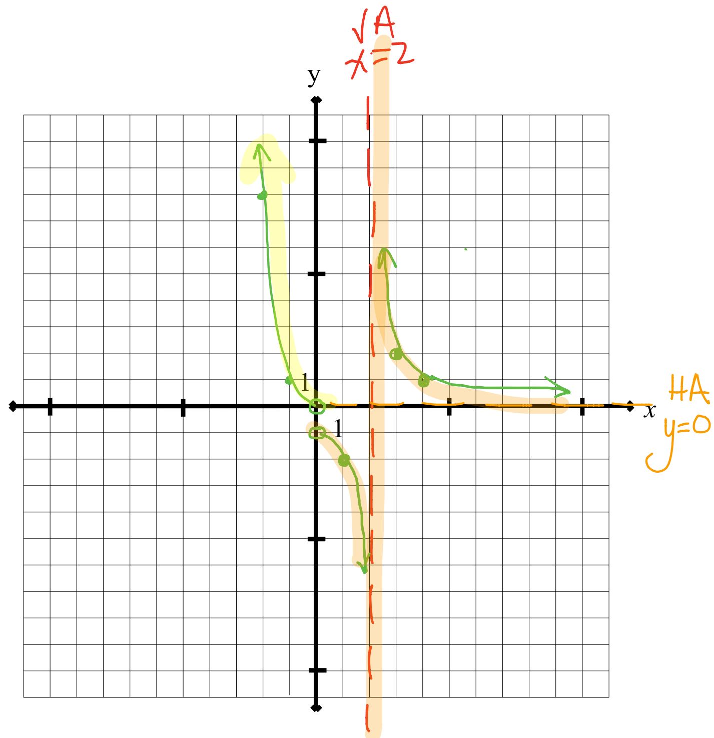
c) $g(x)=\begin{cases}-x^3 & \text{if } x\in(-\infty,0)\\ \dfrac{2}{x-2} & \text{if } x\in(0,\infty)\end{cases}$

①

x	$g(x)$
0	0 open
−1	1
−2	8

②

x	$g(x)$
0	−1 open
1	−2
2	X V.A.
3	2
4	1



∴ $g(x)$ is discontinuous at $x=0$ (jump) and at $x=2$ (infinite)

d) $f(x)=\begin{cases}\sqrt{3-x} & \text{if } x \quad x<-1 \\ 5 & \text{if } x \quad x=-1 \\ -2|x|+4 & \text{if } x>-1\end{cases}$ ① ② ③

① If $x<-1$, $f(x)=\sqrt{3-x}$

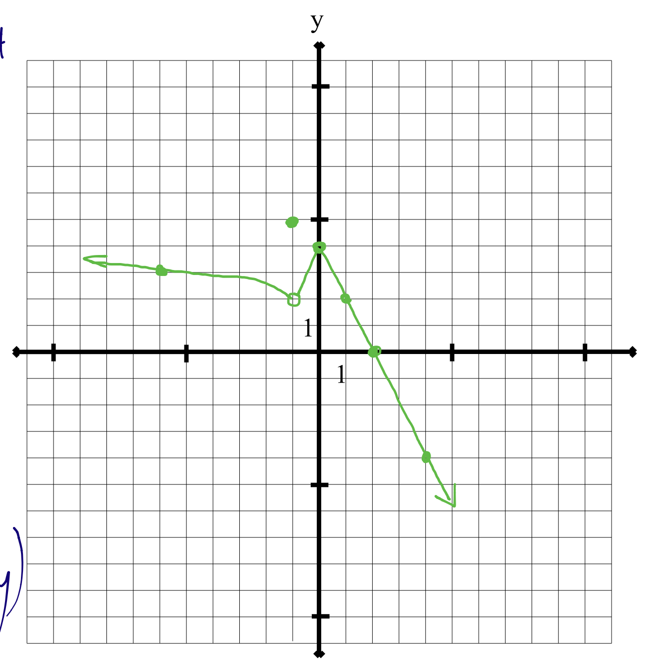
x	$f(x)$
open −1	2
−6	3

②

x	$f(x)$
−1	5 closed

③ If $x>-1$, $f(x)=-2|x|+4$

x	$f(x)$
open −1	2
0	4
1	2
2	0
4	−4



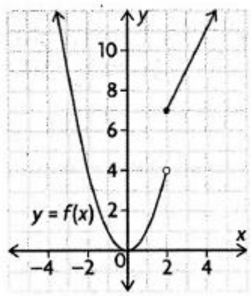
∴ $f(x)$ is discontinuous at $x=-1$.
(removable discontinuity)

HW. Exercise 2.3

Date:_______________ **2.4 Piecewise Functions and Continuity Continued**

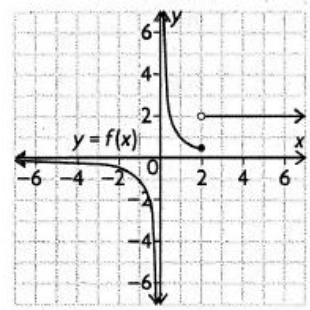
Ex. 1. Write an algebraic representation of each piecewise function, using function notation.

a)



$$f(x) = \begin{cases} x^2, & x < 2 \\ 2x+3, & x \geq 2 \end{cases}$$

b)



$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \leq 2 \quad (x \neq 0) \\ 2, & \text{if } x > 2 \end{cases}$$

Ex. 2. Without graphing determine if the function below is continuous or discontinuous. If it is discontinuous, state where it is discontinuous.

$$g(x) = \begin{cases} x+1 & \text{if } x \leq 0 \quad ① \\ 2x+1 & \text{if } 0 < x < 3 \quad ② \\ 4-x^2 & \text{if } x \geq 3 \quad ③ \end{cases}$$

For continuity at $x=0$

$g_1(0) = g_2(0)$

$g_1(0) = (0)+1$
$= 1$

$g_2(0) = 2(0)+1$
$= 1$

$\because g_1(0) = g_2(0)$

$\therefore g(x)$ is continuous at $x=0$.

For continuity at $x=3$:

$g_2(3) = g_3(3)$

$g_2(3) = 2(3)+1$
$= 7$

$g_3(3) = 4-(3)^2$
$= -5$

$\because g_2(3) \neq g_3(3)$

$\therefore g(x)$ is discontinuous at $x=3$.

Ex. 3. Given $f(x) = \begin{cases} 5-x^2 & \text{if } x \in (-\infty, -1) \\ ax+b & \text{if } x \in [-1, 1) \\ 2x^2 & \text{if } x \in [1, \infty) \end{cases}$, determine the values of a and b so that the function is continuous for all $x \in (-\infty, \infty)$.

For continuity at $x=-1$

$f_1(-1) = f_2(-1)$

$5-(-1)^2 = a(-1)+b$

$\boxed{4 = -a+b}$

For continuity at $x=1$

$f_2(1) = f_3(1)$

$a(1) + b = 2(1)^2$

$\boxed{a+b = 2}$

Solve: $-a+b = 4$

$a+b = 2$

Add $2b = 6$

$\boxed{\therefore b = 3}$

Sub $b=3$ into $a+b=2$:

$a+3 = 2$

$\boxed{\therefore a = -1}$

$\therefore$ for continuity for all $x \in \mathbb{R}$

$a=-1$, $b=3$

2. Rewrite the following functions involving absolute value as piecewise functions and then graph.

a) $f(x) = |4 - x|$

$\therefore f(x) = \begin{cases} x-4, & x > 4 \\ -x+4, & x \leq 4 \end{cases}$

Case ①:

If $4 - x < 0$ then $f(x) = -(4-x)$

$4 < x$ $\quad f(x) = -4 + x$

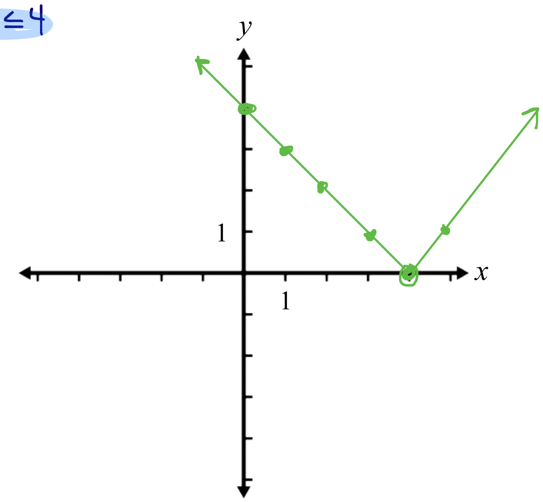
$x > 4$ $\quad f(x) = x - 4$

Case ②:

If $4 - x \geq 0$ then $f(x) = +(4-x)$

$4 \geq x$ $\quad f(x) = 4 - x$

$x \leq 4$ $\quad f(x) = -x + 4$



b) $f(x) = \dfrac{x^2|x+2|}{x+2}$ $\quad x+2 \neq 0$, $x \neq -2$

$\therefore f(x) = \begin{cases} -x^2, & x < -2 \\ x^2, & x > -2 \end{cases}$

Case ①:

If $x + 2 < 0$ then $f(x) = \dfrac{x^2[-(x+2)]}{x+2}$

$x < -2$

$f(x) = \dfrac{-x^2\cancel{(x+2)}^{1}}{\cancel{x+2}_{1}}$

$f(x) = -x^2$

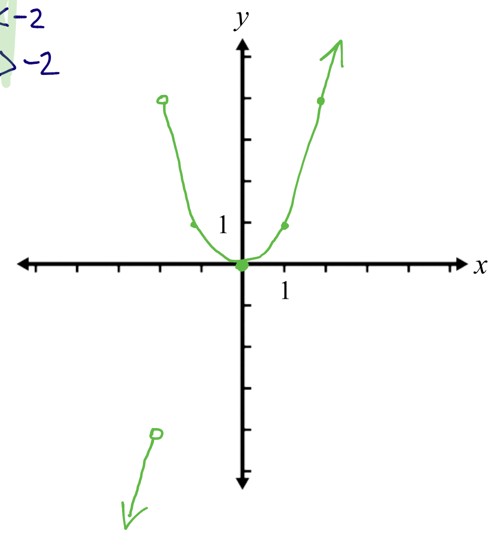
Case ②:

If $x + 2 > 0$ then $f(x) = \dfrac{x^2[+(x+2)]}{x+2}$

$x > -2$

$f(x) = \dfrac{x^2\cancel{(x+2)}^{1}}{\cancel{x+2}_{1}}$

$f(x) = x^2$



c) $f(x) = \dfrac{x^2 + |x-1| - 1}{|x-1|}$ $\quad x - 1 \neq 0$, $x \neq 1$

Case ①:

If $x - 1 < 0$ then $f(x) = \dfrac{x^2 + [-(x-1)] - 1}{-(x-1)}$

$\boxed{x < 1}$

$f(x) = \dfrac{x^2 - x + 1 - 1}{-(x-1)}$

$f(x) = \dfrac{x^2 - x}{-(x-1)}$

$f(x) = \dfrac{x\cancel{(x-1)}^{1}}{-\cancel{(x-1)}_{1}}$

$\boxed{f(x) = -x}$

Case ②:

If $x - 1 > 0$ then $f(x) = \dfrac{x^2 + (x-1) - 1}{x-1}$

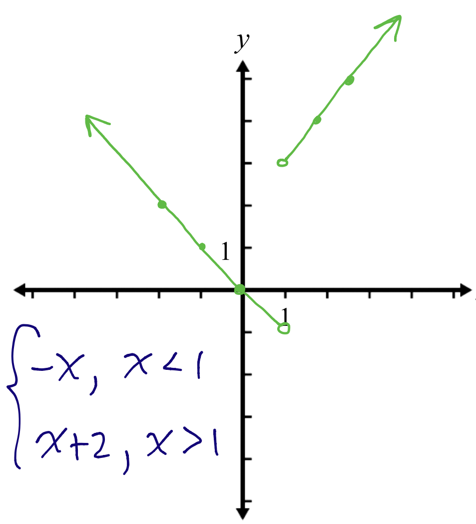
$\boxed{x > 1}$

$f(x) = \dfrac{x^2 + x - 2}{x-1}$

$f(x) = \dfrac{(x+2)\cancel{(x-1)}^{1}}{\cancel{(x-1)}_{1}}$

$\boxed{f(x) = x + 2}$

$\therefore f(x) = \begin{cases} -x, & x < 1 \\ x+2, & x > 1 \end{cases}$



HW. Exercise 2.4