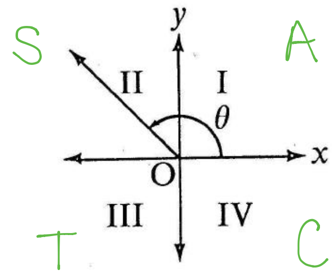
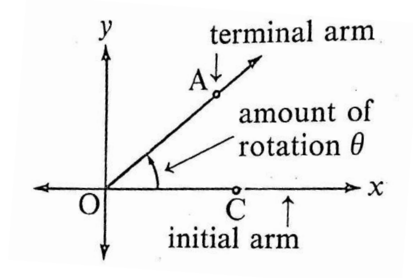
Date: Oct 20/14 **UNIT 3: TRIGONOMETRIC FUNCTIONS AND EQUATIONS**

The Definitions of Trigonometry

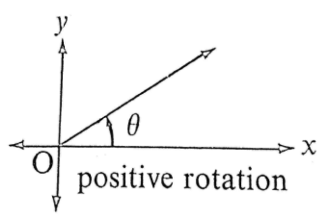
The Cartesian plane is fundamental in the study of trigonometry.
Angles are related to the co-ordinate axes and associated with rotation.
The xy plane is divide into four quadrants numbered, for convenience as shown.



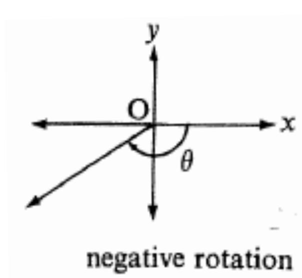
*An angle is in **standard position** if its vertex is at the origin in the xy plane, its initial arm is on the positive x-axis, and its terminal arm is a rotation of the initial arm about the origin.*



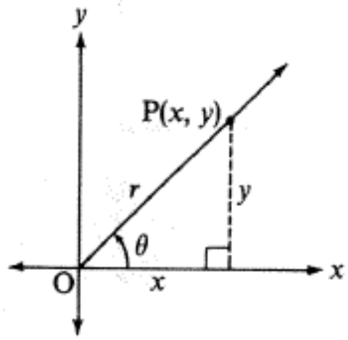
*If the rotation is counterclockwise, the angle has **positive** measure.*
The rotation is usually indicated by a directed arrow starting from the positive x-axis.



*If the rotation is clockwise, the angle has **negative** measure.*



If $P(x, y)$ is a point on the terminal arm of an angle θ, a circle with centre the origin can be drawn through $P(x, y)$ with the following radius:



$r^2 = x^2 + y^2$

or

$r = \sqrt{x^2 + y^2}$

$r > 0$

The following definitions the form the basis of trigonometry.

Primary Trigonometric Values/Ratios

sine $\theta = \frac{y}{r}$ cosine $\theta = \frac{x}{r}$ tangent $\theta = \frac{y}{x}$

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

By writing the reciprocals of the above, other trigonometric values are defined.

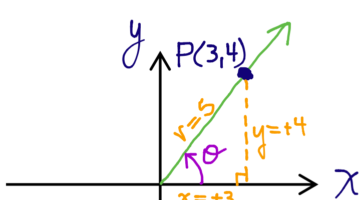
Reciprocal Trigonometric Values/Ratios

cosecant $\theta = \frac{r}{y}$ secant $\theta = \frac{r}{x}$ cotangent $\theta = \frac{x}{y}$

$\csc\theta = \frac{r}{y}$ $\sec\theta = \frac{r}{x}$ $\cot\theta = \frac{x}{y}$

To calculate the trigonometric values you only need to find a point on the terminal arm.

Ex. 1. The point $(3, 4)$ is on the terminal arm of angle θ in standard position. Calculate the **primary** trigonometric ratios. Include a diagram.



$r = \sqrt{(3)^2 + (4)^2}$

$r = \sqrt{25}$

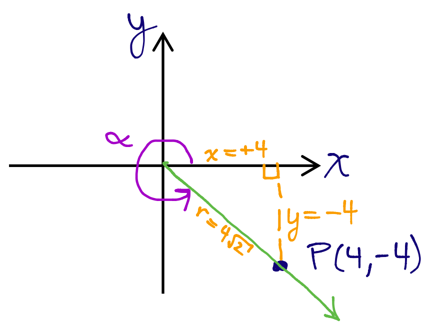
$\therefore r = 5$

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

$\therefore \sin\theta = \frac{4}{5}$ $\therefore \cos\theta = \frac{3}{5}$ $\therefore \tan\theta = \frac{4}{3}$

"alpha" α

Ex. 2. $P(4,-4)$ is a point on the terminal arm of angle α in standard position. Calculate the **reciprocal** trigonometric values. Include a diagram.



$\csc\alpha = \frac{r}{y}$, $\sec\alpha = \frac{r}{x}$, $\cot\alpha = \frac{x}{y}$

$\csc\alpha = \frac{4\sqrt{2}}{-4}$ $\quad \sec\alpha = \frac{4\sqrt{2}}{4}$ $\quad \cot\alpha = \frac{4}{-4}$

$\therefore \csc\alpha = -\sqrt{2}$, $\therefore \sec\alpha = \sqrt{2}$, $\therefore \cot\alpha = -1$

$r = \sqrt{(4)^2 + (-4)^2}$

$r = \sqrt{32}$

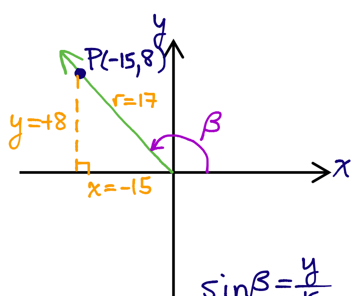
$\therefore r = 4\sqrt{2}$

Ex. 3. Draw a sketch of each angle in standard position. Calculate the **primary** trigonometric ratios for **a)** and the **reciprocal** trigonometric ratios for **b).**

"Beta"

$\cos\beta = \frac{x}{r}$ $\quad$ **a)** $\cos\beta = -\frac{15}{17}$, β is in quadrant II $\qquad$ **b)** $\tan\theta = \frac{1}{\sqrt{3}}$, θ is in quadrant III

$x = -15$, $r = 17$



$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y^2 = (17)^2 - (-15)^2$

$y^2 = 289 - 225$

$y^2 = 64$

$y = +8$ (QII)

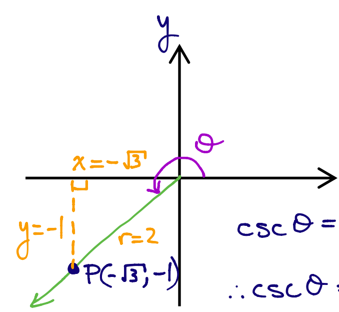
$\sin\beta = \frac{y}{r}$

$\therefore \sin\beta = \frac{8}{17}$

$\cos\beta = -\frac{15}{17}$ (given)

$\tan\beta = \frac{y}{x}$

$\therefore \tan\beta = -\frac{8}{15}$



$r = \sqrt{(-\sqrt{3})^2 + (-1)^2}$

$r = \sqrt{3+1}$

$r = \sqrt{4}$

$r = 2$

$\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$

$\therefore \csc\theta = \frac{2}{-1}$ $\quad \sec\theta = \frac{2}{-\sqrt{3}}$

$\boxed{\therefore \csc\theta = -2}$ $\quad \boxed{\therefore \sec\theta = -\frac{2}{\sqrt{3}}}$

$\cot\theta = \frac{x}{y}$

$\cot\theta = \frac{-\sqrt{3}}{-1}$

$\boxed{\therefore \cot\theta = \sqrt{3}}$

Ex. 4. For any angle θ, show that:

a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad$ **b)** $\csc^2\theta = 1 + \cot^2\theta$

$LS = \tan\theta$

$= \frac{y}{x}$

$RS = \frac{\sin\theta}{\cos\theta}$

$= \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)}$

$= \frac{y}{r} \div \frac{x}{r}$

$= \frac{y}{\cancel{r}_1} \times \frac{\cancel{r}^1}{x}$

$= \frac{y}{x}$

$\boxed{\begin{array}{l} \because LS = RS \\ \therefore \tan\theta = \frac{\sin\theta}{\cos\theta} \\ \text{Q.E.D.} \end{array}}$

$LS = \csc^2\theta$

$= (\csc\theta)^2$

$= \left(\frac{r}{y}\right)^2$

$= \frac{r^2}{y^2}$

$RS = 1 + \cot^2\theta$

$= 1 + \left(\frac{x}{y}\right)^2$

$= 1 + \frac{x^2}{y^2}$

$= \frac{y^2}{y^2} + \frac{x^2}{y^2}$

$= \frac{y^2 + x^2}{y^2}$

$= \frac{r^2}{y^2}$

$\because LS = RS.$

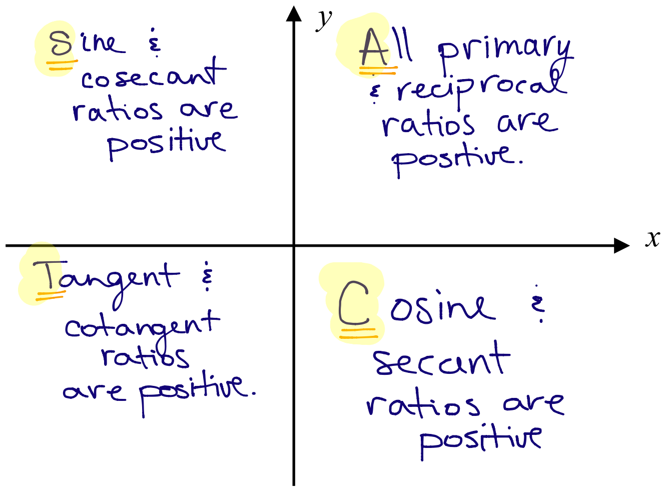
$\therefore \csc^2\theta = 1 + \cot^2\theta$ $\square$

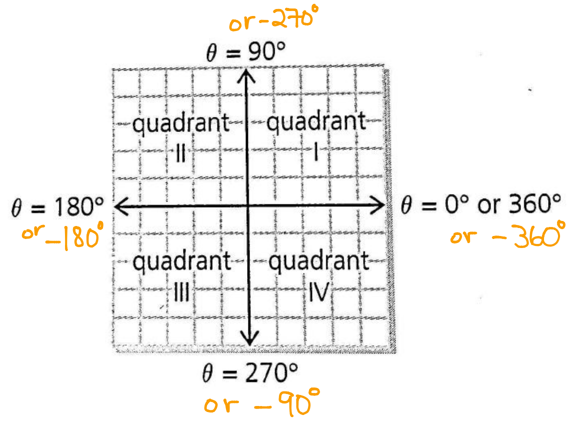
HW. Worksheet on the Definitions of Trigonometry #1 cf, 2 ab, 3, 4, 7, 8, 10, 11, 12 de

Date: Oct 21/14

ANGLES AND QUADRANTS

Recall: The **CAST** rule





Definitions:

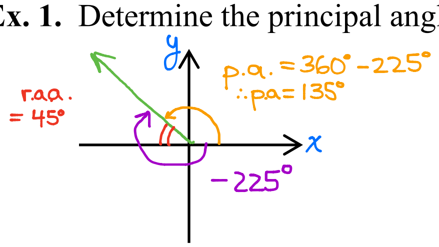
Coterminal angles *share the* *same terminal arm* *and the* *same initial arm*.

The ***principal angle*** *is the angle* *between* 0° *and* 360°. p.a.

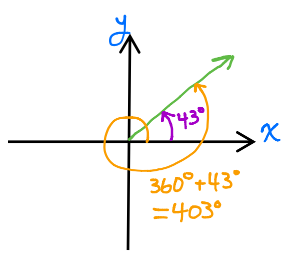
The ***related acute angle*** *is the angle formed by the terminal arm of an angle in standard position and the x-axis. The related acute angle is* *always positive* *and lies* *between* 0° *and* 90°. (QI)

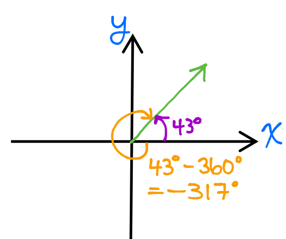
r.a.a.

Ex. 1. Determine the principal angle and the related acute angle for $\theta = -225^\circ$.



Ex. 2. Determine the next two positive and negative coterminal angles for 43°.





∴ the next two positive coterminal angles are 403° and 763°.

∴ the next two coterminal negative angles are −317° and −677°.

Ex. 3. In which possible quadrant(s) does the terminal arm of θ lie if:

a) $\sin\theta$ is positive? I & II

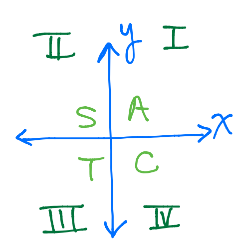
b) $\cos\theta$ is negative? II & III

c) $\tan\theta$ is positive? I & III

d) $\csc\theta$ is negative? III & IV

e) $\cot\theta$ is negative? II & IV

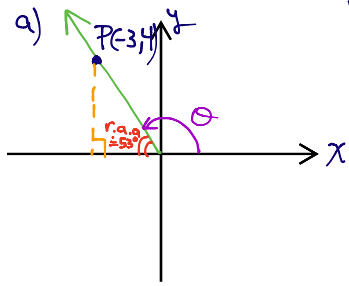
f) $\sec\theta$ is positive? I & IV



Ex. 4. Point $P(-3,4)$ is on the terminal arm of an angle in standard position.

a) Sketch the principal angle, θ.
b) Determine the value of the related acute angle to the nearest degree.
c) What is the measure of θ to the nearest degree?

a)



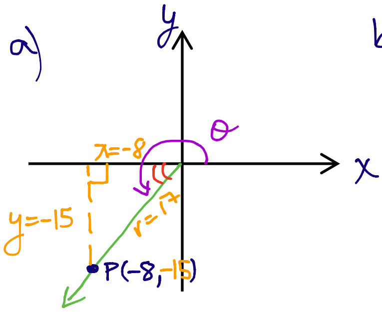
b) $r.a.a = \tan^{-1}\left(+\frac{4}{3}\right)$
$\doteq 53^\circ$

c) $\theta = 180^\circ - 53^\circ$
$\therefore \theta = 127^\circ$

Ex. 5. A positive angle θ is in the third quadrant and $\sec\theta = -\frac{17}{8}$. $\cos\theta = -\frac{8}{17}$, $x=-8$, $r=17$

a) Sketch the principal angle, θ.
b) Determine the value of the related acute angle to the nearest degree.
c) What is the measure of θ to the nearest degree?
d) Find exact values for $\tan\theta$ and $\csc\theta$.

a)



b) $raa = \cos^{-1}\left(+\frac{8}{17}\right)$
$\doteq 62^\circ$

c) $\theta = 180^\circ + 62^\circ$
$\therefore \theta = 242^\circ$

d) Find y:
$y^2 = (17)^2 - (-8)^2$
$y^2 = 225$
$y = -\sqrt{225}$
$\therefore y = -15$

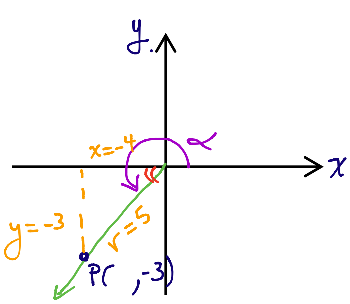
$\tan\theta = \frac{y}{x}$
$\therefore \tan\theta = \frac{15}{8}$

$\csc\theta = \frac{r}{y}$
$\therefore \csc\theta = -\frac{17}{15}$

Ex. 6. Given that $\csc\alpha = -\frac{5}{3}$ and $0^\circ \le \alpha \le 360^\circ$,

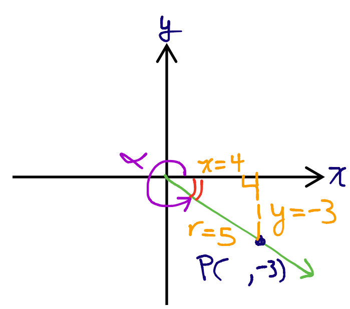
a) Find α. (Include diagrams)
b) Determine exact values for $\cos\alpha$ and $\cot\alpha$.

Q III



$r^2 = x^2 + y^2$
$x^2 = r^2 - y^2$
$x^2 = (5)^2 - (-3)^2$
$x^2 = 16$
$\boxed{x = \pm 4}$

Q IV



b) In Q III:
$\cos\alpha = -\frac{4}{5}$
$\cot\alpha = \frac{4}{3}$

b) In Q IV
$\cos\alpha = \frac{4}{5}$
$\cot\alpha = -\frac{4}{3}$

HW. Worksheet on Angles and Quadrants Part A #1, 2, 5, 6, & 8 even parts; 9, 12; Part B #7, 8, 9

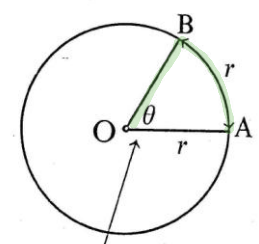
b, d, f, ...

Date: Oct 22/14

RADIAN MEASURE

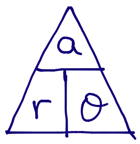
Recall:

*A **radian** is the measure of the angle subtended at the center of the circle by an arc equal in length to the radius of the circle.*



The measure of θ is defined to be 1 radian.

&

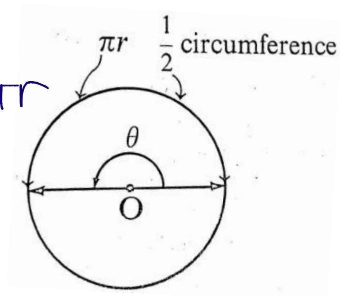


$$\theta_{in\,radians} = \frac{arc\,length}{radius} \quad or \quad \theta = \frac{a}{r} \quad or \quad a = r\theta$$

*Angles can be measured in **degrees** or **radians**.*

Ex. 1. Determine the relationship between degrees and radians.

$C = 2\pi r$



In radians,

$\theta = \frac{a}{r}$

$\theta = \frac{\pi r}{r}$

$\theta = \pi$

In degrees,

$\theta = 180°$

$$\boxed{\pi\ rad = 180^\circ}$$

Ex. 2. Change each radian measure to degree measure. Round to the nearest degree, if necessary.

Hint: $\pi\ rad = 180^\circ$ *or* $1\ rad = \frac{180^\circ}{\pi}$ *: To change radian measure to degree measure, multiply the number of radians by* $\frac{180^\circ}{\pi}$.

a) $\frac{\pi}{6}$

$= \frac{\cancel{\pi}^1}{6} \cdot \frac{180°}{\cancel{\pi}_1}$

$= 30°$

b) $\frac{5\pi}{4}$

$= \frac{5\cancel{\pi}^1}{\cancel{4}_1} \cdot \frac{\cancel{180°}^{45}}{\cancel{\pi}_1}$

$= 225°$

c) $-\frac{3\pi}{2}$

$= -\frac{3\cancel{\pi}^1}{\cancel{2}_1} \cdot \frac{\cancel{180°}^{90}}{\cancel{\pi}_1}$

$= -270°$

d) 2.2

$= 2.2 \times \frac{180°}{\pi}$

$\doteq 126°$

Ex. 3. Find the **exact** radian measure, in terms of π, for each of the following.

Hint: $180^\circ = \pi\, rad$ *or* $1^\circ = \dfrac{\pi}{180} rad$ *: To change degree measure to radian measure, multiply the number of degrees by* $\dfrac{\pi}{180} rad$.

a) 45°

$= \cancel{45^\circ}^{\,1} \cdot \dfrac{\pi}{\cancel{180^\circ}_{\,4}}$

$= \dfrac{\pi}{4}$

b) 60°

$= \cancel{60^\circ}^{\,1} \cdot \dfrac{\pi}{\cancel{180^\circ}_{\,3}}$

$= \dfrac{\pi}{3}$

c) -210°

$= -\cancel{210^\circ}^{\,7} \cdot \dfrac{\pi}{\cancel{180^\circ}_{\,6}}$

$= -\dfrac{7\pi}{6}$

d) -720°

$= -\cancel{720}^{\,4} \cdot \dfrac{\pi}{\cancel{180^\circ}_{\,1}}$

$= -4\pi$

Ex. 4. Change each degree measure to radian measure, to 4 decimal places.

a) 30°

$= 30^\circ \cdot \dfrac{\pi}{180^\circ}$

$= \dfrac{\pi}{6}$

$\doteq 0.5236$

b) -230°

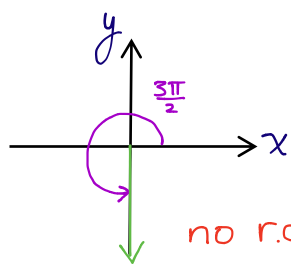
$= -230^\circ \cdot \dfrac{\pi}{180^\circ}$

$= -\dfrac{23\pi}{18}$

$\doteq -4.0143$

Ex. 5. Sketch each angle in standard position.

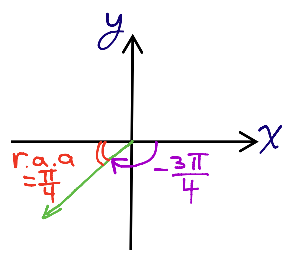
a) $\dfrac{3}{2}\pi$ **b)** $\dfrac{-3\pi}{4}$ **c)** $-\dfrac{5}{3}\pi$

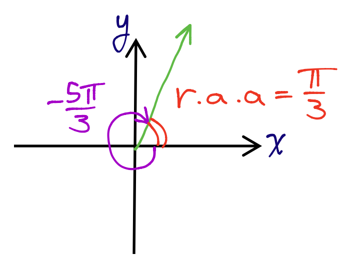


no r.a.a.

$0 < r.a.a < 90$

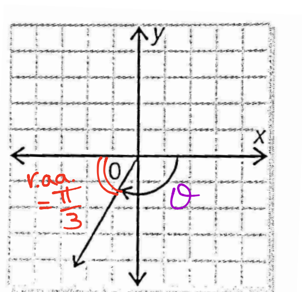
$0 < r.a.a. < \frac{\pi}{2}$





Ex. 6. Write the value of θ in exact radian measure with the given related acute angle for the following:

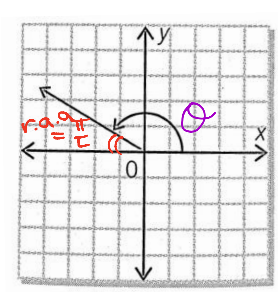
a) $r.a.a = \dfrac{\pi}{3}$ **b)** $r.a.a = 45^\circ$ **c)** $r.a.a = \dfrac{\pi}{6}$



$\theta = -\pi + \dfrac{\pi}{3}$

$\theta = -\dfrac{3\pi}{3} + \dfrac{\pi}{3}$

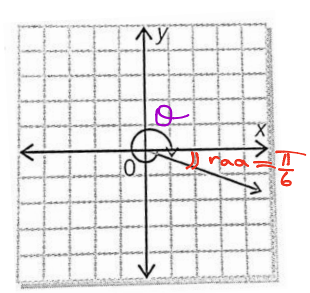
$\theta = -\dfrac{2\pi}{3}$



$\theta = \pi - \dfrac{\pi}{4}$

$\theta = \dfrac{4\pi}{4} - \dfrac{\pi}{4}$

$\theta = \dfrac{3\pi}{4}$



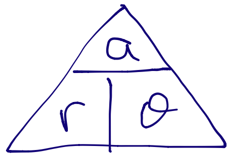
$\theta = -2\pi - \dfrac{\pi}{6}$

$\theta = -\dfrac{12\pi}{6} - \dfrac{\pi}{6}$

$\theta = -\dfrac{13\pi}{6}$

Ex. 7. Sector angles are drawn in a unit circle. Find the measure of the arc of the circle that subtends an angle measuring 60°. *Label the given diagram.*

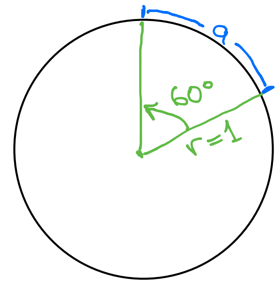
Note: $60^\circ = \frac{\pi}{3}$ radians



$a = r\theta$

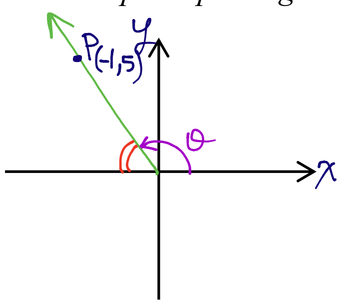
$a = (1)\left(\frac{\pi}{3}\right)$

$\therefore a = \frac{\pi}{3}$



$\therefore$ the arc measures exactly $\frac{\pi}{3}$ units or approximately 1.05 units.

Ex. 8. $P(-1,5)$ is a point on the terminal arm of angle θ in standard position. Calculate the measure of the *principal angle* in radians to 1 decimal place. *Include a diagram.*



$raa = \tan^{-1}\left(+\frac{5}{1}\right)$

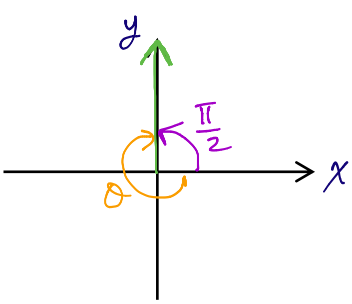
$r.a.a \doteq 1.37$ radians

$\theta = \pi - 1.37$

$\therefore \theta \doteq 1.8$

Ex. 9. The radian measures of angles are shown. Write the measure of the *coterminal angle* θ for $-2\pi \le \theta \le 2\pi$. *Include a diagram.*

a) $\frac{\pi}{2}$

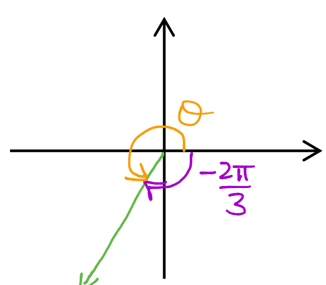


$\theta = \frac{\pi}{2} - 2\pi$

$\theta = \frac{\pi}{2} - \frac{4\pi}{2}$

$\theta = -\frac{3\pi}{2}$

b) $-\frac{2\pi}{3}$



$\theta = \frac{-2\pi}{3} + 2\pi$

$\theta = -\frac{2\pi}{3} + \frac{6\pi}{3}$

$\therefore \theta = \frac{4\pi}{3}$

HW. Worksheet on Radian Measure #1 to 13 all

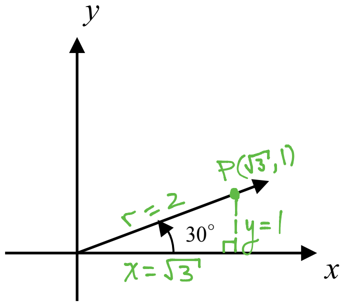
Date: Oct 23/14

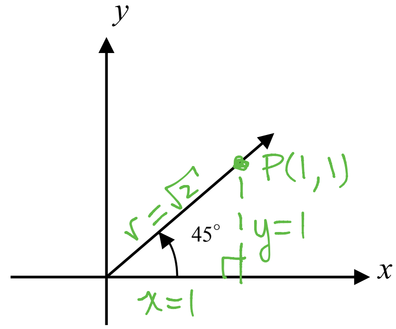
EXACT TRIG RATIOS FOR SPECIAL ANGLES

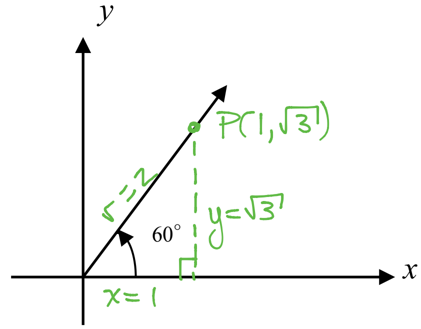
Recall:

Special Angles *of* 30°, 45°, *and* 60° *or* $\frac{\pi}{6}$, $\frac{\pi}{4}$, *and* $\frac{\pi}{3}$ *radians*

i) **ii)** **iii)**







i) $\sin\frac{\pi}{6} = \frac{1}{2}$ **ii)** $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ **iii)** $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\cos\frac{\pi}{3} = \frac{1}{2}$

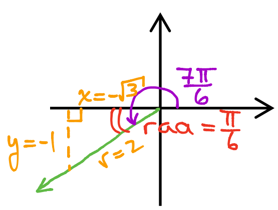
$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\tan\frac{\pi}{4} = 1$ $\tan\frac{\pi}{3} = \sqrt{3}$

S | A
T | C

Ex. 1. Find the exact value of each trigonometric ratio.

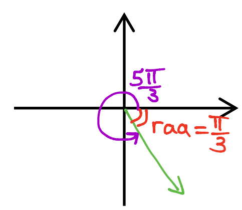
a) $\cos\frac{7}{6}\pi$

$= -\frac{\sqrt{3}}{2}$



b) $\csc\frac{5}{3}\pi$

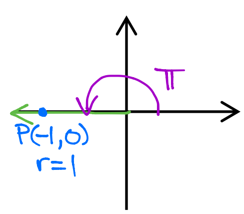
$= -\frac{2}{\sqrt{3}}$



c) $\tan\pi$

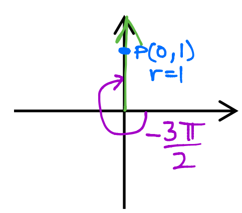
no raa

$= 0$



d) $\sin\left(-\frac{3}{2}\pi\right)$

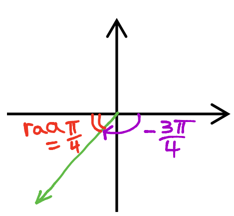
$= 1$



no raa

e) $\sec\left(-\frac{3}{4}\pi\right)$

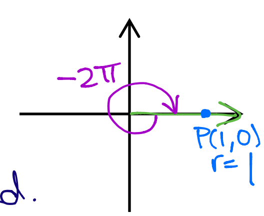
$= -\sqrt{2}$



f) $\cot(-2\pi)$

$= \frac{1}{0}$

$\therefore \cot(-2\pi)$ is undefined.



Ex. 2. Calculate the exact value of $\sin(-135^\circ)\cdot\cos 315^\circ + \cos 390^\circ \div \cot(-300^\circ)$.

$$\overset{\text{III}}{\sin(-135^\circ)}\cdot\overset{\text{IV}}{\cos(315^\circ)}+\overset{\text{I}}{\cos(390^\circ)}\div\overset{\text{I}}{\cot(-300^\circ)}$$

$$=(-\sin 45^\circ)\cdot(+\cos 45^\circ)+(+\cos 30^\circ)\div(+\cot 60^\circ)$$

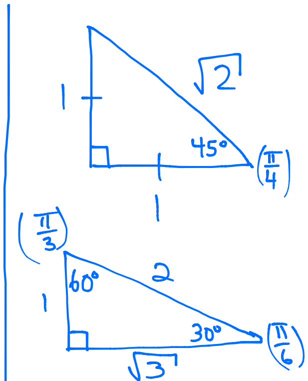
$$=-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\div\frac{1}{\sqrt{3}}$$

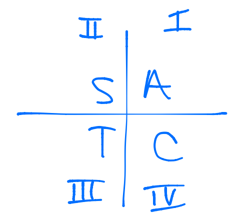
$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{1}$$

$$=-\frac{1}{2}+\frac{3}{2}$$

$$=\frac{2}{2}$$

$$=1.$$



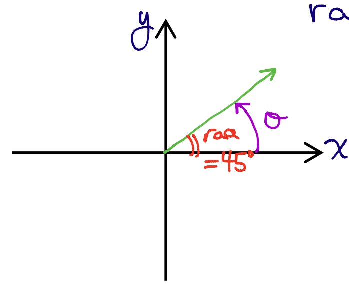


Ex. 3. If $0^\circ \le \theta \le 360^\circ$, find possible values of θ for $\sec\theta = \sqrt{2}$. Include diagrams.

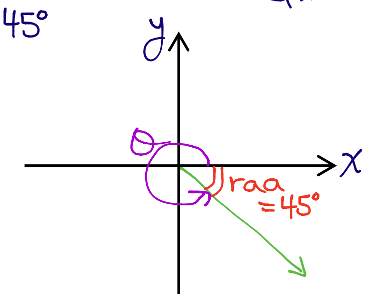
$\cos\theta = \frac{1}{\sqrt{2}}$

raa = 45°

QI



QIV



$\therefore \theta = 45^\circ$ or $\theta = 315^\circ$

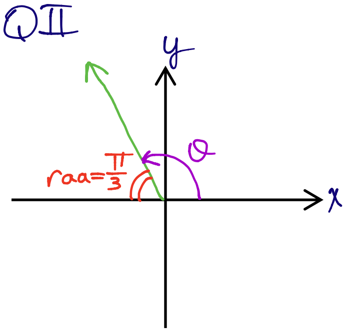
Ex. 4. If $0 \le \theta \le 2\pi$, find possible values of θ for $\cot\theta = -\dfrac{1}{\sqrt{3}}$. Include diagrams.

$\cot\theta = -\frac{1}{\sqrt{3}}$

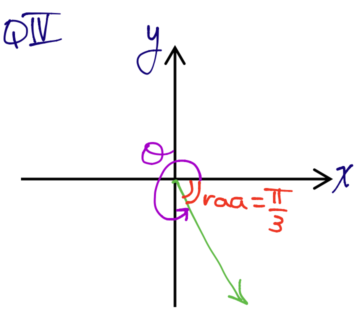
$\therefore \tan\theta = -\sqrt{3}$

$\boxed{\text{raa} = \frac{\pi}{3}}$

QII



QIV



$\theta = \pi - \frac{\pi}{3}$

$\theta = \frac{3\pi}{3} - \frac{\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$

$\theta = 2\pi - \frac{\pi}{3}$

$\theta = \frac{6\pi}{3} - \frac{\pi}{3}$

$\therefore \theta = \frac{5\pi}{3}$

HW. Worksheet on Exact Trig Ratios for Special Angles #5 to 13 all