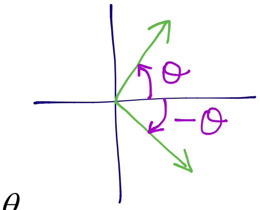
Date: Oct 24/14

The Addition, Subtraction and Double-Angle Formulas



Warmup

1. Given θ is an acute angle, express each quantity in terms of either $\sin\theta$, $\cos\theta$ or $\tan\theta$.

a) $\sin(-\theta)$
$= -\sin\theta$

b) $\cos(-\theta)$
$= +\cos\theta$

c) $\tan(-\theta)$
$= -\tan\theta$

2. Given the identity $\sin^2\theta + \cos^2\theta = 1$, express:

a) $\sin^2\theta$ in terms of $\cos^2\theta$
$\sin^2\theta = 1 - \cos^2\theta$

b) $\cos^2\theta$ in terms of $\sin^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$

Example 1.
Develop the addition formula for sine, ie. $\sin(a+b)$, in terms of angles a and b using the given diagram.

In ΔTUP, $\angle TUP = \frac{\pi}{2} - a$ and in ΔRUQ, $\angle RUQ = \pi - \left(\frac{\pi}{2} - a\right) - \frac{\pi}{2}$

$= \frac{\pi}{2} - \left(\frac{\pi}{2} - a\right)$

$= a$

In ΔTRS, $\sin(a+b) = \frac{RS}{1}$

$\sin(a+b) = RS$

$\sin(a+b) = QP$

$\boxed{\sin(a+b) = UP + QU}$

In ΔTRU,

$\cos b = \frac{TU}{1}$ and $\sin b = \frac{RU}{1}$

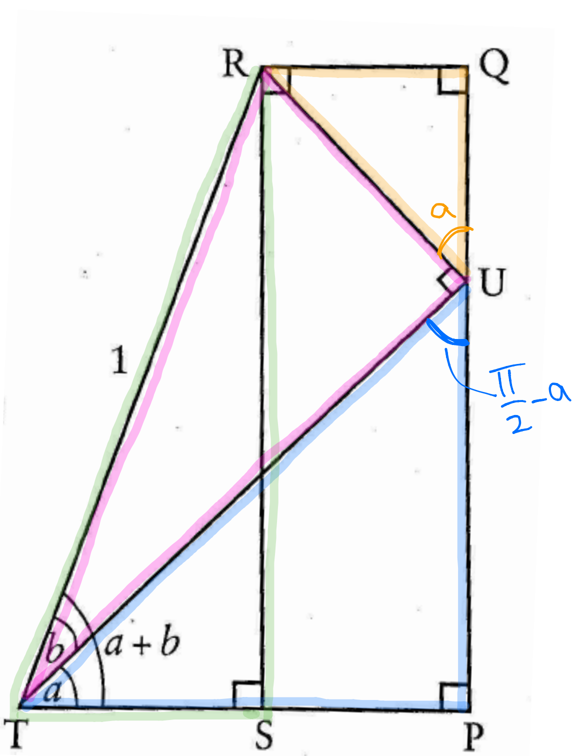
$\boxed{\therefore TU = \cos b}$ $\boxed{\therefore RU = \sin b}$

In ΔTUP, and in ΔRUQ,

$\sin a = \frac{UP}{TU}$ $\cos a = \frac{QU}{RU}$

$\frac{\sin a}{1} = \frac{UP}{\cos b}$ $\frac{\cos a}{1} = \frac{QU}{\sin b}$

$\boxed{\therefore UP = \sin a \cos b}$ $\boxed{\therefore QU = \sin b \cos a}$



$\because \sin(a+b) = UP + QU$

$\therefore \sin(a+b) = \sin a \cos b + \sin b \cos a$

BONUS for Monday

Note:
By following a similar strategy and using the fact that $TS = TP - SP$, the following addition formula for cosine can be determined.

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\boxed{\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ &\&\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B\end{aligned}}$$

Warm-Up

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = +\cos\theta$; θ is acute.

$\tan(-\theta) = -\tan\theta$

Ex. 2. Using the addition formulas for sine and cosine, develop the subtraction formulas, $\sin(A-B)$ and $\cos(A-B)$.

$$\begin{aligned}&\sin(A-B)\\ &= \sin[A+(-B)]\\ &= \sin A\cos(-B) + \sin(-B)\cos A\\ &= \sin A\cos B - \sin B\cos A\end{aligned}$$

$$\boxed{\therefore \sin(A-B) = \sin A\cos B - \sin B\cos A}$$

$$\begin{aligned}&\cos(A-B)\\ &= \cos[A+(-B)]\\ &= \cos A\cos(-B) - \sin A\sin(-B)\\ &= \cos A\cos B + \sin A\sin B\end{aligned}$$

$$\boxed{\therefore \cos(A-B) = \cos A\cos B + \sin A\sin B}$$

Ex. 3. Using the addition formulas for sine and cosine, develop the double-angle formulas, $\sin 2A$ and $\cos 2A$.

$$\begin{aligned}&\sin 2A\\ &= \sin(A+A)\\ &= \sin A\cos A + \sin A\cos A\\ &= 2\sin A\cos A\end{aligned}$$

$$\boxed{\therefore \sin 2A = 2\sin A\cos A}$$

$$\begin{aligned}&\cos 2A\\ &= \cos(A+A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

$$\boxed{\therefore \cos 2A = \cos^2 A - \sin^2 A}$$

In terms of $\sin^2 A$:

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= (1-\sin^2 A) - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\boxed{\therefore \cos 2A = 1 - 2\sin^2 A}$$

In terms of $\cos^2 A$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - (1-\cos^2 A)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

$$\boxed{\therefore \cos 2A = 2\cos^2 A - 1}$$

Ex. 4. Using the addition formulas for sine and cosine, develop the addition and subtraction formulas for tangent, $\tan(A+B)$ and $\tan(A-B)$. Also develop the double-angle formula, $\tan 2A$.

$$\tan(A+B)$$
$$=\frac{\sin(A+B)}{\cos(A+B)}$$
$$=\frac{\sin A\cos B+\sin B\cos A}{\cos A\cos B-\sin A\sin B}\div\frac{\cos A\cos B}{\cos A\cos B}$$
$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{\dfrac{\cos B}{\cos B}-\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}$$
$$=\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}$$
$$\boxed{\therefore \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$

$$\tan(A-B)$$
$$=\tan[A+(-B)]$$
$$=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$$
$$=\frac{\tan A-\tan B}{1+\tan A\tan B}$$
$$\boxed{\therefore \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

$$\tan 2A$$
$$=\tan(A+A)$$
$$=\frac{\tan A+\tan A}{1-\tan A\tan A}$$
$$=\frac{2\tan A}{1-\tan^2 A}$$
$$\boxed{\therefore \tan 2A=\frac{2\tan A}{1-\tan^2 A}}$$

Key Concepts for Compound Angle Formulas

- *Addition, subtraction and double-angle identities for sine, cosine, and tangent*

$\sin(A+B) = \sin A\cos B + \sin B\cos A$

$\sin(A-B) = \sin A\cos B - \sin B\cos A$

$\sin 2A \quad = 2\sin A\cos A$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\cos 2A \quad = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

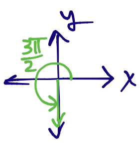
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\tan 2A \quad = \frac{2\tan A}{1 - \tan^2 A}$$

HW. ***Memorize these identities.***

1. Given θ is an acute angle, express each quantity in terms of either $\sin\theta$ or $\cos\theta$ using an appropriate compound angle identity.

a) $\sin\left(\theta + \dfrac{3\pi}{2}\right)$ $\quad \sin(A+B)$



$= \sin\theta\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}\cos\theta$

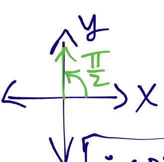
$= \sin\theta(0) + (-1)\cos\theta$

$\boxed{\therefore \sin\left(\theta+\frac{3\pi}{2}\right) = -\cos\theta}$

b) $\cos(\pi+\theta)$

$= -\cos\theta$

c) $\cos\left(\dfrac{\pi}{2} - \theta\right)$ $\quad \cos(A-B)$



$= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$

$= (0)\cos\theta + (1)\sin\theta$

$\boxed{\therefore \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta}$

d) $\sin(\theta - 2\pi)$

$= \sin\theta$

2. Use an appropriate compound angle identity to determine an exact value for $\tan 15^\circ$.

$\tan 15^\circ$

$= \tan(45^\circ - 30^\circ)$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1)\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

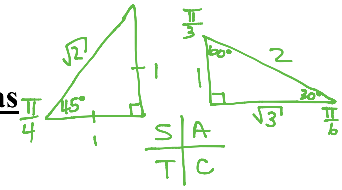
$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$\boxed{\therefore \tan 15^\circ = 2 - \sqrt{3}}$

Date: Oct 27/14 **Using the Addition, Subtraction and Double-Angle Formulas**



Ex. 1. Find the **exact** value of each of the following.

a) $\sin 15^\circ$

$= \sin(60^\circ - 45^\circ)$

$= \sin 60^\circ \cos 45^\circ - \sin 45^\circ \cos 60^\circ$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

b) $\cos \frac{7\pi}{12}$

$= \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$

$= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

c) $\tan \frac{19\pi}{12}$

$= \tan\left(\frac{16\pi}{12} + \frac{3\pi}{12}\right)$

$= \tan\left(\underset{}{\overset{\text{Q III}}{\frac{4\pi}{3}}} + \frac{\pi}{4}\right)$

$= \frac{\tan\frac{4\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{4\pi}{3}\tan\frac{\pi}{4}}$

$= \frac{\sqrt{3}+1}{1-(\sqrt{3})(1)}$

$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{1+2\sqrt{3}+3}{1-3}$

$= \frac{4+2\sqrt{3}}{-2}$

$= -2-\sqrt{3}$

Ex. 2. Express $\tan\left(\frac{\pi}{4} - x\right)$ in terms of $\tan x$.

$\tan\left(\frac{\pi}{4} - x\right)$

$= \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$

$= \frac{1-\tan x}{1+\tan x}$

Ex. 3. If $\sin\theta = \frac{5}{8}$ and $\sin\beta = \frac{4}{5}$ where θ and β are both acute angles, find an exact value for

$\cos(\theta+\beta) - \cos(\theta-\beta)$.

$= (\cos\theta\cos\beta - \sin\theta\sin\beta) - (\cos\theta\cos\beta + \sin\theta\sin\beta)$

$= \cos\theta\cos\beta - \sin\theta\sin\beta - \cos\theta\cos\beta - \sin\theta\sin\beta$

$= -2\sin\theta\sin\beta$

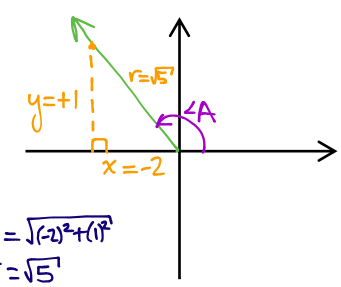
$= -2\left(\frac{\cancel{5}^{1}}{\cancel{8}_{2}}\right)\left(\frac{\cancel{4}^{1}}{\cancel{5}_{1}}\right)$

$= -1$

Ex. 4. If $\tan A = -\frac{1}{2}$, and A lies in the interval $\frac{\pi}{2} \le A \le \pi$ (QII), determine the following:

a) exact values for **i)** $\sin 2A$ **ii)** $\cos 2A$ **iii)** $\cos 4A$

b) the quadrant in which angle [redacted] $2A$ lies



$r = \sqrt{(-2)^2 + (1)^2}$

$r = \sqrt{5}$

$\sin A = \frac{1}{\sqrt{5}}$

$\cos A = -\frac{2}{\sqrt{5}}$

$\tan A = -\frac{1}{2}$

a) i) $\sin 2A$

$= 2\sin A\cos A$

$= 2\left(\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right)$

$\boxed{\therefore \sin 2A = -\frac{4}{5}}$

ii) $\cos 2A$

$= \cos^2 A - \sin^2 A$

$= \left(\frac{-2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2$

$= \frac{4}{5} - \frac{1}{5}$

$\boxed{\therefore \cos 2A = \frac{3}{5}}$

iii) $\cos 4A$

$= \cos[2(2A)]$

$= \cos^2(2A) - \sin^2(2A)$

$= \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2$

$= \frac{9}{25} - \frac{16}{25}$

$= -\frac{7}{25}$

b) $\frac{\pi}{2} \le A \le \pi$

x2) $\pi \le 2A \le 2\pi$ (QIII or QIV)

$\because \cos 2A = +\frac{3}{5}$ & $\sin 2A = -\frac{4}{5}$

$\therefore$ Angle $2A$ lies in QIV.

HW. Worksheet on Using the Addition, Subtraction and Double-Angle Formulas #1-4, 6, 7, 9, 10

Date: Oct 28/14 **The Addition, Subtraction and Double-Angle Formulas Continued**

Ex. 1. Express as a single trigonometric function, and then evaluate.

a) $\cos 45^\circ \cos 15^\circ - \sin 45^\circ \sin 15^\circ$

$= \cos(45^\circ + 15^\circ)$

$= \cos 60^\circ$

$= \frac{1}{2}$

b) $\dfrac{\tan\frac{\pi}{12} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{12}\tan\frac{\pi}{6}}$

$= \tan\left(\frac{\pi}{12} + \frac{\pi}{6}\right)$

$= \tan\left(\frac{3\pi}{12}\right)$

$= \tan\left(\frac{\pi}{4}\right)$

$= 1$

c) $1 - 2\sin^2\left(\frac{5\pi}{8}\right)$ A

$= \cos\left[\frac{2}{1}\left(\frac{5\pi}{8}\right)\right]$

$= \cos\left(\frac{5\pi}{4}\right)$ raa $= \frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}}$

Ex. 2. Express as a single sine or cosine function.

a) $40\sin x\cos x$

$= 20(\underline{2\sin x\cos x})$

$= 20\sin 2x$

b) $\cos^2 3x - \sin^2 3x$

$= \cos[2(3x)]$

$= \cos 6x$

c) $8\sin\frac{x}{2}\cos\frac{x}{2}$

$= 4\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)$

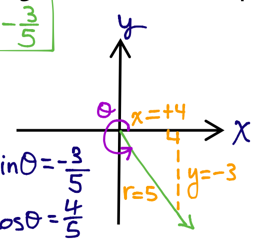
$= 4\sin\left[2\left(\frac{x}{2}\right)\right]$

$= 4\sin x$

QIV QII

Ex. 3. If θ is in the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, α is in the interval $\left[-\frac{3\pi}{2}, -\pi\right]$, $\boxed{\csc\theta = -\frac{5}{3}}$, and $\tan\alpha = \frac{-3}{4}$, determine each value. Include two detailed diagrams.

$\boxed{\sin\theta = -\frac{3}{5}}$



$\sin\theta = -\frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = -\frac{3}{4}$

a) $\sin(\theta - \alpha)$

$= \sin\theta\cos\alpha - \sin\alpha\cos\theta$

$= \left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$

$= \frac{12}{25} - \frac{12}{25}$

$= 0$

c) $\tan 2\theta$

$= \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$= \dfrac{2\left(-\frac{3}{4}\right)}{1 - \left(-\frac{3}{4}\right)^2}$

$= \dfrac{-\frac{3}{2}}{1 - \frac{9}{16}}$

$= -\frac{3}{2} \div \frac{7}{16}$

$= -\frac{3}{\cancel{2}_1} \times \frac{\cancel{16}\,8}{7}$

$= -\frac{24}{7}$

b) $\cot 2\theta$

$= \dfrac{1}{\tan 2\theta}$

$= \dfrac{1}{\left(-\frac{24}{7}\right)}$

$= -\frac{7}{24}$

d) $\cos\frac{\theta}{2}$ **(***)**

$\cos\theta = \cos 2\left(\frac{\theta}{2}\right)$

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

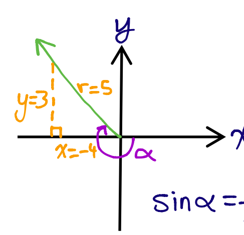
$\frac{4}{5} = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$\frac{1}{2} \times \frac{9}{5} = \frac{1}{2} \times 2\cos^2\left(\frac{\theta}{2}\right)$

$\frac{9}{10} = \cos^2\left(\frac{\theta}{2}\right)$

$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{9}{10}}$

$\cos\left(\frac{\theta}{2}\right) = \pm\frac{3}{\sqrt{10}}$



$\sin\alpha = \frac{3}{5}$

$\cos\alpha = -\frac{4}{5}$

$\tan\alpha = -\frac{3}{4}$

Note: $\frac{3\pi}{2} \le \theta \le \pi$

$\frac{1}{2}$) $\frac{3\pi}{4} \le \frac{\theta}{2} \le \frac{\pi}{2}$

$\therefore \frac{\theta}{2}$ is in QII

so $\cos\left(\frac{\theta}{2}\right) = -\frac{3}{\sqrt{10}}$

HW. Compound Angle Formulas Worksheet #1 to 6 all