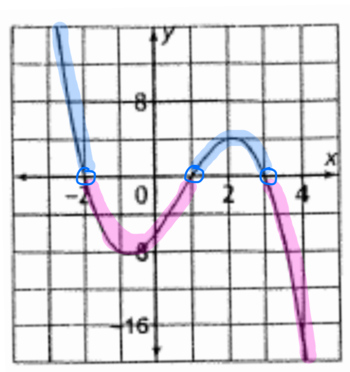
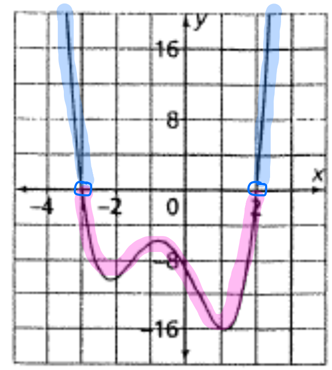
**Date:**______________ **2.8 Solving Polynomial Inequalities Graphically**

**Ex. 1.** Use the graphs of the following functions to state when **i)** $f(x) > 0$ **ii)** $f(x) < 0$
Answer using ***algebraic notation***.

**a)**




**b)**




a) **i)** $-2 > x$ or $1 < x < 3$

a) **ii)** $-2 < x < 1$ or $x > 3$

b) **i)** $x < -3$ or $x > 2$

b) **ii)** $-3 < x < 2$

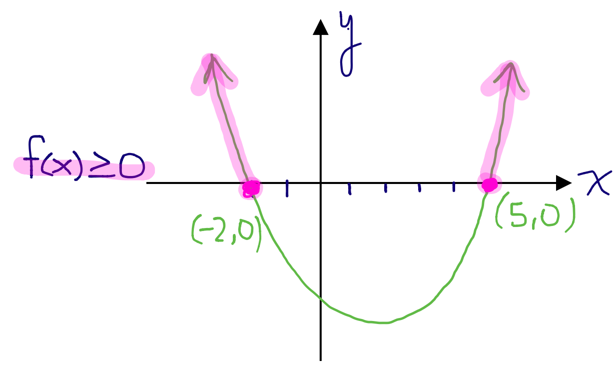
**Ex. 2.** Solve each of the following graphically where, $x \in R$. Answer using a ***solution set***.

**a)** $x^2 - 3x - 10 \geq 0$

Let $f(x) = x^2 - 3x - 10$
& graph

$f(x) = (x-5)(x+2)$

$\therefore$ x-ints are 5 and $-2$ (both single)




$\therefore$ The solution set:
$S.S. = \{x \in \mathbb{R} \mid x \leq -2 \text{ or } x \geq 5\}$

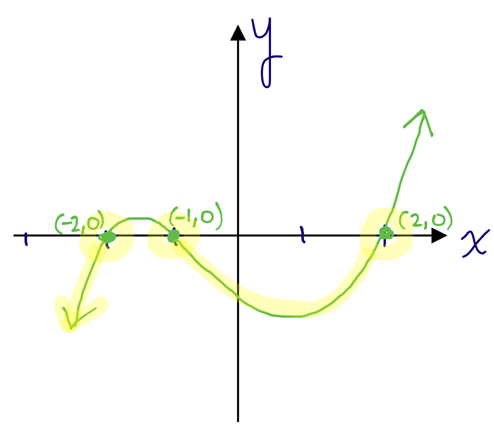
**b)** $x^3 + x^2 - 4x - 4 < 0$

Let $f(x) = x^3 + x^2 - 4x - 4$
$f(x) = x^2(x+1) - 4(x+1)$
$f(x) = (x+1)(x^2-4)$
$f(x) = (x+1)(x-2)(x+2)$

$\therefore$ x-ints are $\underbrace{-2, -1, 2}_{\text{all single}}$

Compare to $y = x$

$f(x) < 0$




$\therefore SS = \{x \in \mathbb{R} \mid x < -2 \text{ or } -1 < x < 2\}$

**Ex. 3.** Solve each of the following graphically where, $x \in R$. Answer using ***interval notation***.

**a)** $x^4 - 10x^2 + 9 \le 0$

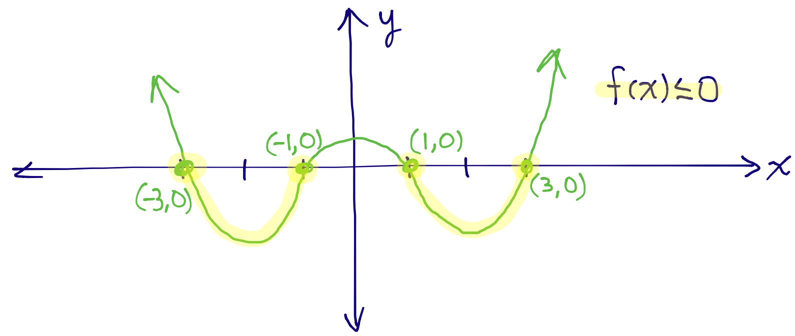
Let $f(x) = x^4 - 10x^2 + 9$

$f(x) = (x^2-9)(x^2-1)$

$f(x) = (x-3)(x+3)(x-1)(x+1)$

$\therefore$ x-ints are $\underbrace{-3, -1, 1 \text{ and } 3}_{\text{all single}}$

Compare to $y = x^2$




$\therefore x \in [-3, -1] \cup [1, 3]$

**b)** $x^5 - 6x^4 + 8x^3 - 2x^2 - 2 > -4x^3 + 6x^2 - 2$

$x^5 - 6x^4 + 12x^3 - 8x^2 > 0$

Let $f(x) = x^5 - 6x^4 + 12x^3 - 8x^2$

$f(x) = x^2(x^3 - 6x^2 + 12x - 8)$ ***

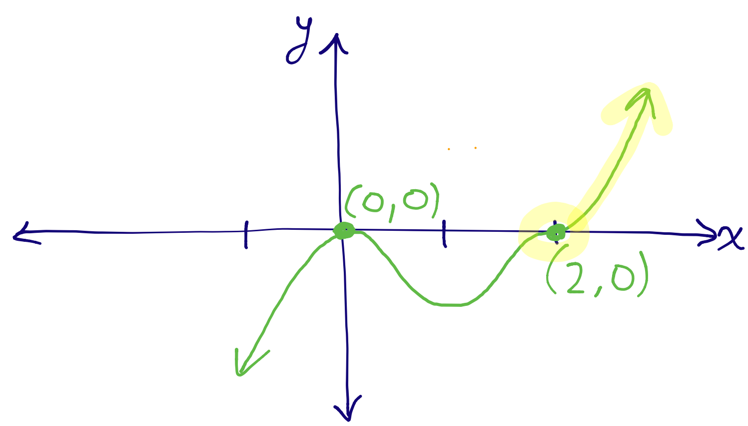
$f(x) = x^2(x-2)(x^2-4x+4)$

$f(x) = x^2(x-2)^3$

$\therefore$ x-ints are 0 and 2

double ↗ ↖ triple

Compare to $y = x$




$\therefore x \in (2, +\infty)$

Long divide:

***

$$
\begin{array}{r}
x^2 - 4x + 4 \\
x-2 \overline{) x^3 - 6x^2 + 12x - 8} \\
\underline{x^3 - 2x^2 \phantom{+12x-8}} \\
-4x^2 + 12x \phantom{-8} \\
\underline{-4x^2 + 8x \phantom{-8}} \\
4x - 8 \\
\underline{4x - 8} \\
0
\end{array}
$$

**HW. Exercise 2.8**

**Date:**________________

# 2.9 Solving Polynomial & Rational Inequalities Using a Number Line Strategy

**Warmup:** Solve the following polynomial inequality graphically.

$$4x^3 + 12x^2 - 3x - 9 \geq 0$$

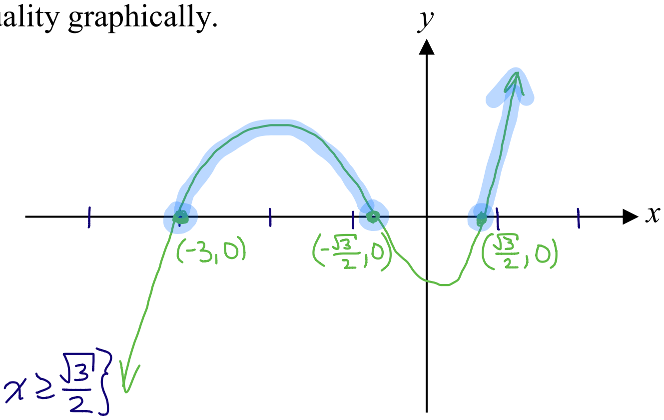
Let $f(x) = 4x^3 + 12x^2 - 3x - 9$ & graph

$f(x) = 4x^2(x+3) - 3(x+3)$

$f(x) = (x+3)(4x^2 - 3)$

x-ints: $-3, -\frac{\sqrt{3}}{2}, +\frac{\sqrt{3}}{2}$

compare to $y = x$




$\therefore$ S.S. $= \{x \in \mathbb{R} \mid -3 \leq x \leq -\frac{\sqrt{3}}{2}, x \geq \frac{\sqrt{3}}{2}\}$ ✓

or I.N.: $x \in \left[-3, -\frac{\sqrt{3}}{2}\right] \cup \left[\frac{\sqrt{3}}{2}, \infty\right)$

**Ex. 1.** Solve the following polynomial inequalities using a ***number line strategy***. State your final answer using ***set notation***.

**a)** $(x+1)(x-2)(x+3)^2 \leq 0$

Let $f(x) = (x+1)(x-2)(x+3)^2$

x-int: $-1, 2, -3$

$\therefore$ SS $= \{x \in \mathbb{R} \mid x = -3, -1 \leq x \leq 2\}$


| | $f(-4)$ | $-3$ | $f(-2)$ | $-1$ | $f(0)$ | $2$ | $f(3)$ |
|---|---|---|---|---|---|---|---|
| | $\Rightarrow (-)(-)(-)^2$ | | $\Rightarrow (-)(-)(+)^2$ | | $\Rightarrow (+)(-)(+)^2$ | | $\Rightarrow (+)(+)(+)^2$ |
| | $\Rightarrow +$ | | $\Rightarrow +$ | | $\Rightarrow -$ | | $\Rightarrow +$ |
| $f(x) \leq 0$ | X | | X | | ✓ | | X |

**b)** $2x^3 + 3x^2 > 17x - 12$

$2x^3 + 3x^2 - 17x + 12 > 0$

$(x-1)(2x^2 + 5x - 12) > 0$

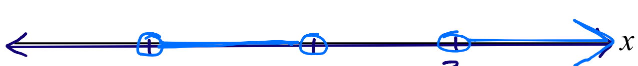
$(x-1)(2x-3)(x+4) > 0$

Let $f(x) = (x-1)(2x-3)(x+4)$

x-ints: $-4, 1, \frac{3}{2}$

$$\begin{array}{r} 2x^2 + 5x - 12 \\ x-1 \enclose{longdiv}{2x^3 + 3x^2 - 17x + 12} \\ \underline{2x^3 - 2x^2} \\ 5x^2 - 17x \\ \underline{5x^2 - 5x} \\ -12x + 12 \\ \underline{-12x + 12} \\ 0 \end{array}$$

$\therefore$ SS $= \{x \in \mathbb{R} \mid -4 < x < 1, x > \frac{3}{2}\}$




| | $f(-5)$ | $-4$ | $f(0)$ | $1$ | $f(1.1)$ | $\frac{3}{2}$ | $f(2)$ |
|---|---|---|---|---|---|---|---|
| | $\Rightarrow -$ | | $\Rightarrow +$ | | $\Rightarrow -$ | | $\Rightarrow +$ |
| $f(x) > 0$: | X | | ✓ | | X | | ✓ |

**Ex. 2.** Solve the following ***rational inequalities*** using a ***number line strategy***. State your final answer using ***interval notation***.

* Never clear fractions.

**a)** $x-2<\frac{8}{x}$

$$\frac{x}{1}-\frac{2}{1}-\frac{8}{x}<0$$

$$\frac{x^2}{x}-\frac{2x}{x}-\frac{8}{x}<0$$

$$\frac{x^2-2x-8}{x}<0$$

$$\frac{(x-4)(x+2)}{x}<0$$

Let $f(x)=\frac{(x-4)(x+2)}{x}$

x-ints: $4, -2$

restrictions: $x\neq 0$

Number line: $-2$, $0$, $4$

$f(x)<0$

| $f(-3)$ | $f(-1)$ | $f(1)$ | $f(5)$ |
|---|---|---|---|
| $\Rightarrow \frac{(-)(-)}{(-)}$ | $\Rightarrow \frac{(-)(+)}{(-)}$ | $\Rightarrow \frac{(-)(+)}{(+)}$ | $\Rightarrow \frac{(+)(+)}{(+)}$ |
| $\Rightarrow -$ | $\Rightarrow +$ | $\Rightarrow -$ | $\Rightarrow +$ |
| ✓ | X | ✓ | X |

$\therefore x\in(-\infty,-2)$ or $x\in(0,4)$

**b)** $\frac{x+3}{x+1}\geq\frac{x-2}{x-3}$

$$\frac{(x-3)(x+3)}{(x-3)(x+1)}-\frac{(x-2)(x+1)}{(x-3)(x+1)}\geq 0$$

$$\frac{(x^2-9)-(x^2-x-2)}{(x+1)(x-3)}\geq 0$$

$$\frac{x^2-9-x^2+x+2}{(x+1)(x-3)}\geq 0$$

$$\frac{x-7}{(x+1)(x-3)}\geq 0$$

Let $f(x)=\frac{x-7}{(x+1)(x-3)}$

x-int: $7$

restrictions: $x\neq -1$, $x\neq 3$

Number line: $-1$, $3$, $7$

$f(x)\geq 0$

| $f(-2)$ | $f(0)$ | $f(4)$ | $f(8)$ |
|---|---|---|---|
| $\Rightarrow \frac{(-)}{(-)(-)}$ | $\Rightarrow \frac{(-)}{(+)(-)}$ | $\Rightarrow \frac{(-)}{(+)(+)}$ | $\Rightarrow \frac{(+)}{(+)(+)}$ |
| $\Rightarrow -$ | $\Rightarrow +$ | $\Rightarrow -$ | $\Rightarrow +$ |
| X | ✓ | X | ✓ |

$\therefore x\in(-1,3)\cup[7,+\infty)$

**HW. Exercise 2.9**

**Date:**_______________ **2.10 Graphing Rational Functions With Horizontal Asymptotes**

A ***rational*** function is of the form $f(x)=\dfrac{p(x)}{q(x)}$ and has:

**i)** a ***vertical asymptote*** at $x=a$ if $q(a)=0$ and $p(a)\neq 0$
For the *vertical asymptote*, set the denominator equal to 0 and solve.
&
**ii)** a ***horizontal asymptote*** at $y=L$ if $f(x)\to L$ as $x\to\pm\infty$
and the degree of $p(x)$ is less than or equal to the degree of $q(x)$
For the *horizontal asymptote*, divide each term in the function's *expanded* numerator and denominator by the highest power of $x$ in the denominator and then examine end behaviour.

**Ex. 1.** Graph the following rational functions by finding and labeling any intercepts, asymptotes and points where the function crosses the horizontal asymptote. Include a table of values for a more accurate graph if appropriate.

**a)** $f(x)=\dfrac{2(x-2)(x-1)}{x^2-2x-3}$

$f(x)=\dfrac{2x^2-6x+4}{x^2-2x-3}$ / $f(x)=\dfrac{2(x-2)(x-1)}{(x-3)(x+1)}$

① For x-ints: let $f(x)=0$

$\dfrac{2(x-2)(x-1)}{x^2-2x-3}=\dfrac{0}{1}$

$2(x-2)(x-1)=0$

$x=1,\ x=2$

$\therefore$ x-ints are 1, 2.

② For y-int: let $x=0$

$f(0)=\dfrac{2(0)^2-6(0)+4}{(0)^2-2(0)-3}$

$f(0)=-\frac{4}{3}$

$\therefore$ y-int is $-\frac{4}{3}$

③ For Vertical Asymptote(s):
(denominator $\neq 0$)

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=-1,\ x=3$

$\therefore$ V.A. at $x=-1,\ x=3$

④ For Horizontal Asymptote:
(end behaviour; let $x\to\pm\infty$)

$f(x)=\dfrac{2x^2-6x+4}{x^2-2x-3}\ \begin{matrix}\div x^2\\ \div x^2\end{matrix}$

$f(x)=\dfrac{2-\frac{6}{x}+\frac{4}{x^2}}{1-\frac{2}{x}-\frac{3}{x^2}}$

As $x\to\pm\infty$, $f(x)\to\dfrac{2-0+0}{1-0-0}$

$f(x)\to 2$

$\therefore$ H.A. is $y=2$.

⑤ Does f(x) cross the H.A.?
Let $f(x)=2$

$\dfrac{2x^2-6x+4}{x^2-2x-3}=\dfrac{2}{1}$

$2x^2-6x+4=2x^2-4x-6$

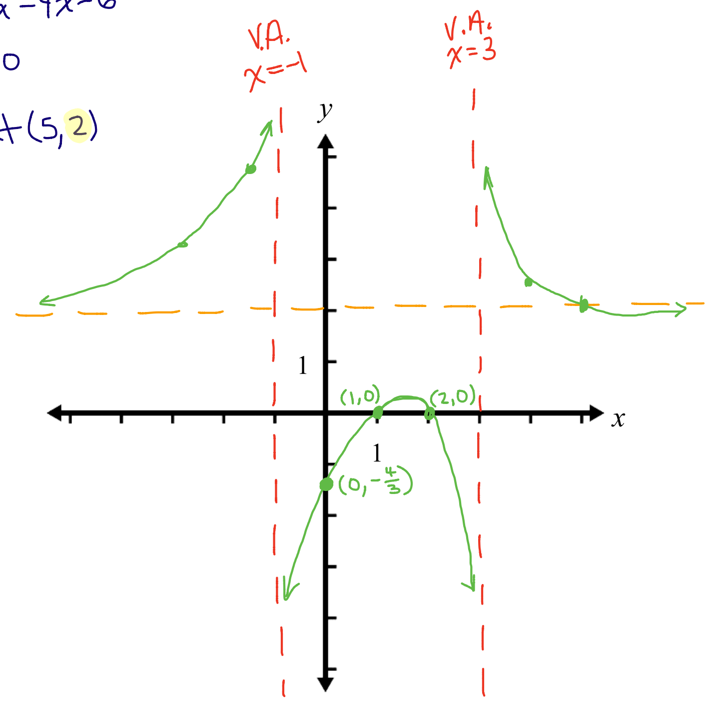
$-2x=-10$

$x=5$

$\therefore$ crosses at $(5,2)$

⑥

| $x$ | $f(x)$ |
|---|---|
| $-2$ | $\frac{24}{5}$ |
| $-3$ | $\frac{10}{3}$ |
| $4$ | $\frac{12}{5}$ |



**b)** $f(x)=\dfrac{4}{x^2+2}$ ← degree 0 ← degree 2

① x-int: $\boxed{\text{no } x\text{-ints.}}$

② y-int: $f(0)=2$

$\boxed{\therefore y\text{-int is } 2}$

③ For VA: $x^2+2=0$
no solution

$\boxed{\therefore \text{no V.A.}}$

④ For H.A: $f(x)=\dfrac{4}{x^2+2}\begin{matrix}\div x^2\\ \div x^2\end{matrix}$

$f(x)=\dfrac{\frac{4}{x^2}}{1+\frac{2}{x^2}}$

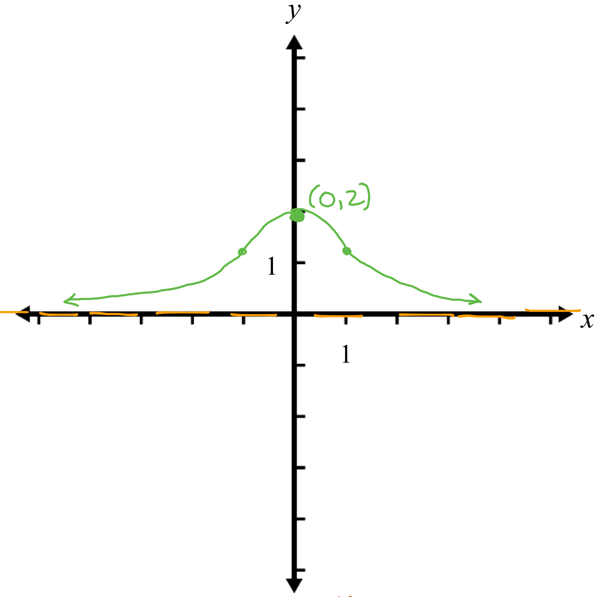
As $x\to\pm\infty$, $f(x)\to 0$

$\boxed{\therefore \text{HA } y=0}$

⑤ f(x) does not cross H.A. since HA is $y=0$ (x-axis) and no x-ints.

⑥

| $x$ | $f(x)$ |
|---|---|
| 1 | $\frac{4}{3}$ |
| −1 | $\frac{4}{3}$ |




**c)** $f(x)=\dfrac{x+2}{x-1}$ ← degree 1 ← degree 1

① x-int: $\boxed{x=-2}$

② y-int: $\boxed{y=-2}$

③ For V.A.: $x-1=0$

$\boxed{\therefore \text{V.A is } x=1}$

④ For HA: $f(x)=\dfrac{x+2}{x-1}\begin{matrix}\div x\\ \div x\end{matrix}$

$f(x)=\dfrac{1+\frac{2}{x}}{1-\frac{1}{x}}$

As $x\to\pm\infty$, $f(x)\to\dfrac{1+0}{1-0}$

$f(x)\to 1$

$\boxed{\therefore \text{H.A. is } y=1}$

⑥ Does f(x) cross the H.A.?

$\dfrac{x+2}{x-1}=\dfrac{1}{1}$

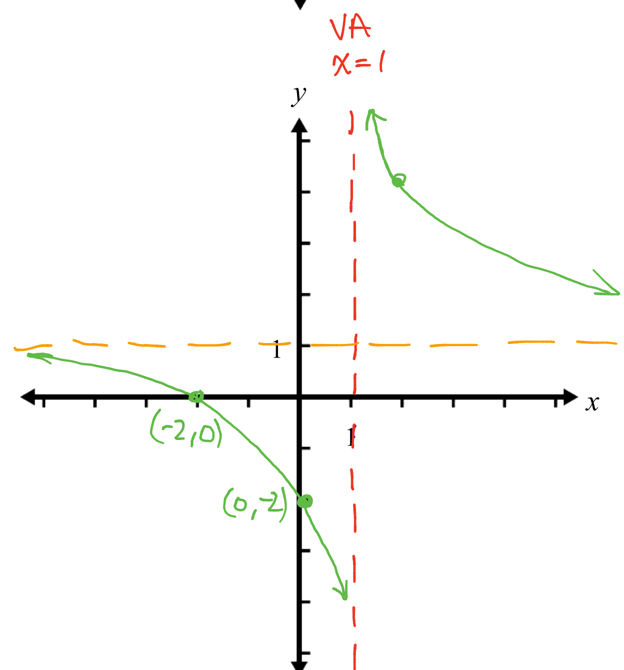
$x+2=x-1$

$0=-3$

∴ f(x) does not cross the H A

⑥

| $x$ | $f(x)$ |
|---|---|
| 2 | 4 |




**d)** $f(x)=\dfrac{1}{4-x^2}\to f(x)=\dfrac{1}{(2-x)(2+x)}$ ← degree 0 ← degree 2

① x-int: none

② y-int: $\frac{1}{4}$

③ For V.A. $(2-x)(2+x)=0$

∴ V.A. at $x=-2, x=2$

④ For HA: $f(x)=\dfrac{1}{4-x^2}\begin{matrix}\div x^2\\ \div x^2\end{matrix}$

$f(x)=\dfrac{\frac{1}{x^2}}{\frac{4}{x^2}-1}$

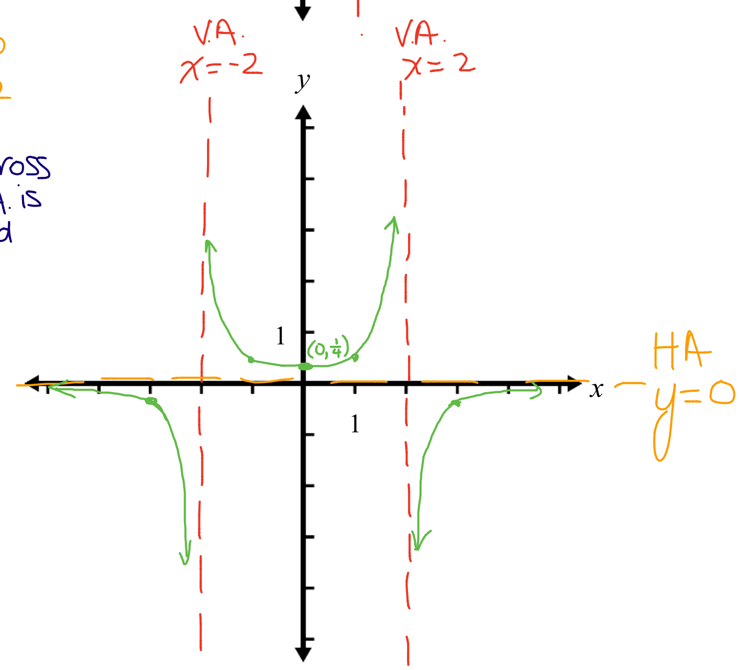
As $x\to\pm\infty$, $f(x)\to 0$

∴ H.A. at $y=0$

⑤ f(x) does not cross H.A. since HA is $y=0$ (x-axis) and no x-ints.

⑥

| $x$ | $f(x)$ |
|---|---|
| −3 | $-\frac{1}{5}$ |
| −1 | $\frac{1}{3}$ |
| 1 | $\frac{1}{3}$ |
| 3 | $-\frac{1}{5}$ |




**HW. Exercise 2.10**

**Date:**_______________ **2.11 Graphing Rational Functions With Oblique Asymptotes**

A ***rational*** function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has:

**i)** a ***vertical asymptote*** at $f(x)=\frac{p(x)}{q(x)}$ if $f(x)=\frac{p(x)}{q(x)}$ and $q(a)=0$
For the *vertical asymptote*, set the denominator equal to 0 and solve.

&

**ii)** a ***linear oblique asymptote*** at $y=mx+b$ if $f(x)\rightarrow mx+b$ as $x\rightarrow\pm\infty$
and the degree of $p(x)$ is exactly one more than the degree of $f(x)\rightarrow L$
For the *linear oblique asymptote*, rewrite the function in mixed rational form using long division and then examine end behaviour.

**Ex. 1.** Graph the following rational function by finding and labeling any intercepts, asymptotes and points where the function crosses the linear oblique asymptote. Include a table of values for a more accurate graph if appropriate.

**a)** $f(x)=\dfrac{9-x^2}{x+1}$ ← deg 2, ← deg 1

$f(x)=\dfrac{(3-x)(3+x)}{x+1}$

① x-ints are $-3, 3$.

② y-int is 9

③ V.A. is $x=-1$

④ For L.O.A $f(x)=(-x+1)+\dfrac{8}{x+1}$

$$\begin{array}{r} -x+1 \\ x+1\,\overline{)\,-x^2+0x+9} \\ \underline{-x^2-x\phantom{+9}} \\ x+9 \\ \underline{x+1} \\ 8 \end{array}$$

As $x\rightarrow\pm\infty$, $f(x)\rightarrow -x+1$

∴ L.O.A is $y=-x+1$

⑤ Does f(x) cross the L.O.A?

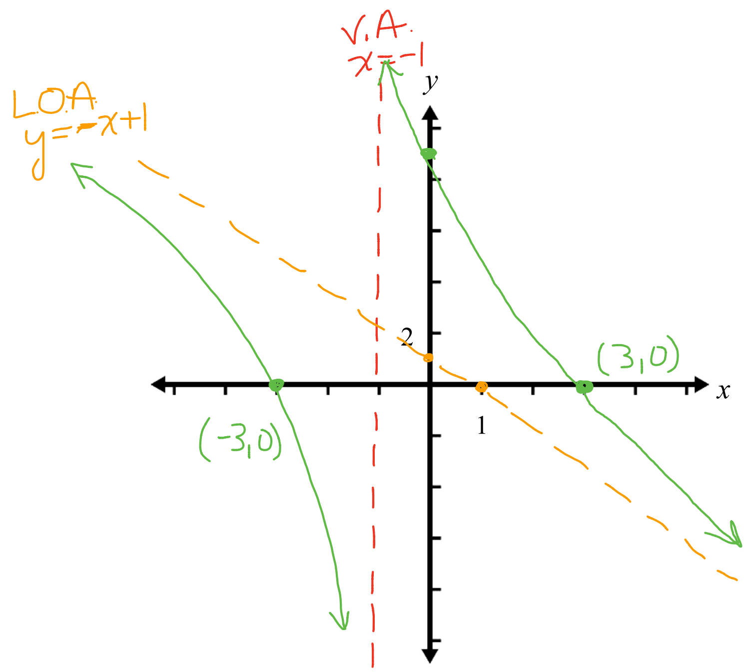
$\dfrac{9-x^2}{x+1}=\dfrac{-x+1}{1}$

$9-x^2=(-x+1)(x+1)$

$9-x^2=1-x^2$

$0=-8$

∴ does not cross.



**c)** $f(x)=\dfrac{x^2+x+1}{x}$ ← deg 2, ← deg 1

① x-int: $x^2+x+1=0$

$\vdots$

$x=\dfrac{-1\pm\sqrt{3}i}{2}$

$\therefore$ no x-ints.

② y-int $f(0)$ is undefined

$\therefore$ no y-ints

③ V.A. is $x=0$

④ For L.O.A. $f(x)=\dfrac{x^2+x+1}{x}$

$=(x+1)+\dfrac{1}{x}$

As $x\to\pm\infty$, $f(x)\to x+1$

$\therefore$ L.O.A is $y=x+1$

⑤ Does f(x) cross the L.OA?

$\dfrac{x^2+x+1}{x}=\dfrac{x+1}{1}$

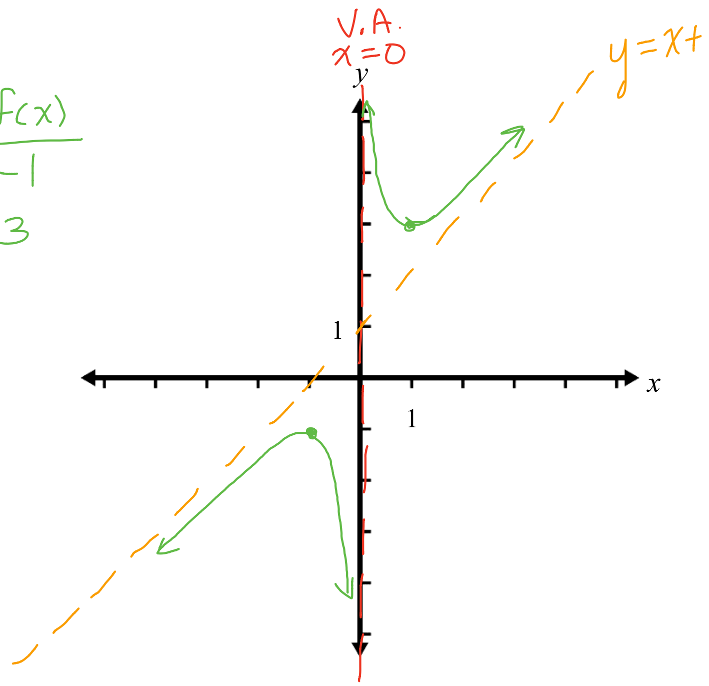
$x^2+x+1=x^2+x$

$1=0$

$\therefore$ does not cross.

⑥

| $x$ | $f(x)$ |
|---|---|
| $-1$ | $-1$ |
| $1$ | $3$ |




**d)** $f(x)=\dfrac{x^3}{x^2-4}$ / $f(x)=\dfrac{x^3}{(x-2)(x+2)}$

① x-int is 0 (triple)

② y-int is 0

③ V.A. are $x=-2$, $x=2$

④ For L.O.A.:

$$\begin{array}{r}x+0\\ x^2+0x-4\,\big)\overline{x^3+0x^2+0x+0}\\ \underline{x^3+0x^2-4x}\\ 0x^2+4x+0\end{array}$$

$f(x)=\dfrac{x^3}{x^2-4}$

$f(x)=x+\dfrac{4x}{x^2-4}\begin{matrix}\div x^2\\ \div x^2\end{matrix}$

$f(x)=x+\dfrac{\frac{4}{x}}{1-\frac{4}{x^2}}$

As $x\to\pm\infty$, $f(x)\to x$

$\boxed{\therefore \text{LOA is } y=x}$

⑤ Does f(x) cross the L.O.A?

$\dfrac{x^3}{x^2-4}=\dfrac{x}{1}$

$x^3=x^3-4x$

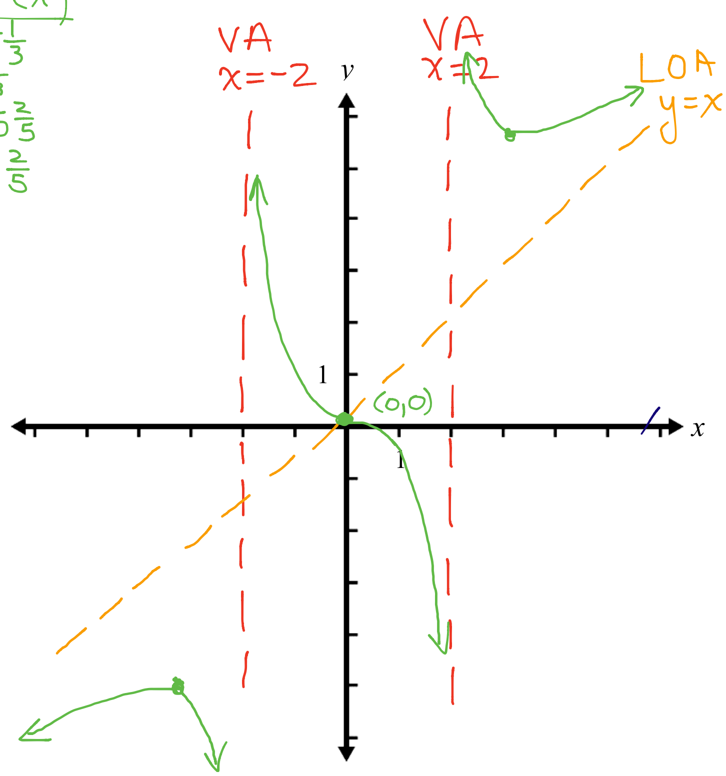
$4x=0$

$x=0$

$\therefore$ f(x) crosses the L.O.A at (0,0)

⑥

| $x$ | $f(x)$ |
|---|---|
| $1$ | $-\frac{1}{3}$ |
| $-1$ | $\frac{1}{3}$ |
| $-3$ | $-5\frac{2}{5}$ |
| $3$ | $5\frac{2}{5}$ |




**HW. Exercise 2.11**