Date:______________ **UNIT 6: LOGARITHMIC FUNCTIONS & LOGARITHMS**

6.1 The Logarithmic Function

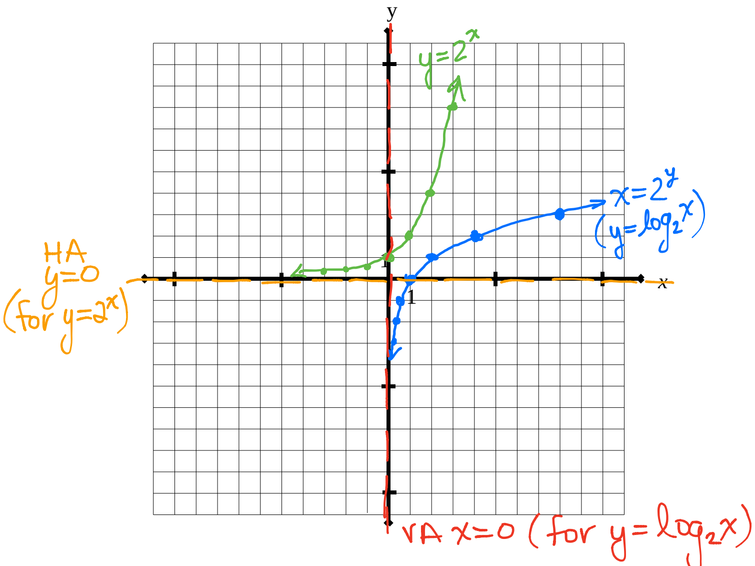
The *logarithmic function* $y = \log_b x$ is the *inverse* of the *exponential function* $y = b^x$.

Read $y = \log_b x$ *as "y is the logarithm of x to the base b".*

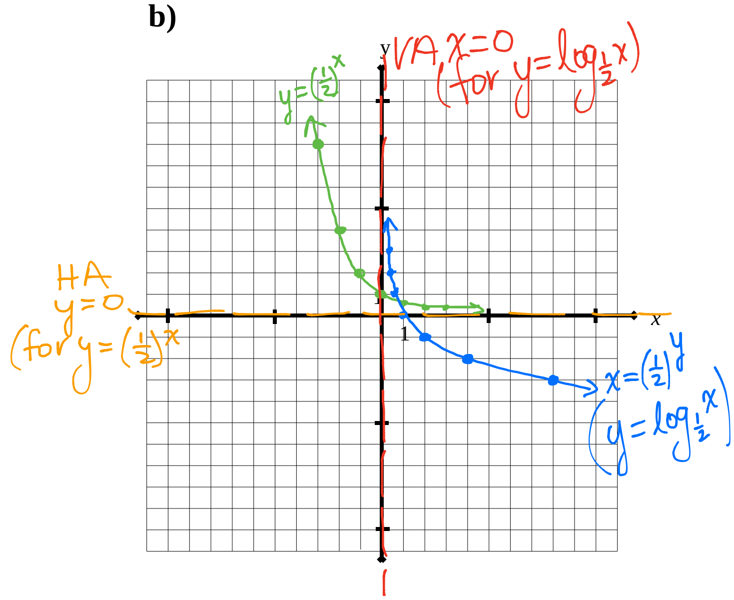
1. Graph **a)** $y = 2^x$ and its inverse & **b)** $y = \left(\frac{1}{2}\right)^x$ and its inverse on the grids below.

$x \leftrightarrow y$

a)



b)



Exponential Form		**Logarithmic Form**	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number *x* with a given base is **the exponent** to which that base must be raised to yield *x*.

John Napier (1550-1617)
Scottish mathematician John Napier invented logarithms. The word *logarithm* is composed of the two words *logic* and *arithmetic*. Napier did not develop the natural logarithmic function, but it is occasionally called the Napierian logarithm, in his honour.

Properties of the Logarithmic Function $y = \log_b x$

- The base *b* is positive.
- The *x*-intercept is 1.
- The *y*-axis is a vertical asymptote.
- The domain is the set of positive real numbers.
- The range is the set of all real numbers.
- The function is increasing if $b > 1$.
- The function is decreasing if $0 < b < 1$.

If $\log_b x = \log_b y$ then $x = y$, since bases are equal.

Exponential Form		Logarithmic Form	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is **the exponent** to which that base must be raised to yield x.

"lawn x"

Note: $\log x = \log_{10} x$ **and** $\ln x = \log_e x$

Ex. 1. Change to exponential form.

a) $\log_2 16 = 4$

$16 = 2^4$

b) $\log_9 3 = \frac{1}{2}$

$9^{\frac{1}{2}} = 3$

c) $\log\left(\frac{1}{100}\right) = -2$

$\log_{10}\left(\frac{1}{100}\right) = -2$

$\frac{1}{100} = 10^{-2}$

Ex. 2. Change to logarithmic form.

a) $3^2 = 9$

$\log_3 9 = 2$

b) $e^0 = 1$

$0 = \log_e 1$

$(0 = \ln 1)$

c) $49^{\frac{1}{2}} = 7$

$\frac{1}{2} = \log_{49} 7$

Ex. 3. Use your calculator to find the value of the following:

a) $\log_{10} 1000$

$= 3$ $(10^3 = 1000)$

b) $\log 500$

$\doteq 2.699$ $(10^{2.699} \doteq 500)$

c) $\ln 0.5 \rightarrow \log_e 0.5$

$\doteq -0.693$ $(e^{-0.693} \doteq 0.5)$

Ex. 4. Simplify

$(a^0 = 1)$ **a)** $\log_a 1 = 0$ **ie. i)** $\log_2 1 = 0$ **ii)** $\log_{0.5} 1 = 0$

$(a^x = a^x)$ **b)** $\log_a a^x = x$ **ie. i)** $\log_3 3^4 = 4$ **ii)** $\log_5 5^{\sqrt{2}} = \sqrt{2}$

c) $a^{\log_a x} = x$ **ie. i)** $10^{\log 0.5} = 0.5$ **ii)** $4^{\log_4 5} = 5$

Ex. 5. Evaluate each of the following.

a) $\log_5 25$

$= 2$

b) $\log_3 27$

$= 3$

c) $\log_2\left(\frac{1}{4}\right)$

$= -2$

d) $\log_{\frac{1}{3}} \sqrt[4]{27}$

$= -\frac{3}{4}$

$*$ Let $x = \log_{\frac{1}{3}} \sqrt[4]{27}$

$\left(\frac{1}{3}\right)^x = \sqrt[4]{27}$

$\left(\frac{1}{3}\right)^x = (3^3)^{\frac{1}{4}}$

$(3^{-1})^x = 3^{\frac{3}{4}}$

$3^{-x} = 3^{\frac{3}{4}}$

Since bases are equal

$-x = \frac{3}{4}$

$x = -\frac{3}{4}$

Exponential Form		Logarithmic Form	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is **the exponent** to which that base must be raised to yield x.

Ex. 6. Solve for x.

a) $\log_2 x = 8$

$x = 2^8$

$x = 256$

b) $\log_9 3\sqrt{3} = x$

$9^x = 3\sqrt{3}$

$3^{2x} = 3^1 \cdot 3^{\frac{1}{2}}$

$3^{2x} = 3^{\frac{3}{2}}$

$\because$ bases are equal

$\therefore 2x = \frac{3}{2}$

$x = \frac{3}{4}$

c) $\log_2 \sqrt[3]{2} = x$

$2^x = \sqrt[3]{2}$

$2^x = 2^{\frac{1}{3}}$

$\because$ bases are equal

$\therefore x = \frac{1}{3}$

d) $\log_x 4 = \frac{1}{2}$

$4 = x^{\frac{1}{2}}$

$(4)^{\frac{2}{1}} = (x^{\frac{1}{2}})^{\frac{2}{1}}$

$16 = x$

$\therefore x = 16$

e) $\log_x 81 = \frac{4}{5}$

$(x^{\frac{4}{5}})^{\frac{5}{4}} = (81)^{\frac{5}{4}}$

$x = (\sqrt[4]{81})^5$

$\therefore x = 243$

f) $e^{\ln 10} = x$

$e^{\log_e 10} = x$

$10 = x$

$\therefore x = 10$

or $e^{\ln 10} = x$:

rewrite in logarithmic form :

$\log_e x = \ln 10$

$\log_e x = \log_e 10$

$\because$ log bases are equal

$\therefore x = 10$

HW. Exercise 6.1

Date:_____________

6.2 Logarithmic Properties, Laws & Transformations

Recall:

Exponential Form		**Logarithmic Form**	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number *x* with a given base is **the exponent** to which that base must be raised to yield *x*.

Properties of Logarithms: **i)** $\log_a 1 = 0$ **ii)** $\log_a a = 1$

iii) $\log_a a^x = x$ ***iv)** $a^{\log_a x} = x$

Exponent Laws:

1. $a^m \cdot a^n = a^{m+n}$ ***multiplication***
2. $a^m \div a^n = a^{m-n}$ ***division***
3. $(a^m)^n = a^{m \cdot n}$ ***power***

Logarithm Laws:

$a, x, y > 0$

*** 1.** $\log_a(x \cdot y) = \log_a x + \log_a y$

***2.** $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

***3.** $\log_a x^n = n \cdot \log_a x$

Note: *We will prove logarithm properties and laws marked with a* *****.

Ex. 1. Use the properties of logarithms to write each of the following as a sum and/or a difference and/or multiple of logarithms.

a) $\log_7(5x)$

$= \log_7 5 + \log_7 x$

b) $\log_3\left(\frac{ab}{cd}\right)$

$= \log_3 ab - \log_3 cd$

$= (\log_3 a + \log_3 b) - (\log_3 c + \log_3 d)$

$= \log_3 a + \log_3 b - \log_3 c - \log_3 d$

c) $\log_8 \sqrt[4]{m}$

$= \log_8 m^{\frac{1}{4}}$

$= \frac{1}{4} \log_8 m$

Ex. 2. Express as a single logarithm and evaluate.

a) $\log_4 192 - \log_4 3$

$= \log_4\left(\frac{192}{3}\right)$

$= \log_4 64$

$= 3$

b) $\log_8 6 - \log_8 3 + \log_8 4$

$= \log_8\left(\frac{6}{3} \cdot 4\right)$

$= \log_8 8$

$= 1$

c) $2\log_4 8$

$= \log_4 8^2$

$= \log_4 64$

$= 3$

Ex. 3. Evaluate exactly.

a) $\log \sqrt[4]{1000}$

$= \log_{10} \sqrt[4]{1000}$

$= \log_{10} 1000^{\frac{1}{4}}$

$= \frac{1}{4} \cdot \log_{10} 1000$

$= \frac{1}{4} \cdot 3$

$= \frac{3}{4}$

b) $3^{-\frac{1}{2}\log_3 49}$

$= 3^{\log_3 49^{-\frac{1}{2}}}$

$= 49^{-\frac{1}{2}}$

$= \frac{1}{7}$

Ex. 4. Solve to two decimal places, using logarithms.

a) $3^x = 11$

Take the logarithm of both sides

$\log 3^x = \log 11$

$x \cdot \log 3 = \log 11$

$x = \frac{\log 11}{\log 3}$

$x \doteq 2.18$

b) $4(7^{x-2}) = 8$

$\frac{4(7^{x-2})}{4} = \frac{8}{4}$

$7^{x-2} = 2$

$\log 7^{x-2} = \log 2$

$(x-2)\log 7 = \log 2$

$x - 2 = \frac{\log 2}{\log 7}$

$x = \frac{\log 2}{\log 7} + 2$

$x \doteq 2.36$

Transformations on the Logarithmic Function $f(x)=\log_b x$ **,where** $f(x)=a\log_b[k(x-d)]+c$

Recall: $(x,y)\rightarrow\left(\frac{1}{k}x+d, ay+c\right)$

Ex. 2. Use the properties of logarithms to write each of the following logarithmic functions as a sum and/or a difference and/or multiple of logarithms. Then, graph, by naming and applying transformations on the indicated parent logarithmic function.

a) $y=\log_2(4x^2)$

$y=\log_2 4+\log_2 x^2$

$y=\log_2 4+2\log_2 x$

$y=2\log_2 x+2$

Transformations on $y=\log_2 x$ *are:*

i) V.E. by a factor of 2

ii) V.T. 2 units up.

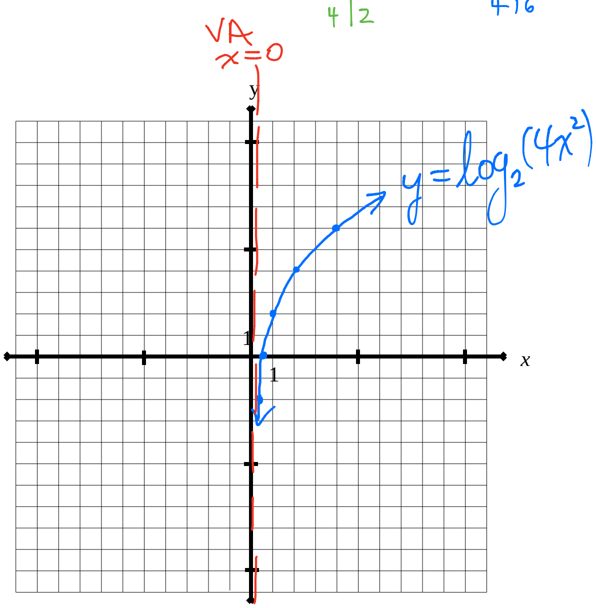
$(x,y)\rightarrow(x, 2y+2)$

$2^y=x$ (inverse of $y=2^x$)

x	y
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

$\rightarrow$

x	y
$\frac{1}{4}$	-2
$\frac{1}{2}$	0
1	2
2	4
4	6



Vertical Asymptote: $x=0$

Domain : $\{x\in\mathbb{R}\mid x>0\}$

Range: $\{y\in\mathbb{R}\}$

***y* – intercept :** none

***x* – intercept :** $\frac{1}{2}$

b) $y=\log_3\left(\frac{x}{3}\right)^3$

$y=3\log_3\left(\frac{x}{3}\right)$

$y=3[\log_3 x-\log_3 3]$

$y=3[\log_3 x-1]$

$y=3\log_3 x-3$

Transformations on $y=\log_3 x$ *are:*

i) V.E. by a factor of 3

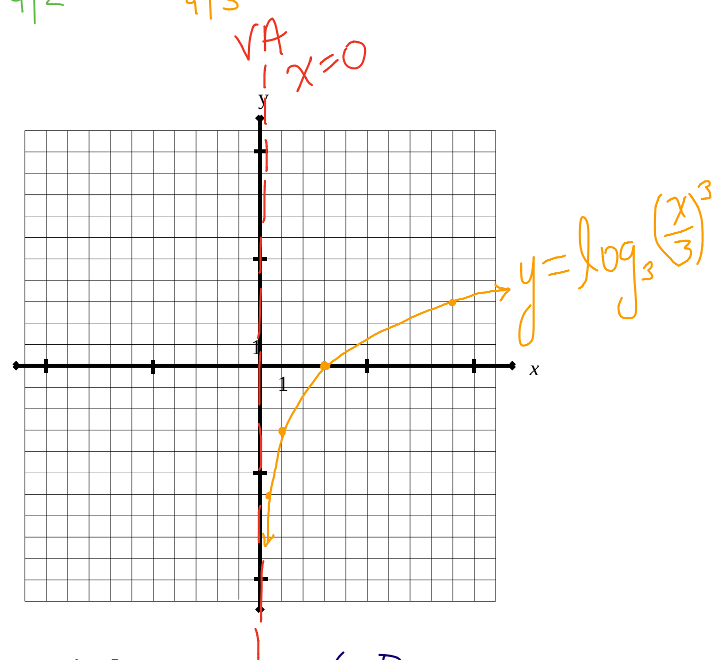
ii) V.T 3 units down

$(x,y)\rightarrow(x, 3y-3)$

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

$\rightarrow$

x	y
$\frac{1}{9}$	-9
$\frac{1}{3}$	-6
1	-3
3	0
9	3



Vertical Asymptote: $x=0$

Domain : $\{x\in\mathbb{R}\mid x>0\}$

Range: $\{y\in\mathbb{R}\}$

***y* – intercept :** none

***x* – intercept :**

Proving Selected Properties and Laws of Logarithms

Recall:

Exponential Form	$\longleftrightarrow$	**Logarithmic Form**	
$x = b^y$		$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is **the exponent** to which that base must be raised to yield x.

1. Prove $a^{\log_a x} = x$.

Let $y = a^{\log_a x}$

Rewrite in logarithmic form:

$\log_a x = \log_a y$

$\because$ log bases are equal

$x = y$

$\because y = a^{\log_a x}$

$\therefore x = a^{\log_a x}$

2. Prove $\log_a x^n = n \log_a x$.

Let $x = a^m$

$L.S. = \log_a x^n$

$= \log_a (a^m)^n$

$= \log_a a^{m \cdot n}$

$= m \cdot n$

$RS = n \log_a x$

$= n \log_a a^m$

$= n \cdot m$

$= m \cdot n$

$\because LS = RS$

$\therefore \log_a x^n = n \log_a x$

3. Prove $\log_a (x \cdot y) = \log_a x + \log_a y$.

Let $x = a^m$, $y = a^n$

$LS = \log_a (x \cdot y)$

$= \log_a (a^m \cdot a^n)$

$= \log_a (a^{m+n})$

$= m+n$

$RS = \log_a x + \log_a y$

$= \log_a a^m + \log_a a^n$

$= m+n$

$\because LS = RS$

$\therefore \log_a (x \cdot y) = \log_a x + \log_a y$

4. Prove $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.

Let $x = a^m$, $y = a^n$

$LS = \log_a \left(\dfrac{a^m}{a^n}\right)$

$= \log_a (a^{m-n})$

$= m-n$

$RS = \log_a a^m - \log_a a^n$

$= m-n$

$\because LS = RS$

$\therefore \log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

HW. Exercise 6.2

Date:_____________ **6.3 Solving Logarithmic Equations**

Recall:

Exponential Form		**Logarithmic Form**	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is **the exponent** to which that base must be raised to yield x.

Properties of Logarithms: **i)** $\log_a 1 = 0$ **ii)** $\log_a a = 1$ **iii)** $\log_a a^x = x$ **iv)** $a^{\log_a x} = x$

For $\log_a f(x)$: $f(x) > 0$ for all x.

Logarithm Laws: **1.** $\log_a(x \cdot y) = \log_a x + \log_a y$ ***multiplication***

$a, x, y > 0$ **2.** $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ ***division***

3. $\log_a x^n = n \log_a x$ ***power***

Ex. 1. Solve and check each of the following.

a) $\log_6 \boxed{x} = 2$

Pre-Check: $x > 0$

$x = 6^2$

$x = 36$ ✓

Post-Check: $36 > 0$

b) $4\log_6 x = \log_6 625$

$\log_6 x^4 = \log_6 625$

$\because$ log bases are equal

$x^4 = 625$

$x = \pm 5$

$\because x > 0 \therefore$ the solution is $x = 5$.

c) $2\log x = \log 8 + \log 2$

$\log x^2 = \log 16$

$\because$ log bases are equal

$\therefore x^2 = 16$

$x = \pm 4$

$\because x > 0$

$\therefore$ the solution is $x = 4$.

Pre-Check:

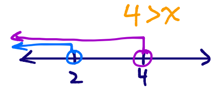
① $2 - x > 0$

$2 > x$

and

② $4 - x > 0$

$4 > x$



$\boxed{\therefore x < 2}$

d) $\log_8(2 - x) + \log_8(4 - x) = 1$

$\log_8[(2-x)(4-x)] = 1$

$\log_8(x^2 - 6x + 8) = 1$

$\log_8(x^2 - 6x + 8) = \log_8 8$

$\because$ log bases are equal

$x^2 - 6x + 8 = 8$

$x^2 - 6x = 0$

$x(x - 6) = 0$

$\therefore x = 0, x = 6$

$\because x < 2 \therefore x = 0$ is the solution.

Post-Check:

If $x = 0$: $2 - x = 2 - 0 = 2 > 0$ ✓

$4 - x = 4 - 0 = 4 > 0$ ✓

If $x = 6$: $2 - x = 2 - 6 = -4 < 0$ ✗

e) $\log_2(x + 1) - \log_2(x - 1) = 1$

$\log_2\left[\frac{(x+1)}{(x-1)}\right] = 1$

$\frac{x+1}{x-1} = \frac{2^1}{1}$

$1(x + 1) = 2(x - 1)$

$x + 1 = 2x - 2$

$3 = x$

$\therefore x = 3$ is the solution

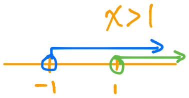
Check:

① $x + 1 > 0$

$x > -1$

② $x - 1 > 0$

$x > 1$



$\boxed{\therefore x > 1}$

Check: $x = 3$: $x + 1 = 3 + 1 = 4 > 0$ ✓

$x - 1 = 3 - 1 = 2 > 0$ ✓

f) $3^x = 23 \longrightarrow \log_3 23 = x$

$\log 3^x = \log 23$

$x \log 3 = \log 23$

$x = \frac{\log 23}{\log 3}$

$x \doteq 2.85$

Ex. 2. On bright sunny days, the amount of bromine in a municipal swimming pool decreases by 10% each hour. If there were 145 g of bromine in the pool at noon on a sunny day, when would the pool contain 102 g?

Let M grams represent the amount of bromine in the pool, t hours after noon.

$M = 145(0.9)^t$

Find t when $M = 102$

$102 = 145(0.9)^t$

$\frac{102}{145} = (0.9)^t$

$\log\left(\frac{102}{145}\right) = \log(0.9)^t$

$\log\left(\frac{102}{145}\right) = t\log(0.9)$

$t = \frac{\log\left(\frac{102}{145}\right)}{\log(0.9)}$

$t \doteq 3.3$

$\therefore$ the pool contains 102g of bromine at approx 3:18pm

Ex. 3. The half-life of a radioactive substance is 23 days. How long is it until the percent remaining is 10 %?

Let t represent the length of time in days until 10% remains:

$10 = 100\left(\frac{1}{2}\right)^{\frac{t}{23}}$

$\frac{1}{10} = \left(\frac{1}{2}\right)^{\frac{t}{23}}$

$\log\left(\frac{1}{10}\right) = \log\left(\frac{1}{2}\right)^{\frac{t}{23}}$

$\log\left(\frac{1}{10}\right) = \frac{t}{23}\log\left(\frac{1}{2}\right)$

$\frac{23\log\left(\frac{1}{10}\right)}{\log\left(\frac{1}{2}\right)} = t$

$t \doteq 76.4$

$\therefore$ 10% remains after approx 76.4 days.

Ex. 4. Solve $3^{2x} - 3^x - 12 = 0$.

$(3^x)^2 - 3^x - 12 = 0$

Let $y = 3^x$

$y^2 - y - 12 = 0$

$(y-4)(y+3) = 0$

$3^x = 4$ or $3^x = -3$

no solution $\because 3^x > 0$ for all x

$\log 3^x = \log 4$

$x\log 3 = \log 4$

$x = \frac{\log 4}{\log 3} \doteq 1.26$

HW. Exercise 6.3

Date:________________ **6.4 Change of Base Formula**

Ex. 1. Determine the value of each of the following logarithms, correct to two decimal places.

a) $\log_4 120$ $\quad 3 < \log_4 120 < 4$

Let $x = \log_4 120$

$4^x = 120$

$\log 4^x = \log 120$

$x \log 4 = \log 120$

$x = \dfrac{\log 120}{\log 4}$

$x \doteq 3.45$

b) $\log_{\frac{1}{3}} 11$ $\quad -3 < \log_{\frac{1}{3}} 11 < -2$

Let $x = \log_{\frac{1}{3}} 11$

$\left(\frac{1}{3}\right)^x = 11$

$\log\left(\frac{1}{3}\right)^x = \log 11$

$x \log\left(\frac{1}{3}\right) = \log 11$

$x = \dfrac{\log 41}{\log\left(\frac{1}{3}\right)}$

$x \doteq -2.18$

Change of Base Formulas

i) $\log_a x = \dfrac{\log_b x}{\log_b a}$ $\qquad$ **ii)** $\log_b x = \dfrac{1}{\log_x b}$

i) Proof: Let $y = \log_a x$

$a^y = x$

Take the logarithm of both sides with respect to base b.

$\log_b a^y = \log_b x$

$y \log_b a = \log_b x$

$y = \dfrac{\log_b x}{\log_b a}$

$\boxed{\therefore \log_a x = \dfrac{\log_b x}{\log_b a}}$

ii) Proof:

$LS = \log_b x$

$RS = \dfrac{1}{\log_x b}$

$= \dfrac{1}{\left(\dfrac{\log_a b}{\log_a x}\right)}$ using change of base (i)

$= \dfrac{\log_a x}{\log_a b}$

$= \log_b x$ change of base (i)

$\therefore LS = RS$

$\therefore$ QED

Ex. 2. In each of the following, use the change of base formula to express the given logarithm in terms of the base *b*, and then use a calculator to evaluate to three decimal places.

a) $\log_{\sqrt{2}} 20, \quad b = 10$

$= \dfrac{\log_{10} 20}{\log_{10} \sqrt{2}}$

$= \dfrac{\log 20}{\log \sqrt{2}}$

$\doteq 8.644$

b) $3\underbrace{\log_4 0.4}, \quad b = e$

$= 3 \cdot \dfrac{\log_e 0.4}{\log_e 4}$

$= 3 \cdot \dfrac{\ln 0.4}{\ln 4}$

$\doteq -1.983$

Ex. 3. Simplify to a single logarithm and evaluate.

a) $\dfrac{\log_5 8}{\log_5 2}$ $= \log_2 8$

$= 3$

b) $\dfrac{\log_{\sqrt{2}} \sqrt{243}}{\log_{\sqrt{2}} 3}$ $= \log_3 \sqrt{243}$

or

$= \log_3 243^{\frac{1}{2}}$

$= \frac{1}{2} \cdot \log_3 243$

$= \frac{1}{2} \cdot 5$

$= \frac{5}{2}$

or

Let $x = \log_3 \sqrt{243}$

$3^x = \sqrt{243}$

$3^x = 3^{\frac{5}{2}}$

$\because$ bases are equal

$\therefore x = \frac{5}{2}$

$= \frac{5}{2}$

Ex. 4. Show that $\dfrac{1}{\log_8 a} - \dfrac{1}{\log_2 a} = \dfrac{1}{\log_4 a}$.

$LS = \dfrac{1}{\log_8 a} - \dfrac{1}{\log_2 a}$

$= \log_a 8 - \log_a 2$

$= \log_a \left(\frac{8}{2}\right)$

$= \log_a 4$

$RS = \dfrac{1}{\log_4 a}$

$= \log_a 4$

$\because LS = RS$

$\therefore$ Q.E.D.

Recall: **Transformations on the Logarithmic Function** $f(x) = \log_b x$ **,where**

$$f(x) = a\log_b[k(x-d)] + c \quad \& \quad (x,y) \rightarrow \left(\frac{1}{k}x + d, ay + c\right)$$

Ex. 5. Graph $y = -2\log_{\frac{1}{3}}(-x+5)$ by naming and applying transformations on $y = \log_{\frac{1}{3}} x$.

Clearly label the equation of the asymptote, state the domain and range of the function and determine any intercepts.

$y = -2\log_{\frac{1}{3}}[-(x-5)]$ $\leftarrow$ for asymptote: $-x+5>0$, $5>x$ $\therefore$ V.A. $x=5$

Transformations on $y=\log_{\frac{1}{3}} x$

i) V.R. in x-axis

ii) V.S. by a factor of 2

iii) H.R. in y-axis

iv) H.T. right 5 units

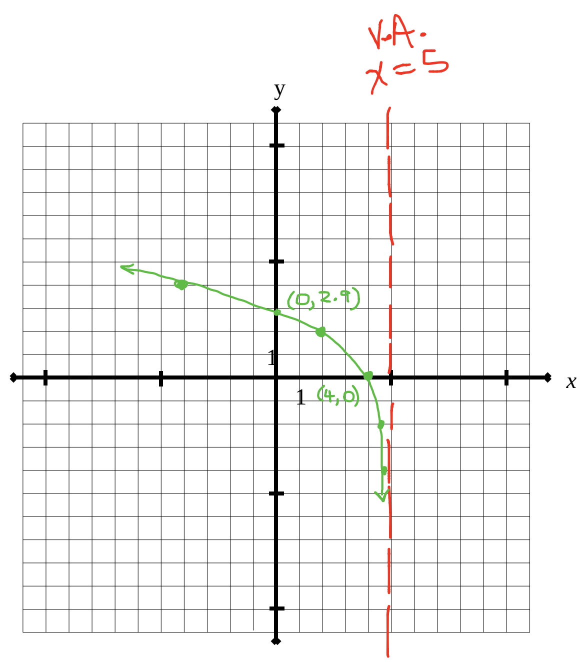
$(x,y) \rightarrow (-x+5, -2y)$

(inverse of $y=(\frac{1}{3})^x$)

$y = \log_{\frac{1}{3}} x$

x	y
9	-2
3	-1
1	0
$\frac{1}{3}$	1
$\frac{1}{9}$	2

x	y
-4	4
2	2
4	0
$4\frac{2}{3}$	-2
$4\frac{8}{9}$	-4



D: $\{x \in \mathbb{R} \mid x < 5\}$

R: $\{y \in \mathbb{R}\}$

x-int: 4

y-int: $-2\log_{\frac{1}{3}}(5) \doteq 2.9$

HW. Exercise 6.4

Date:_______________

6.5 Working With Natural Logarithms

The truth about "e"… Euler's amazing death-defying constant

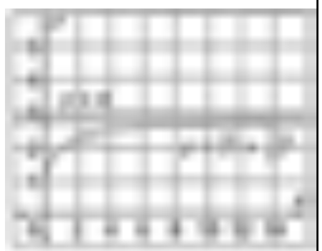
The letter ***e*** honours the Swiss mathematician **Leonard Euler** (1707-1783), whose last name is pronounced *oiler*. Euler also developed the symbols π and i.

Two definitions of ***e*** are as follows:

First Definition of e

$$\left(1+\frac{1}{x}\right)^x \to e, \text{ as } x\to\infty$$

Let $y=\left(1+\frac{1}{x}\right)^x$ and graph.



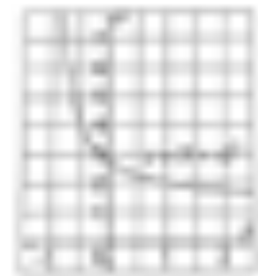
x	$y=\left(1+\frac{1}{x}\right)^x$
1	2
2	2.25
3	2.3703
…	…
100	2.7048
1 000	2.7169
100 000	2.7183

In both the table and graph, as $x\to\infty$, $y\to e$.

Second Definition of e

$$(1+x)^{\frac{1}{x}} \to e, \text{ as } x\to 0$$

Let $y=(1+x)^{\frac{1}{x}}$ and graph.

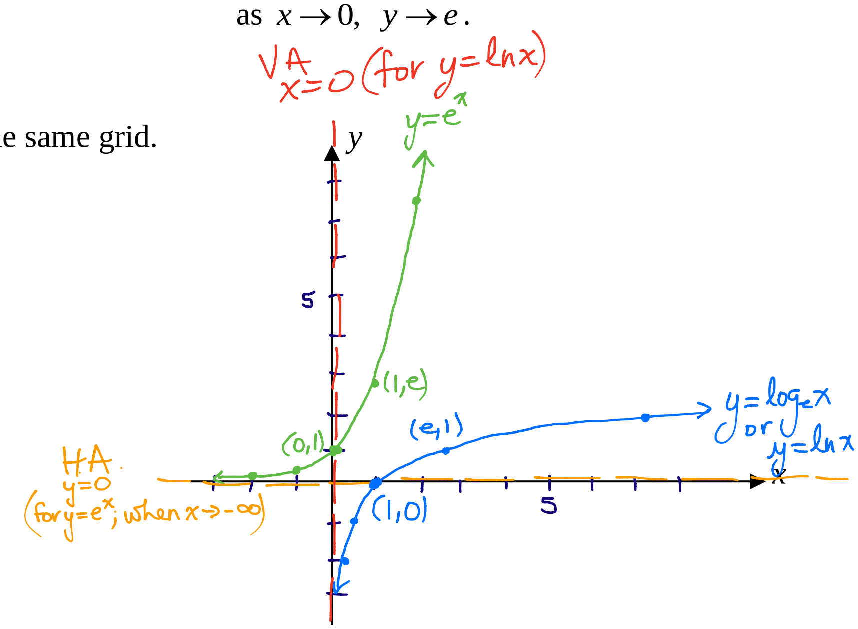


x	$y=(1+x)^{\frac{1}{x}}$
-1.0	undefined
-0.1	2.8680
-0.01	2.7320
-0.001	2.7196
-0.0001	2.7184
0	Undefined
0.00001	2.7183
0.0001	2.7181
0.001	2.7169
0.01	2.7048
0.1	2.5937
1.0	2.0

In both the table and graph, as $x\to 0$, $y\to e$.

Ex. 1. Graph $y=e^x$ and its inverse on the same grid.

x	y
-2	$\frac{1}{e^2}\doteq 0.1$
-1	$\frac{1}{e}\doteq 0.4$
0	$e^0=1$
1	$e^1\doteq 2.7$
2	$e^2\doteq 7.4$



Ex. 2. Sketch $f(x) = \ln(-x+3)$. State the domain, range and exact values of any intercepts.

$f(x) = \ln[-(x-3)]$

Transformations on $y = \ln x$
i) H.R. in y-axis
ii) H.T. 3 units right

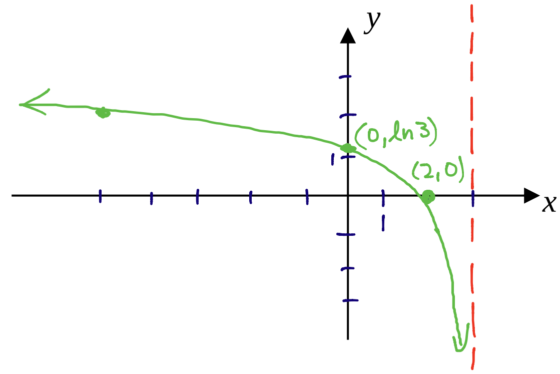
$* \quad -x+3>0$
$3>x$
$x<3$
$\therefore$ V.A. at $x=3$

For x-int:
$f(x)=0$
$0=\ln(-x+3)$
$e^0=-x+3$
$1=-x+3$
$\boxed{\therefore x=2}$

For y-int:
$f(0)=\ln 3$ ← exact
$\doteq 1.1$

x	$f(x)$
-5	$\ln 8 \doteq 2.1$

V.A. $x=3$



Recall:

Properties of Natural Logarithms: i) $\ln 1 = 0$ **ii)** $\ln e = 1$ **iii)** $\ln e^x = x$ **iv)** $e^{\ln x} = x$

$\ln x = \log_e x$

Natural Logarithm Laws: 1. $\ln(x \cdot y) = \ln x + \ln y$ ***multiplication***

$x, y > 0$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ ***division***

3. $\ln x^n = n \cdot \ln x$ ***power***

Ex. 3. Use the properties of logarithms to write each expression as a sum, difference, and/or multiple of logarithms.

a) $\ln\left[\frac{x^3(x-2)^4}{\sqrt{x^2+1}}\right]$

$= \ln[x^3(x-2)^4] - \ln\sqrt{x^2+1}$
$= \ln x^3 + \ln(x-2)^4 - \ln(x^2+1)^{\frac{1}{2}}$
$= 3\ln x + 4\ln(x-2) - \frac{1}{2}\ln(x^2+1)$

b) $\ln \cot\theta$

$= \ln\left(\frac{\cos\theta}{\sin\theta}\right)$
$= \ln(\cos\theta) - \ln(\sin\theta)$

Ex. 4. Evaluate each of the following.

a) $e^{-2\ln 3}$
$= e^{\ln 3^{-2}}$
$= 3^{-2}$
$= \frac{1}{9}$

b) $\ln e^{\sin^2 x + \cos^2 x}$
$= \ln e^1$
$= 1$

$(\ln x)^2 \neq \ln x^2$

Ex. 5. Solve each of the following.

a) $-e^{-t} + 3e^{-3t} = 0$

$-e^{-3t}(e^{2t}-3)=0$

$-e^{-3t}=0$ or $e^{2t}-3=0$

$-\frac{1}{e^{3t}}=\frac{0}{1}$ $\quad e^{2t}=3$

$\therefore$ no solution $(-e^{-3t}<0)$

$\ln e^{2t}=\ln 3$
$2t\ln e=\ln 3$
$t=\frac{\ln 3}{2\ln e}=\frac{1}{2}\ln 3=\ln\sqrt{3}$
$(t\doteq 0.55)$

b) $(\ln x)^2 + \ln x^2 = 0$

$(\ln x)^2+2\ln x=0$
$\ln x(\ln x+2)=0$

$\ln x=0$ or $\ln x=-2$
$e^0=x$ $\quad e^{-2}=x$
$\therefore x=1$ $\quad x=\frac{1}{e^2}$
$(x\doteq 0.14)$

HW. Exercise 6.5

Date:__________________ **6.6 Logarithmic Scales & Their Applications**

Logarithmic scales *are useful for measuring quantities that can have a very large range, because logarithms enable us to make large or small numbers more manageable to work with.*

Examples of logarithmic scales include the ***Richter scale****, which measures earthquakes, the* ***decibel scale****, which measures sound, and the* ***pH scale****, which measures acidity.*

Logarithms and Earthquakes

The formula Richter used to define the magnitude of an earthquake is

$$M = \log\left(\frac{I}{I_0}\right), \text{ where}$$

I is the intensity of the earthquake being measured,
I_0 is the intensity of a reference earthquake, and
M is the Richter number used to measure the intensity of earthquakes.

On the Richter scale, the energy of the earthquake increases by powers of 10 in relation to the Richter magnitude number. Earthquakes below magnitude 4 usually cause no damage, and quakes below 2 cannot be felt. A magnitude 6 earthquake is strong, while one of magnitude 7 or higher causes major damage.

Ex. 1. An earthquake of magnitude 7.5 on the Richter scale struck Guatemala on February 4, 1976, killing 23 000 people. On October 2, 1993, an earthquake of magnitude 6.4 killed 20 000 in Maharashtra, India. Compare the intensities of the two earthquakes.

Let I_G and I_M represent the intensities of the Guatemalan and the Maharashtran earthquakes, respectively.

① $7.5 = \log\left(\frac{I_G}{I_0}\right)$

solve for I_G:

$10^{7.5} = \frac{I_G}{I_0}$

$I_G = 10^{7.5} I_0$

② $6.4 = \log\left(\frac{I_M}{I_0}\right)$

solve for I_M:

$10^{6.4} = \frac{I_M}{I_0}$

$I_M = 10^{6.4} I_0$

Compare: $\frac{I_G}{I_M} = \frac{10^{7.5} I_0}{10^{6.4} I_0}$

$\frac{I_G}{I_M} = 10^{1.1}$

$\frac{I_G}{I_M} = 12.6$

$I_G = 12.6 I_M$

∴ the intensity of the Guatemalan earthquake was 12.6 times that of the Indian earthquake.

Logarithms and Sound

> The formula used to compare sounds is
> $$L = 10\log\left(\frac{I}{I_0}\right)\text{, where}$$
> I is the intensity of the sound being measure,
> I_0 is the intensity of a sound at the threshold of hearing, and
> L is the loudness measured in decibels.

The loudness of any sound is measured relative to the loudness of sound at the threshold of hearing. Sounds at this level are the softest that can still be heard. At the threshold of hearing, the loudness of sound is zero decibels (0 dB).

Ex. 2. A sound is 1000 times more intense than a sound you can just hear. What is the measure of its loudness in decibels?

$L = 10\log\left(\frac{I}{I_o}\right)$ where I_o is the intensity of a sound that can just be heard. & $I = 1000 I_o$

$L = 10\log\left(\frac{1000 I_o}{I_o}\right)$

$L = 10\log 1000$

$L = 10 \times 3$

$L = 30$

∴ the loudness of the sound is 30 dB.

Exposure to sound levels of 85 dB during a 35 h work week will eventually cause damage to most ears. The 120 dB volume of the average rock concert will cause the same damage in less than half an hour. The higher the level, the less time it takes before sound-receptor cells start dying and permanent hearing damage occurs. At sound levels of 130 dB, after 75 s you are at risk of suffering permanent damage to your hearing.

Ex. 3. How many more times intense is the sound of normal conversation (60 dB) than the sound of a whisper (30 dB)?

Let I_n and I_w represent the intensities of normal conversation and of a whisper, respectively.

① $60 = 10\log\left(\frac{I_n}{I_o}\right)$

Solve for I_n:

$6 = \log\left(\frac{I_n}{I_o}\right)$

$10^6 = \frac{I_n}{I_o}$

$I_n = 10^6 I_o$

② $30 = 10\log\left(\frac{I_w}{I_o}\right)$

Solve for I_w:

$3 = \log\left(\frac{I_w}{I_o}\right)$

$10^3 = \frac{I_w}{I_o}$

$I_w = 10^3 I_o$

Compare:

$\frac{I_n}{I_w} = \frac{10^6 I_o}{10^3 I_o}$

$\frac{I_n}{I_w} = 10^3$

$I_n = 1000 I_w$

∴ the intensity of normal conversation is 1000 times that of a whisper.

Logarithms and Chemistry

> Chemists define the acidity of a liquid on a pH scale,
> $pH = -\log[H^+]$, where
> $[H^+]$ is the concentration of the hydrogen ion in moles per litre.
> A solution with a *pH* lower than 7 is *acidic*.
> A solution with a *pH* equal to 7 is *distilled water*.
> A solution with a *pH* greater than 7 is *basic*.

Ex. 4. Find the *pH* of a swimming pool with a hydrogen ion concentration of 6.1×10^{-8} mol/L (*pH* is given to two decimal places.)

$pH = -\log[H^+]$

$pH = -\log(6.1\times10^{-8})$

$pH \doteq 7.21$

$\therefore$ the pH of the pool is 7.21

Ex. 5. The *pH* of a fruit juice is 3.10. What is the hydrogen ion concentration of the fruit juice?

$pH = -\log[H^+]$

$3.10 = -\log[H^+]$

$-3.10 = \log[H^+]$

$H^+ = 10^{-3.10}$

$H^+ = 0.00079$

$\therefore$ the hydrogen ion concentration is 7.9×10^{-4} mol/L.

HW. Exercise 6.6
For Unit 6 Test: do Unit 6 Review Exercise