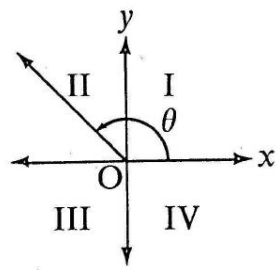
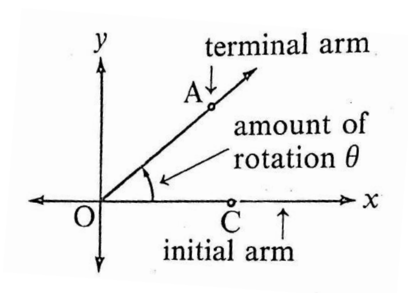
Date:______________ **UNIT 3: TRIGONOMETRIC FUNCTIONS & EQUATIONS**

3.1 The Definitions of Trigonometry

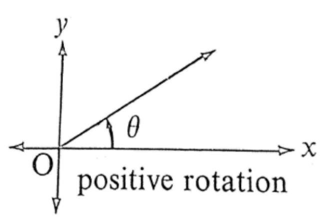
The Cartesian plane is fundamental in the study of trigonometry.
Angles are related to the co-ordinate axes and associated with rotation.
The xy plane is divide into four quadrants numbered, for convenience as shown.



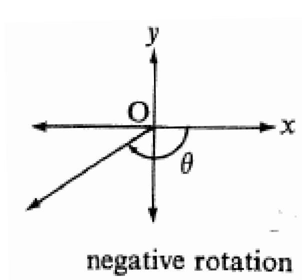
An angle is in ***standard position*** *if its vertex is at the origin in the xy plane, its initial arm is on the positive x-axis, and its terminal arm is a rotation of the initial arm about the origin.*



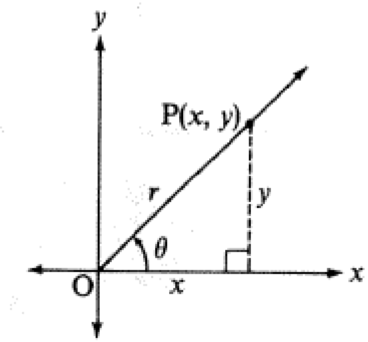
If the rotation is counterclockwise, the angle has ***positive*** *measure.*
The rotation is usually indicated by a directed arrow starting from the positive x-axis.



If the rotation is clockwise, the angle has ***negative*** *measure.*



If $P(x, y)$ is a point on the terminal arm of an angle θ, a circle with centre the origin can be drawn through $P(x, y)$ with the following radius:



The following definitions the form the basis of trigonometry.

Primary Trigonometric Values/Ratios

$\text{sine}\,\theta = \frac{y}{r}$ $\quad$ $\text{cosine}\,\theta = \frac{x}{r}$ $\quad$ $\text{tangent}\;\theta = \frac{y}{x}$

$\sin\theta = \frac{y}{r}$ $\quad$ $\cos\theta = \frac{x}{r}$ $\quad$ $\tan\theta = \frac{y}{x}$

By writing the reciprocals of the above, other trigonometric values are defined.

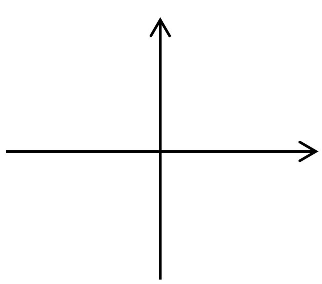
Reciprocal Trigonometric Values/Ratios

$\text{cosecant}\,\theta = \frac{r}{y}$ $\quad$ $\text{secant}\,\theta = \frac{r}{x}$ $\quad$ $\text{cotangent}\;\theta = \frac{x}{y}$

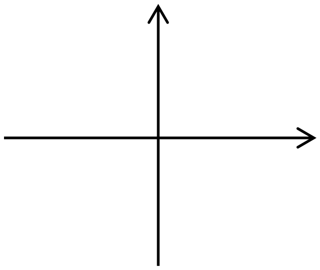
$\csc\theta = \frac{r}{y}$ $\quad$ $\sec\theta = \frac{r}{x}$ $\quad$ $\cot\theta = \frac{x}{y}$

To calculate the trigonometric values you only need to find a point on the terminal arm.

Ex. 1. The point $(3, 4)$ is on the terminal arm of angle θ in standard position.
Calculate the **primary** trigonometric ratios. Include a diagram.



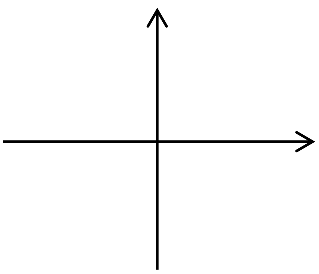
Ex. 2. $P(4,-4)$ is a point on the terminal arm of angle α in standard position.
Calculate the **reciprocal** trigonometric values. Include a diagram.

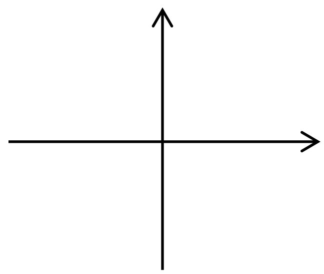


Ex. 3. Draw a sketch of each angle in standard position. Calculate the **primary** trigonometric ratios for **a)** and the **reciprocal** trigonometric ratios for **b).**

a) $\cos\beta = -\frac{15}{17}$**,** β is in quadrant II

b) $\tan\theta = \frac{1}{\sqrt{3}}$**,** θ is in quadrant III





Ex. 4. For any angle θ, show that:

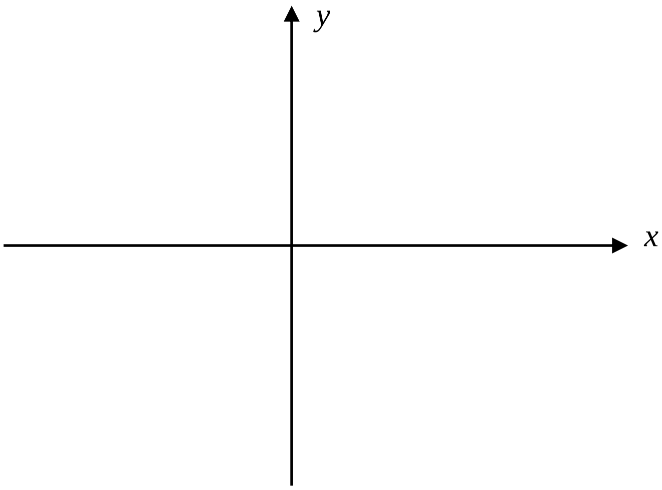
a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

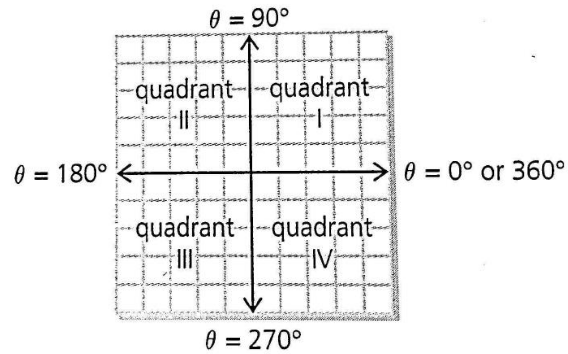
b) $\csc^2\theta = 1 + \cot^2\theta$

HW. Exercise 3.1

Date:______________ **3.2 Angles and Quadrants**

Recall: The **CAST** rule





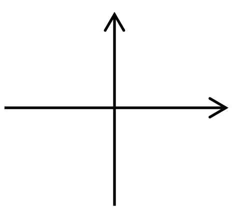
Definitions:

Coterminal angles *share the same terminal arm and the same initial arm.*

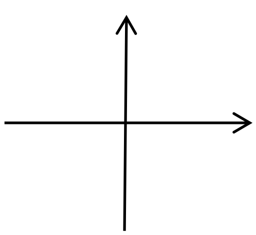
The ***principal angle*** *is the angle between* 0° *and* 360°.

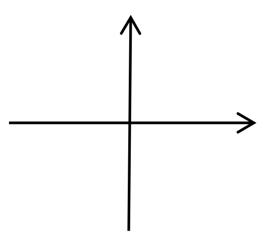
The ***related acute angle*** *is the angle formed by the terminal arm of an angle in standard position and the x-axis. The related acute angle is always positive and lies between* 0° *and* 90°.

Ex. 1. Determine the principal angle and the related acute angle for $\theta = -225^\circ$.



Ex. 2. Determine the next two positive and negative coterminal angles for 43°.





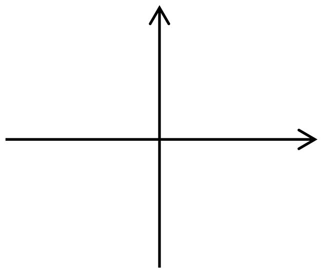
Ex. 3. In which possible quadrant(s) does the terminal arm of θ lie if:

a) $\sin\theta$ is positive? **b)** $\cos\theta$ is negative? **c)** $\tan\theta$ is positive?

d) $\csc\theta$ is negative? **e)** $\cot\theta$ is negative? **f)** $\sec\theta$ is positive?

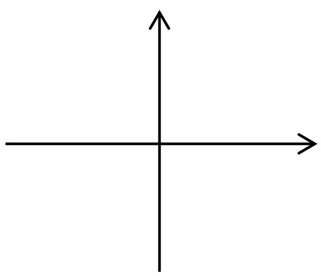
Ex. 4. Point $P(-3, 4)$ is on the terminal arm of an angle in standard position.

- **a)** Sketch the principal angle, θ.
- **b)** Determine the value of the related acute angle to the nearest degree.
- **c)** What is the measure of θ to the nearest degree?



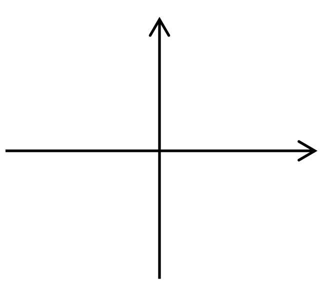
Ex. 5. A positive angle θ is in the third quadrant and $\sec\theta = -\frac{17}{8}$.

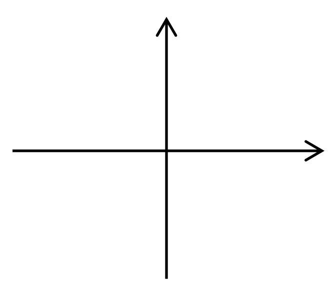
- **a)** Sketch the principal angle, θ.
- **b)** Determine the value of the related acute angle to the nearest degree.
- **c)** What is the measure of θ to the nearest degree?
- **d)** Find exact values for $\tan\theta$ and $\csc\theta$.



Ex. 6. Given that $\csc\alpha = -\frac{5}{3}$ and $0^\circ \leq \alpha \leq 360^\circ$,

- **a)** Find α. (Include diagrams)
- **b)** Determine exact values for $\cos\alpha$ and $\cot\alpha$.





HW. Exercise 3.2

Date:_______________

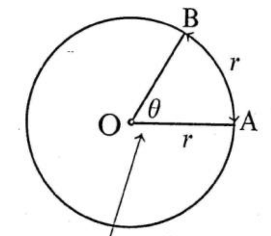
3.3 Radian Measure

Recall:

*A **radian** is the measure of the angle subtended at the center of the circle by an arc equal in length to the radius of the circle.*

&

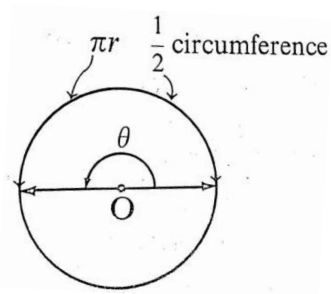
$$\theta_{in\,radians} = \frac{arc\,length}{radius} \quad or \quad \theta = \frac{a}{r} \quad or \quad a = r\theta$$



The measure of θ is defined to be 1 radian.

*Angles can be measured in **degrees** or **radians**.*

Ex. 1. Determine the relationship between degrees and radians.



In radians,

$\theta =$

In degrees,

$\theta =$

$$\boxed{\pi\ rad = 180^\circ}$$

Ex. 2. Change each radian measure to degree measure. Round to the nearest degree, if necessary.

Hint: $\pi\ rad = 180^\circ$ *or* $1\ rad = \frac{180^\circ}{\pi}$ *: To change radian measure to degree measure, multiply the number of radians by* $\frac{180^\circ}{\pi}$.

a) $\frac{\pi}{6}$

b) $\frac{5\pi}{4}$

c) $-\frac{3\pi}{2}$

d) 2.2

Ex. 3. Find the **exact** radian measure, in terms of π, for each of the following.

Hint: $180^\circ = \pi\, rad$ *or* $1^\circ = \frac{\pi}{180} rad$: *To change degree measure to radian measure, multiply the number of degrees by* $\frac{\pi}{180} rad$.

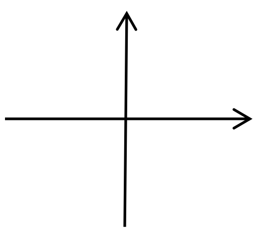
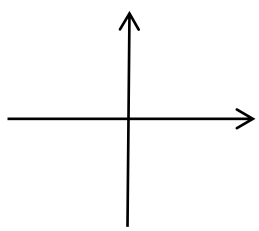
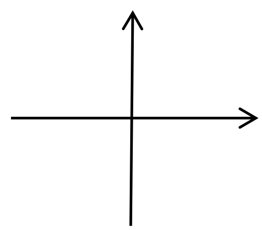
a) 45° **b)** 60° **c)** -210° **d)** -720°

Ex. 4. Change each degree measure to radian measure, to 4 decimal places.

a) 30° **b)** -230°

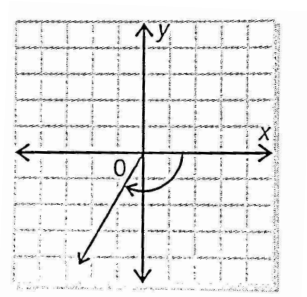
Ex. 5. Sketch each angle in standard position.

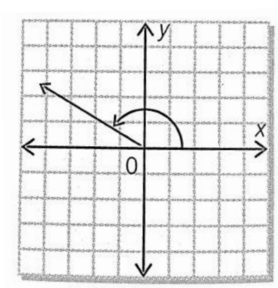
a) $\frac{3}{2}\pi$ **b)** $\frac{-3\pi}{4}$ **c)** $-\frac{5}{3}\pi$

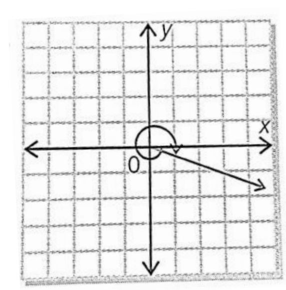




Ex. 6. Write the value of θ in exact radian measure with the given related acute angle for each of the following:

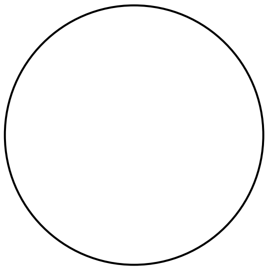
a) $r.a.a = \frac{\pi}{3}$ **b)** $r.a.a = 45^\circ$ **c)** $r.a.a = \frac{\pi}{6}$



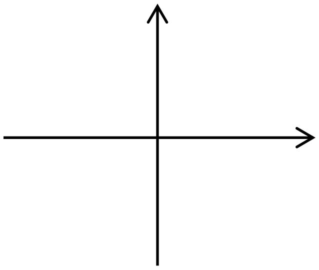




Ex. 7. Sector angles are drawn in a ***unit*** circle. Find the measure of the arc of the circle that subtends an angle measuring 60°. *Label the given diagram.*

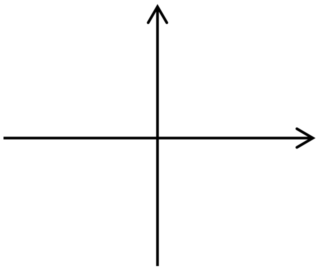


Ex. 8. $P(-1,5)$ is a point on the terminal arm of angle θ in standard position. Calculate the measure of the *principal angle* in radians to 1 decimal place. *Include a diagram.*

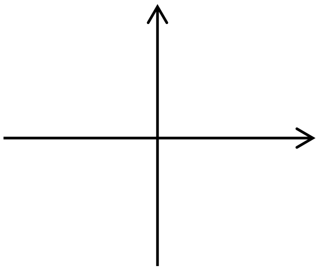


Ex. 9. The radian measures of angles are shown. Write the measure of the *coterminal angle* θ for $-2\pi \le \theta \le 2\pi$. *Include a diagram.*

a) $\frac{\pi}{2}$



b) $-\frac{2\pi}{3}$



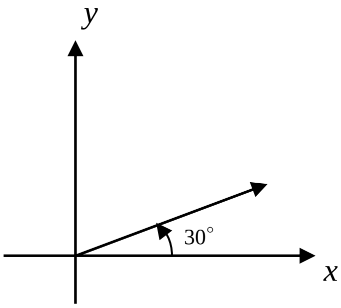
HW. Exercise 3.3

Date:__________________

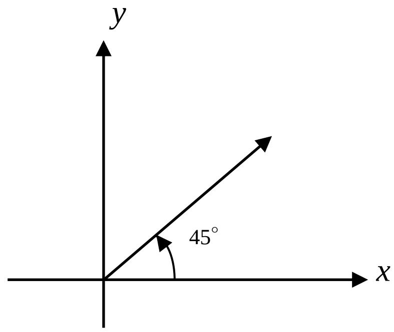
3.4 Exact Trigonometric Ratios For Special Angles

Special Angles *of* 30°, 45°, *and* 60° *or* $\frac{\pi}{6}$, $\frac{\pi}{4}$, *and* $\frac{\pi}{3}$ *radians*

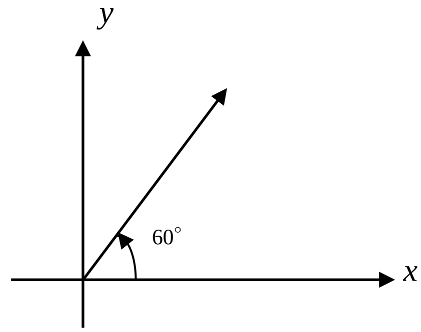
i)



ii)



iii)



i) $\sin\frac{\pi}{6} =$

$\cos\frac{\pi}{6} =$

$\tan\frac{\pi}{6} =$

ii) $\sin\frac{\pi}{4} =$

$\cos\frac{\pi}{4} =$

$\tan\frac{\pi}{4} =$

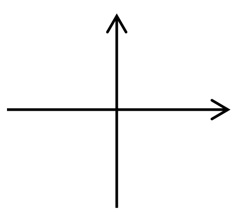
iii) $\sin\frac{\pi}{3} =$

$\cos\frac{\pi}{3} =$

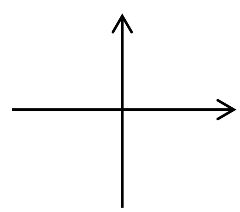
$\tan\frac{\pi}{3} =$

Ex. 1. Find the exact value of each trigonometric ratio.

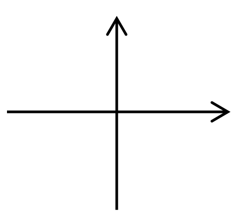
a) $\cos\frac{7}{6}\pi$



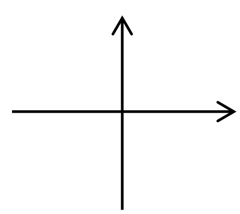
b) $\csc\frac{5}{3}\pi$



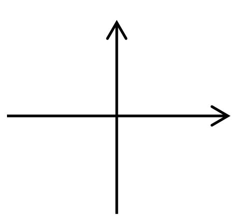
c) $\tan\pi$



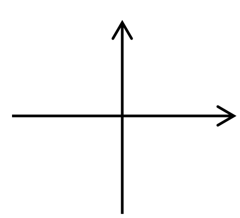
d) $\sin\left(-\frac{3}{2}\pi\right)$



e) $\sec\left(-\frac{3}{4}\pi\right)$

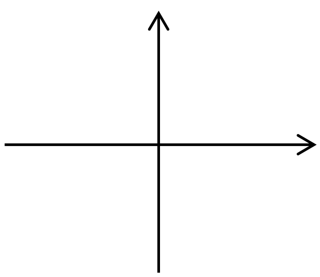


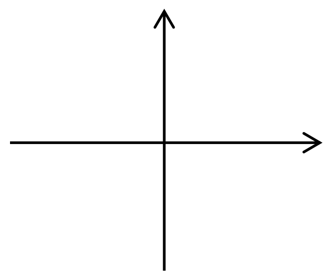
f) $\cot(-2\pi)$



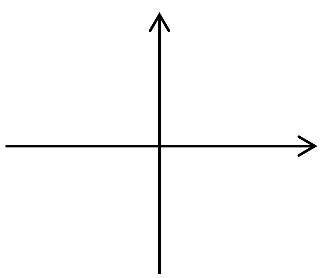
Ex. 2. Calculate the exact value of $\sin(-135^\circ)\cdot\cos 315^\circ + \cos 390^\circ \div \cot(-300^\circ)$.

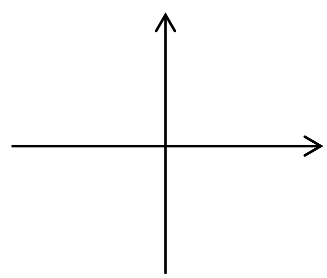
Ex. 3. If $0^\circ \le \theta \le 360^\circ$, find possible values of θ for $\sec\theta = \sqrt{2}$. Include diagrams.





Ex. 4. If $0 \le \theta \le 2\pi$, find possible values of θ for $\cot\theta = -\frac{1}{\sqrt{3}}$. Include diagrams.





HW. Exercise 3.4

Date:________________ **3.5 Addition, Subtraction and Double-Angle Formulas**

Warmup

1. Given θ is an acute angle, express each quantity in terms of either $\sin\theta$, $\cos\theta$ or $\tan\theta$.

a) $\sin(-\theta)$ **b)** $\cos(-\theta)$ **c)** $\tan(-\theta)$

2. Given the identity $\sin^2\theta + \cos^2\theta = 1$, express:

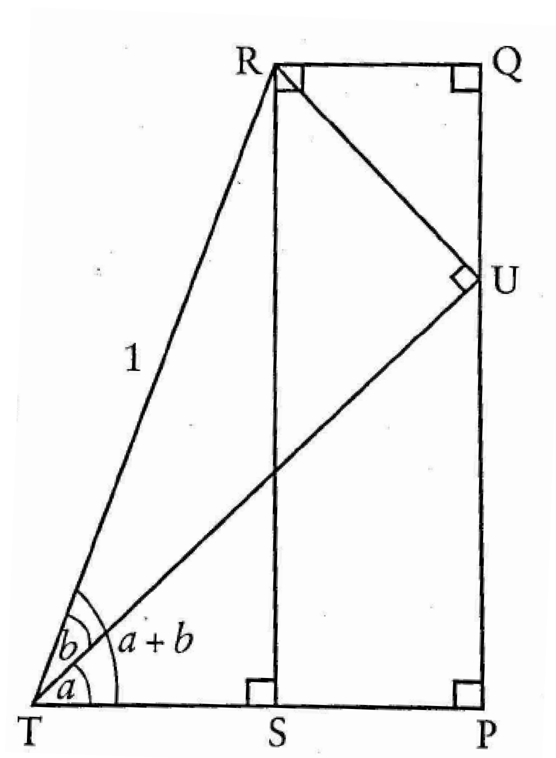
a) $\sin^2\theta$ in terms of $\cos^2\theta$ **b)** $\cos^2\theta$ in terms of $\sin^2\theta$

Example 1.

Develop the addition formula for sine, ie. $\sin(a+b)$, in terms of angles a and b using the given diagram.

In ΔTUP, $\angle TUP =$ and in ΔRUQ, $\angle RUQ =$

In ΔTRS, $\sin(a+b) =$



In ΔTRU,

$\cos b =$ and $\sin b =$

In ΔTUP, and in ΔRUQ,

$\sin a =$ $\cos a =$

$\because \sin(a+b) = UP + QU$

$\therefore \sin(a+b) =$

Note:

By following a similar strategy and using the fact that $TS = TP - SP$, the following addition formula for cosine can be determined.

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
&
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Ex. 2. Using the addition formulas for sine and cosine, develop the subtraction formulas, $\sin(A-B)$ and $\cos(A-B)$.

Ex. 3. Using the addition formulas for sine and cosine, develop the double-angle formulas, $\sin 2A$ and $\cos 2A$.

Ex. 4. Using the addition formulas for sine and cosine, develop the addition and subtraction formulas for tangent, $\tan(A+B)$ and $\tan(A-B)$. Also develop the double-angle formula, $\tan 2A$.

Key Concepts for Compound Angle Formulas

- *Addition, subtraction and double-angle identities for sine, cosine, and tangent*

$\sin(A+B) =$
$\sin(A-B) =$
$\sin 2A \quad =$

$\cos(A+B) =$
$\cos(A-B) =$
$\cos 2A \quad =$
$\quad =$
$\quad =$

$\tan(A+B) =$

$\tan(A-B) =$

$\tan 2A \quad =$

HW. ***Memorize these identities.***

1. Given θ is an acute angle, express each quantity in terms of either $\sin\theta$ or $\cos\theta$ using an appropriate compound angle identity.

a) $\sin\left(\theta + \frac{3\pi}{2}\right)$

b) $\cos(\pi + \theta)$

c) $\cos\left(\frac{\pi}{2} - \theta\right)$

d) $\sin(\theta - 2\pi)$

2. Use an appropriate compound angle identity to determine an exact value for $\tan 15^\circ$.

Date:______________ **3.6 Using the Addition, Subtraction and Double-Angle Formulas**

Ex. 1. Find the **exact** value of each of the following.

a) $\sin 15^{\circ}$

c) $\tan \frac{19\pi}{12}$

b) $\cos \frac{7\pi}{12}$

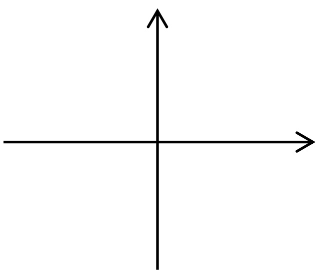
Ex. 2. Express $\tan\left(\frac{\pi}{4} - x\right)$ in terms of tan *x*.

Ex. 3. If $\sin\theta = \frac{5}{8}$ and $\sin\beta = \frac{4}{5}$ where θ and β are both acute angles, find an exact value for $\cos(\theta+\beta) - \cos(\theta-\beta)$.

Ex. 4. If $\tan A = -\frac{1}{2}$, and A lies in the interval $\frac{\pi}{2} \le A \le \pi$, determine the following:

a) exact values for **i)** $\sin 2A$ **ii)** $\cos 2A$ **iii)** $\cos 4A$

b) the quadrant in which angle angle $2A$ lies



HW. Exercise 3.6

Date:______________ **3.7 Addition, Subtraction and Double-Angle Formulas Continued**

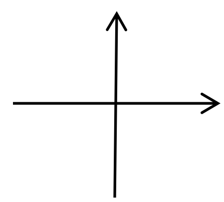
Ex. 1. Express as a single trigonometric function, and then evaluate.

a) $\cos 45^\circ \cos 15^\circ - \sin 45^\circ \sin 15^\circ$ **b)** $\dfrac{\tan\frac{\pi}{12} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{12}\tan\frac{\pi}{6}}$ **c)** $1 - 2\sin^2 \dfrac{5\pi}{8}$

Ex. 2. Express as a single sine or cosine function.

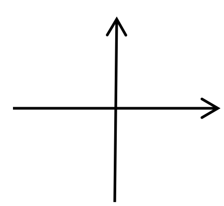
a) $40\sin x \cos x$ **b)** $\cos^2 3x - \sin^2 3x$ **c)** $8\sin\dfrac{x}{2}\cos\dfrac{x}{2}$

Ex. 3. If θ is in the interval $\left[\dfrac{3\pi}{2}, 2\pi\right]$, α is in the interval $\left[-\dfrac{3\pi}{2}, -\pi\right]$, $\csc\theta = -\dfrac{5}{3}$, and $\tan\alpha = \dfrac{-3}{4}$, determine each value. Include two detailed diagrams.



a) $\sin(\theta - \alpha)$ **c)** $\cot 2\theta$

b) $\tan 2\theta$ **d)** $\cos\dfrac{\theta}{2}$ **(***)**



HW. Exercise 3.7

Date:_______________

3.8 Trigonometric Identities

In terms of $\sin\theta$ and $\cos\theta$:

$\tan\theta =$ $\cot\theta =$ $\csc\theta =$ $\sec\theta =$

$\sin^2\theta + \cos^2\theta =$ $\sin^2\theta =$ $\cos^2\theta =$

Ex. 1. Prove the following identities.

a) $\dfrac{1+\cot\theta}{\csc\theta} = \sin\theta + \cos\theta$

b) $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$

c) $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

d) $\dfrac{\sin t - \cos t}{\cos t} + \dfrac{\sin t + \cos t}{\sin t} = \sec t \csc t$

e) $\dfrac{1}{1 + \sin \phi} = \sec^2 \phi - \dfrac{\tan \phi}{\cos \phi}$

HW. Exercise 3.8

Date:_____________ **3.9 Trigonometric Identities Involving Compound Angle Formulas**

Ex. 1. Prove the following trigonometric identities.

a) $\frac{\sin 2x}{1+\cos 2x}=\tan x$

b) $\tan x+\tan y=\frac{\sin(x+y)}{\cos x\cos y}$

c) $\sin 3\theta + \sin\theta = 2\sin 2\theta \cos\theta$

d) $\tan\frac{\theta}{2} + \cot\frac{\theta}{2} = 2\csc\theta$

HW. Exercise 3.9
For Unit 3 Test: do Unit 3 Review Exercise