Date:_____ **UNIT 6: LOGARITHMIC FUNCTIONS & LOGARITHMS**

6.1 The Logarithmic Function

Exponential Form		**Logarithmic Form**	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is **the exponent** to which that base must be raised to yield x.

1. Write each of the following in logarithmic form.

a) $3^2 = 9$ **b)** $9^0 = 1$ **c)** $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

d) $36^{\frac{1}{2}} = 6$ **e)** $27^{\frac{2}{3}} = 9$ **f)** $2^{-3} = \frac{1}{8}$

2. Write each of the following in exponential form.

a) $\log_5 125 = 3$ **b)** $\log_7 1 = 0$ **c)** $\log_5\left(\frac{1}{25}\right) = -2$

d) $\log_7\left(\frac{1}{7}\right) = -1$ **e)** $\log_{\frac{1}{3}} 9 = -2$ **f)** $\log_9 27 = \frac{3}{2}$

3. **i)** Without a calculator determine between what pair of consecutive integers each of the following logarithms lie.
ii) Use your calculator to determine approximate values accurate to four decimal places.

a) $\log_{10} 37$ **b)** $\log 0.24$ **c)** $\log 20\,000$

d) $\log_e 18$ **e)** $\ln 2$ **f)** $\ln 0.2$

4. On four separate grids, sketch the graphs of the following:

a) $y = 5^x$ and $y = \log_5 x$ **b)** $y = \left(\frac{1}{5}\right)^x$ and $y = \log_{\frac{1}{5}} x$

c) $y = 10^x$ and $y = \log x$ **d)** $y = e^x$ and $y = \ln x$

5. In each of the following, use the definition of a logarithm as an exponent and determine the value of each of the following:

a) $\log_5 5$ **b)** $\log_3 1$ **c)** $\log_7 7$ **d)** $\log_4 1$

e) $\log_2 2^5$ **f)** $\log_3 3^4$ **g)** $\log 10^{3.6}$ **h)** $\ln e^{5.78}$

i) $5^{\log_5 25}$ **j)** $4^{\log_4 64}$ **k)** $10^{\log 6}$ **l)** $e^{\ln 97}$

m) $\log_b 1$ **n)** $\log_b b$ **o)** $\log_b b^x$ **p)** $b^{\log_b x}$

6. Evaluate each of the following:

a) $\log_2 8$ **b)** $\log_5 25$ **c)** $\log_3 81$ **d)** $\log_7 49$

e) $\log_2\left(\frac{1}{8}\right)$ **f)** $\log_3\left(\frac{1}{27}\right)$ **g)** $\log_5\sqrt{5}$ **h)** $\log_2 4^2$

i) $\log_6 36 - \log_5 25$ **j)** $\log_9\left(\frac{1}{3}\right) + \log_3\left(\frac{1}{9}\right)$ **k)** $\log_6\sqrt{36} - \log_{25} 5$ **l)** $\log_2\sqrt[4]{32}$

m) $\log_3\sqrt[4]{27}$ **n)** $\log_3\left(9\times\sqrt[5]{9}\right)$ **o)** $\log_2 16^{\frac{1}{3}}$ **p)** $\log_3 3 + \log_5 1$

q) $\log_4 4^4 + \log_3 3^3$ **r)** $\log_3 9^3 - \log_5 125^4$ **s)** $4\log_4 2 - \log_4 1$ **t)** $\frac{3}{2}\log 100 - \log 0.001$

7. Use your knowledge of logarithms to solve each of the following equations for *x*. State any restrictions on *x*.

a) $\log_5 x = 3$ **b)** $\log_4 x = 2$ **c)** $\log_x 27 = \frac{3}{2}$ **d)** $\log_4\left(\frac{1}{64}\right) = x$

e) $\log_x 0.04 = -2$ **f)** $\log_{\frac{1}{4}} x = -0.5$ **g)** $\log_{\sqrt{2}} x = -8$ **h)** $\log_{27}\left(\frac{1}{3}\right) = x$

Answers:

1. a) $\log_3 9 = 2$ **b)** $\log_9 1 = 0$ **c)** $\log_{\frac{1}{2}} \frac{1}{4} = 2$ **d)** $\log_{36} 6 = \frac{1}{2}$ **e)** $\log_{27} 9 = \frac{2}{3}$ **f)** $\log_2 \frac{1}{8} = -3$

2. a) $5^3 = 125$ **b)** $7^0 = 1$ **c)** $5^{-2} = \frac{1}{25}$ **d)** $7^{-1} = \frac{1}{7}$ **e)** $\left(\frac{1}{3}\right)^{-2} = 9$ **f)** $9^{\frac{3}{2}} = 27$

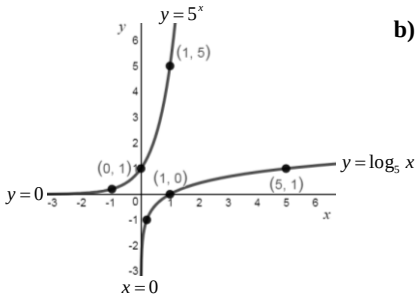
3. a) i) $1 < \log_{10} 37 < 2$ **b) i)** $-1 < \log 0.24 < 0$ **c) i)** $4 < \log 20000 < 5$

ii) $\log_{10} 37 \doteq 1.5682$ **ii)** $\log 0.24 \doteq -0.6198$ **ii)** $\log 20000 \doteq 4.3010$

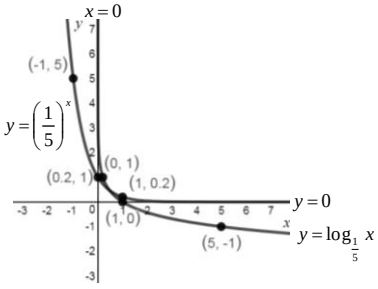
d) i) $2 < \log_e 18 < 3$ **e) i)** $0 < \ln 2 < 1$ **f) i)** $-2 < \ln 0.2 < -1$

ii) $\ln 18 \doteq 2.8904$ **ii)** $\ln 2 \doteq 0.6931$ **ii)** $\ln 0.2 \doteq -1.6094$

4. a)



b)



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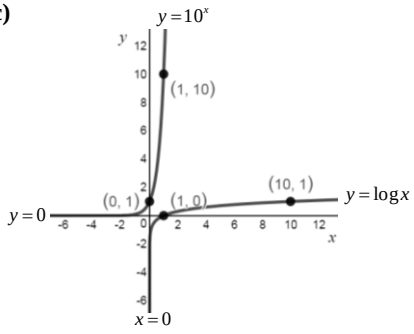
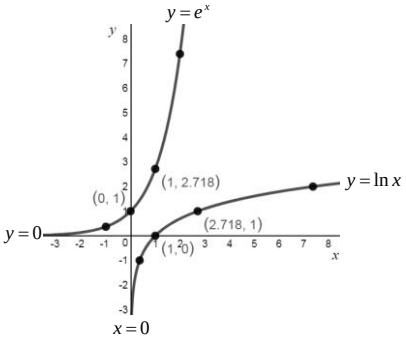


d)



5. a) 1 **b)** 0 **c)** 1 **d)** 0 **e)** 5 **f)** 4 **g)** 3.6 **h)** 5.78 **i)** 25 **j)** 64 **k)** 6 **l)** 97 **m)** 0 **n)** 1 **o)** x **p)** x

6. a) 3 **b)** 2 **c)** 4 **d)** 2 **e)** −3 **f)** −3 **g)** $\frac{1}{2}$ **h)** 4 **i)** 0 **j)** $-\frac{5}{2}$ **k)** $\frac{1}{2}$ **l)** $\frac{5}{4}$ **m)** $\frac{3}{4}$ **n)** $\frac{12}{5}$ **o)** $\frac{4}{3}$ **p)** 1 **q)** 7 **r)** −6 **s)** 2 **t)** 6

7. a) 125, $x > 0$ **b)** 16, $x > 0$ **c)** 9, $x > 0, x \neq 1$ **d)** −3 **e)** 5, $x > 0, x \neq 1$ **f)** 2, $x > 0$ **g)** $\frac{1}{16}$, $x > 0$ **h)** $-\frac{1}{3}$

Date:_____ **6.2 Logarithmic Properties, Laws & Transformations**

1. Write each of the following as a sum of logarithms.

a) $\log_a(xy)$ **b)** $\log_m(pq)$

2. Write each of the following as a difference of logarithms.

a) $\log_a\left(\frac{x}{y}\right)$ **b)** $\log_a\left(\frac{r}{s}\right)$

3. Write each of the following in the form $m\log_a p$.

a) $\log_6 13^4$ **b)** $\log_5 1.3^{-2}$ **c)** $\log_7 x^{\frac{1}{3}}$ **d)** $\log_a 6^{-\frac{3}{4}}$

4. Use the properties of logarithms to write each of the following as a sum and/ or a difference and/or a multiple of logarithms.

a) $\log_b\left(\frac{xy}{z}\right)$ **b)** $\log_a\left(\frac{x}{yz}\right)$ **c)** $\log_a \sqrt[3]{x^6y^9}$

d) $\log_a\left(\frac{x^2}{y^6}\right)^{\frac{1}{4}}$ **e)** $\log_a \sqrt{\frac{x^3y^2}{z}}$ **f)** $\log_a \frac{x^3y^4}{\sqrt{x^{\frac{1}{4}}y^{\frac{2}{3}}}}$

5. Use the properties of logarithms to write each of the following as a single logarithm.

a) $\log_a x + \log_a w$ **b)** $\log_a s - \log_a r$ **c)** $7\log_a x + 4\log_a y$

d) $4\log_5 x - 2\log_5 y - 3\log_5 z$ **e)** $\frac{1}{3}\log_b x + \frac{1}{4}\log_b y - \frac{2}{5}\log_b z$

6. Use the properties of logarithms to express each of the following as a single logarithm and then evaluate.

a) $\log_{\frac{1}{3}} 162 - \log_{\frac{1}{3}} 2$ **b)** $\log_{12} 4 + \log_{12} 36$ **c)** $\log_8 640 - \log_8 10$

d) $\log_2 640 - \log_2 80$ **e)** $\log_2 40 + \log_2\left(\frac{4}{5}\right)$ **f)** $\log_4 192 - \log_4 3$

g) $\log_5(0.1) + \log_5(0.4)$ **h)** $\frac{2}{3}\log_3 216 - 2\log_3 6$ **i)** $\log_3 36 + \log_3\left(\frac{3}{4}\right)$

j) $\log_3\left(\frac{2}{3}\right) - \log_3 18$ **k)** $\log 40 + \log\left(\frac{5}{2}\right)$ **l)** $\log_2\left(\frac{1}{14}\right) - \log_2\left(\frac{8}{7}\right)$

m) $\log_{25} 20 - \log_{25} 100$ **n)** $\log_6 4 + \log_6 54$ **o)** $\log_8(1.2) + \log_8 5 - \log_8 3$

7. Write each of the following as a single logarithm.

a) $4\log_{10} x + 2\log_{10}(x-1)$ **b)** $3\log(x+2) - \log(2x-3)$

c) $\log_e(x+1) + 2\log_e(2x+1) - \log_e(x-3)$ **d)** $2x\ln(x) + x\ln(3x-4)$

8. Solve each of the following. Answer to four decimal places if appropriate.

a) $10^{-x} = 31.46$ **b)** $(0.6)^{4x} = 0.734$

c) $2(7^{x+9}) = 112$ **d)** $125 + 3^{5-2x} = 1000$

9. Use the properties of logarithms to write each of the second logarithmic functions as a sum and/or a difference and/or a multiple of logarithms. Then, graph, by naming and applying transformations on the first function.

a) $y = \log x$ and $y = \log(10x)$ **b)** $y = \log_2 x$ and $y = \log_2\left(\frac{x^2}{8}\right)$

c) $y = \log_3 x$ and $y = \log_3\left(\frac{3}{x}\right)^3$ **d)** $y = \log_{\frac{1}{2}} x$ and $y = \log_{\frac{1}{2}} \sqrt{x}$

Answers:

1. a) $\log_a x + \log_a y$ **b)** $\log_m p + \log_m q$ **2. a)** $\log_a x - \log_a y$ **b)** $\log_a r - \log_a s$

3. a) $4\log_6 13$ **b)** $-2\log_5 1.3$ **c)** $\frac{1}{3}\log_7 x$ **d)** $-\frac{3}{4}\log_a 6$ **4. a)** $\log_b x + \log_b y - \log_b z$ **b)** $\log_b x - \log_b y - \log_b z$

c) $2\log_a x + 3\log_a y$ **d)** $\frac{1}{2}\log_a x - \frac{3}{2}\log_a y$ **e)** $\frac{3}{2}\log_a x + \log_a y - \frac{1}{2}\log_a z$ **f)** $\frac{23}{8}\log_a x + \frac{11}{3}\log_a y$

5. a) $\log_a(xw)$ **b)** $\log_a\left(\frac{s}{r}\right)$ **c)** $\log_a(x^7 y^4)$ **d)** $\log_5 \frac{x^4}{y^2 z^3}$ **e)** $\log_a \frac{x^{\frac{1}{3}} y^{\frac{1}{4}}}{z^{\frac{2}{5}}}$

6. a) $\log_{\frac{1}{3}} 81 = -4$ **b)** $\log_{12} 144 = 2$ **c)** $\log_8 64 = 2$ **d)** $\log_2 8 = 3$ **e)** $\log_2 32 = 5$ **f)** $\log_4 64 = 3$ **g)** $\log_5\left(\frac{1}{25}\right) = -2$ **h)** $\log_3 1 = 0$

i) $\log_3 27 = 3$ **j)** $\log_3\left(\frac{1}{27}\right) = -3$ **k)** $\log 100 = 2$ **l)** $\log_2\left(\frac{1}{16}\right) = -4$ **m)** $\log_{25} \frac{1}{5} = -\frac{1}{2}$ **n)** $\log_6 216 = 3$ **o)** $\log_8 2 = \frac{1}{3}$

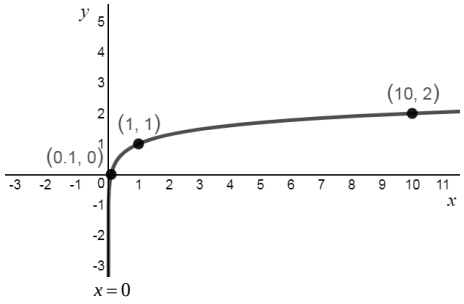
7. a) $\log_{10}[x^4(1-x)^2]$ **b)** $\log\frac{(x+2)^3}{2x-3}$ **c)** $\log_e \frac{(x+1)(2x+1)^2}{x-3}$ **d)** $\ln[x^{2x}(3x-4)^x]$

8. a) –1.4978 **b)** 0.1513 **c)** –6.9314 **d)** –0.5831

9. a) $y = \log_{10} x + 1$

Transformation on $y = \log_{10} x$ *is:*

i) *vertical translation up* 1 *unit*

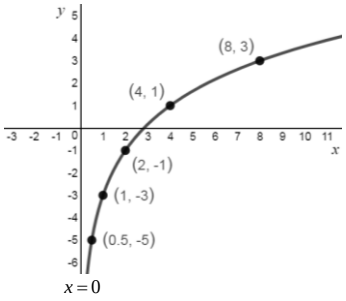


b) $y = 2\log_2 x - 3$

Transformations on $y = \log_2 x$ *are:*

i) *vertical expansion by a factor of* 2

ii) *vertical translation down* 3 *units*



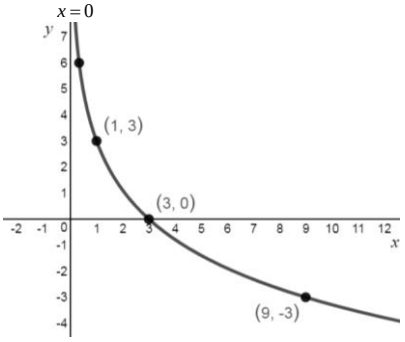
c) $y = -3\log_3 x + 3$

Transformations on $y = \log_3 x$ *are:*

i) *vertical reflection in the x-axis*

ii) *vertical expansion by a factor of* 3

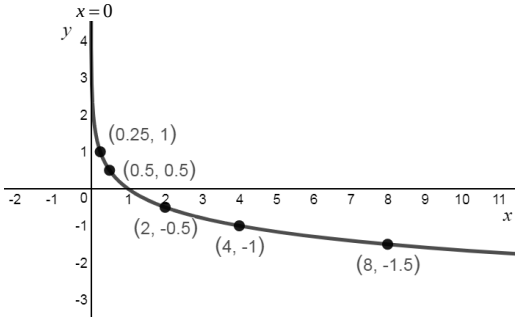
iii) *vertical translation up* 3 *units*



d) $y = \frac{1}{2}\log_{\frac{1}{2}} x$

Transformation on $y = \log_{\frac{1}{2}} x$ *is:*

i) *vertical compression by a factor of* $\frac{1}{2}$



Date:_____ **6.3 Solving Logarithmic Equations**

1. Solve by first expressing each side of the equation as *simplified logarithms* of the same base.

a) $\log_2 x = 2\log_2 4$ **b)** $4\log_3 x = \log_3 16$

c) $\log_3 x^4 = 2\log_3 4$ **d)** $2\log_5 x = \log_5 36$

e) $2\log_{10} x = 4\log_{10} 7$ **f)** $\log_2 x = \log_2 5 + \log_2 10$

g) $\log x = \log 84 + \log 5 - \log 7$ **h)** $\log_e 3 - \log_e 4 = \log_e 1 - \log_e x$

i) $\ln x = 2\ln 3 + 3\ln 2$ **j)** $\log_6 x = 2\log_6 4 + 3\log_6 3$

k) $\log_5 x - \log_5 8 = \log_5 6 + 3\log_5 2$ **l)** $2\log x = \log 32 + \log 2$

m) $2\log x - \log 3 = \log 27$ **n)** $\log x^2 = 3\log 4 - 2\log 2$

2. Solve by first expressing each side of the equation as *simplified logarithms* of the same base. State any restrictions on the variable.

a) $\log\sqrt{x} = \log 1 - 2\log 3$ **b)** $\log x^{\frac{1}{2}} - \log x^{\frac{1}{3}} = \log 2$

c) $\log_5(2x+2) - \log_5(x-1) = \log_5(x+1)$ **d)** $\log_3 x + \log_3(x-5) = \log_3 x + \log_3 8$

e) $\ln(x+5) = 2\ln(x-1)$ **f)** $\log\sqrt{x^2+48x} = \log 10$

g) $\log_a(x+2) + \log_a(x-1) = \log_a(8-2x)$ **h)** $\log_4(x+2) = \log_4 9 - \log_4(x-3)$

3. Solve by first expressing the equation in simplified logarithmic form and then rewriting the equation in exponential form. State any restrictions on the variable.

a) $\log_6(x+1) + \log_6(x+2) = 1$ **b)** $\log_4 x + \log_4(x+6) = 2$ **c)** $\log_7(x+2) + \log_7(x-4) = 1$

d) $\log_4(x+2) - \log_4(x-1) = 1$ **e)** $\log_2 x - \log_2(x+2) = 3$ **f)** $\log_2(x+2) = 3 - \log_2 x$

g) $2 - \log_2(x-2) = \log_2(x-5)$ **h)** $\log(x+5) = 1 + \log(x-4)$ **i)** $\log_6(x-1) = 2 - \log_6(x+4)$

4. Solve the following exponential equations. Give the answer correct to two decimal places.

a) $225(1.05)^{2x} = 450$ **b)** $3^{x+2} + 3^x = 350$ **c)** $4^{x-1} - 4^{x-2} = 18$

d) $15 = \left(\frac{1}{2}\right)^{\frac{x}{4}}$ **e)** $2^{2x} - 2^x - 6 = 0$ **f)** $7^x = 10(7^{-x}) - 3$

5. a) The solution to the equation $5^{3x} = 4^{x+1}$ is given below, but in jumbled order. Copy the solution into your notebook, with the steps rearranged in the correct order and a write a brief explanation beside each step.

$x(3\log 5 - \log 4) = \log 4$

$5^{3x} = 4^{x+1}$

$3x\log 5 - x\log 4 = \log 4$

$\log 5^{3x} = \log 4^{x+1}$

$x \doteq 0.40$

$3x\log 5 = (x+1)\log 4$

$3x\log 5 = x\log 4 + \log 4$

$x = \dfrac{\log 4}{3\log 5 - \log 4}$

b) Solve the equation $4^{2x-1} = 3^{x+2}$. Answer with both exact and approximate values to 2 decimal places.

6. Consider the exponential equation $8^{2x} - 2(8)^x - 5 = 0$.

First use the quadratic formula to solve for exact values of 8^x and then solve for x with both exact and approximate values to two decimal places.

7. A car depreciates at 15% per year. How long is it until it is worth half its original value?

8. An investment of \$2500 grows at a rate of 4.8% per year, compounded annually. How long will it take for the investment to be worth \$4000?

Recall that the formula for compound interest is $A = P(1+i)^n$.

9. If you invested \$500 in an account that pays 12% per annum compounded weekly, how long will it take for your deposit to triple?

10. Polonium-218 is a radioactive substance that spontaneously decays into lead-214. One minute after a 100-mg sample of polonium-218 is placed into a nuclear chamber, only 80 mg remains. Determine the the half-life of polonium-218 to one decimal place.

11. An isotope of cobalt, Co^{60}, is used in medical therapy. When the radioisotope activity has decreased to 45% of its initial level, the exposure times required are too long and the hospital needs to replace the cobalt. How often does the cobalt need to be replaced?
(The half-life of Co^{60} is 5.24 years.)

12. A man wants to sell an old piece of wood to a museum. He claims it came from the stable in which Christ was born 2000 years ago. The museum tests the wood and finds that it contains 4.2×10^{10} atoms of C^{14} per gram. Carbon from present-day wood contains 5.0×10^{10} atoms of C^{14} per gram. Determine the approximate age of the wood. Do you think the relic is authentic?
(The half-life of C^{14} is 5760 years.)

Answers:

1. a) 16 **b)** 2 **c)** ± 2 **d)** 6 **e)** 49 **f)** 50 **g)** 60 **h)** $\frac{4}{3}$ **i)** $\frac{9}{8}$ **j)** 432 **k)** 384 **l)** 8 **m)** 9 **n)** ± 4

2. a) $\frac{1}{81}$, $x>0$ **b)** 64, $x>0$ **c)** 3, $x>1$ **d)** 13, $x>5$ **e)** 4, $x>1$ **f)** $-50, 2$, $x<-48$ *or* $x>0$ **g)** 2, $1<x<4$ **h)** $\frac{1+\sqrt{61}}{2}$, $x>3$

3. a) 1, $x>-1$ **b)** 2, $x>0$ **c)** 5, $x>4$ **d)** 2, $x>1$ **e)** *no solution*, $x>0$ **f)** 2, $x>0$ **g)** 6, $x>5$ **h)** 5, $x>4$ **i)** 5, $x>1$

4. a) 7.10 **b)** 3.24 **c)** 3.29 **d)** −15.63 **e)** 1.58 **f)** 0.36

5. a)

$5^{3x} = 4^{x+1}$

$\log 5^{3x} = \log 4^{x+1}$ *Take the logarithm of both sides.*

$3x\log 5 = (x+1)\log 4$ *Apply the power law of logarithms.*

$3x\log 5 = x\log 4 + \log 4$ *Apply the distributive property.*

$3x\log 5 - x\log 4 = \log 4$ *Collect variable terms on one side of the equation.*

$x(3\log 5 - \log 4) = \log 4$ *Factor x on the left side.*

$x = \frac{\log 4}{3\log 5 - \log 4}$ *Solve for an exact value of x.*

$x \doteq 0.40$ *Approximate x with a calculator.*

5. b) $x = \frac{2\log 3 + \log 4}{2\log 4 - \log 3}$ *or* $x \doteq 2.14$

6. $8^x = 1-\sqrt{6}$ *or* $8^x = 1+\sqrt{6}$, $x = \frac{\log(1+\sqrt{6})}{\log 8}$, $x \doteq 0.60$

7. 4 years and 3 months
8. 10.025 years
9. 477.9 weeks or 9.2 years
10. 3.1 minutes
11. 6 years
12. 1450 years, No

Date:_____ **6.4 Change of Base Formula**

1. Use the ***change of base formula*** , $\log_a x = \dfrac{\log x}{\log a}$, to determine the value of each of the following with your *calculator* to two decimal places.

a) $\log_2 12$ **b)** $\log_3 \dfrac{1}{2}$ **c)** $\log_3 8$ **d)** $\log_4 6$ **e)** $\log_{\frac{1}{2}} 15$

2. Graph each function below using transformations. Find and label any asymptotes and intercepts and state the domain and range.

a) $y = \log_4(x-3)+2$ **b)** $f(x) = 2\log_3(x+4)-3$ **c)** $y = -3\log_2(x-1)+4$

d) $y = \log_{\frac{1}{2}}[-(x+3)]-5$ **e)** $g(x) = -2\log_2(-2x+4)-1$ **f)** $y = -\dfrac{1}{2}\log_{\frac{1}{3}}\left(3-\dfrac{1}{2}x\right)$

3. Simplify to a single logarithm and evaluate.

a) $\log_5 7 - \log_5 35$ **b)** $2\log 5 + \dfrac{1}{2}\log 16$ **c)** $\dfrac{\log_6 256}{\log_6 4}$ **d)** $\dfrac{\log_4 3}{\log_4 27}$ **e)** $\dfrac{\log 8}{\log 32}$

4. Show that each of the following statements is true.

a) $\dfrac{1}{\log_5 a} + \dfrac{1}{\log_3 a} = \dfrac{1}{\log_{15} a}$ **b)** $\dfrac{2}{\log_9 a} - \dfrac{1}{\log_3 a} = \dfrac{3}{\log_3 a}$

5. Solve for *x*. State any restrictions on the variable.

a) $x = \log_7 7^{\sqrt{2}}$ **b)** $x = \log_5 225 - 2\log_5 3$

c) $x - 3\log_3 243 = 4\log_2 \sqrt{512}$ **d)** $\log_{\frac{1}{3}}(2x+5) = -1$

e) $\log(x-4) = 1$ **f)** $5^{8\log_5 \sqrt{3}} = x$

g) $\log_3(4x+1)^2 = 4$ **h)** $3\log x - \log 2 = \log(2x)$

i) $\log_7(x-2) + \log_7(x+4) = 1$ **j)** $(\log x)^2 + 3\log x - 10 = 0$

k) $2^x = 7$ **l)** $2^x - 2^{-x} = 4$ ***Hint:*** *multiply by* 2^x

m) $\log_5(\log_3 x) = 0$ ***Hint:*** *Let* $y = \log_3 x$ **n)** $\log_2(\log_4 x) + 1 = 0$

6. Determine all points of intersection, (x, y)**,** where the two curves $y = \log_{10}(x^4)$ and $y = (\log_{10} x)^3$.

7. If $\log_a b = p^3$ and $\log_b a = \dfrac{4}{p^2}$, solve for *p* and state the restrictions on all the variables.

Hint: *Use the change of base reciprocal identity for logarithms.*

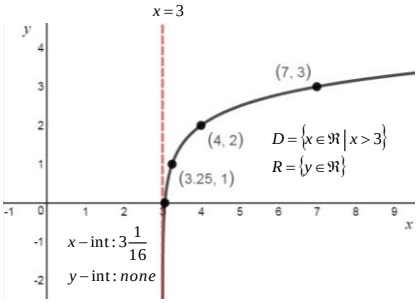
8. If $\log_{\sin x}(\cos x) = \dfrac{1}{2}$ and $0 < x < \dfrac{\pi}{2}$**,** solve for $\sin x$ exactly, and *x* to the nearest hundredth of a radian.

Answers:

1. a) 3.58 **b)** −0.63 **c)** 1.89 **d)** 1.29 **e)** −3.91

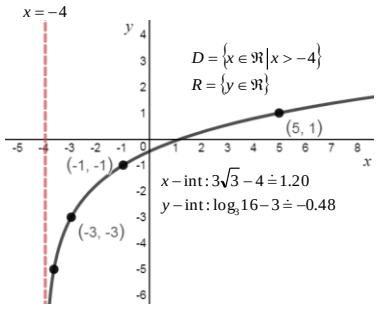
2. a) *Transformation on* $y=\log_4 x$ *are:*

i) *horizontal translation right* 3 *units*
ii) *vertical translation up* 2 *units*



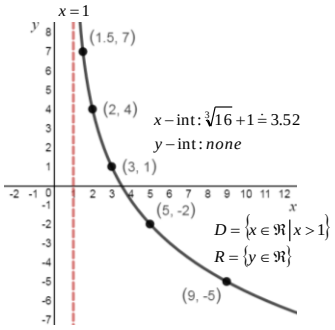
b) *Transformations on* $y=\log_3 x$ *are:*

i) *vertical expansion by a factor of* 2
ii) *horizontal translation left* 4 *units*
iii) *vertical translation down* 3 *units*



c) *Transformations on* $y=\log_2 x$ *are:*

i) *vertical reflection in the x-axis*
ii) *vertical expansion by a factor of* 3
iii) *horizontal translation right* 1 *unit*
iv) *vertical translation up* 4 *units*



d) *Transformation on* $y=\log_{\frac{1}{2}} x$ *are:*

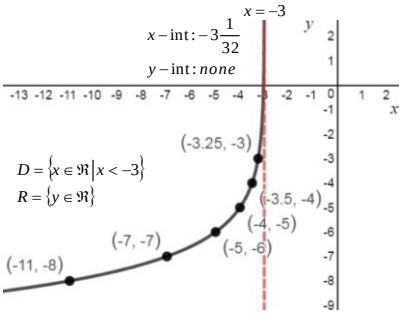
i) *horizontal reflection in the y-axis*
ii) *horizontal translation left* 3 *units*
iii) *vertical translation down* 5 *units*

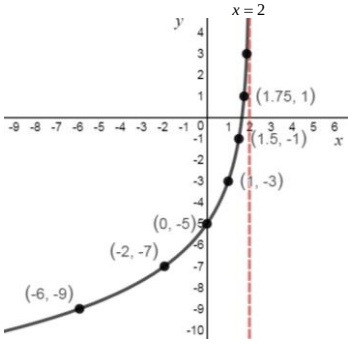
e) *Transformations on* $y=\log_2 x$ *are:*

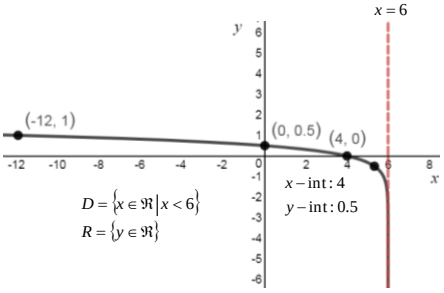
i) *vertical reflection in the x-axis*
ii) *vertical expansion by a factor of* 2
iii) *horizontal reflection in the y-axis*
iv) *horizontal compression by a factor of* $\frac{1}{2}$
v) *horizontal translation right* 2 *units*
vi) *vertical translation down* 1 *unit*

f) *Transformations on* $y=\log_{\frac{1}{3}} x$ *are:*

i) *vertical reflection in the x-axis*
ii) *vertical compression by a factor of* $\frac{1}{2}$
iii) *horizontal reflection in the y-axis*
iv) *horizontal expansion by a factor of* 2
v) *horizontal translation right* 6 *units*







3. a) $\log_5 \frac{1}{5}=-1$ **b)** $\log 100=2$ **c)** $\log_4 256=4$ **d)** $\log_{27} 3=\frac{1}{3}$ **e)** $\log_{32} 8=\frac{3}{5}$

5. a) $\sqrt{2}$ **b)** 2 **c)** 33 **d)** $-1,\ x>-\frac{5}{2}$ **e)** 14, $x>4$ **f)** 81 **g)** $-\frac{5}{2}, 2,\ x\neq -\frac{1}{4}$ **h)** 2 , $x>0$ **i)** 3 , $x>2$ **j)** $\frac{1}{100000}, 100,\ x>0$

k) $\frac{\log 7}{\log 2}\doteq 2.81$ **l)** $\frac{\log(2+\sqrt{5})}{\log 2}\doteq 2.08$ **m)** 3, $x>0$ **n)** 2, $x>0$

6. $\left(\frac{1}{100}, -8\right)$, (1,0), (100,8)

7. $p=\frac{1}{4},\ a,b>0, a,b\neq 1,\ p\neq 0$

8. $\sin x=\frac{-1+\sqrt{5}}{2},\ x\doteq 0.67$

Date:_____ **6.5 Working With Natural Logarithms**

1. Sketch the graph of the logarithmic function $y = \ln(x+2) - 1$, using transformations. Find and label any asymptotes and intercepts and state the domain and range.

2. Determine the equation of the inverse of the logarithmic function $y = \ln(x+2) - 1$ by switching *x* and *y* in the equation and solving for *y*. Sketch the graph of this exponential function, using transformations. Find and label any asymptotes and intercepts and state the domain and range.

3. Find the domain of each function.

a) $f(x) = \ln(\ln x)$ **b)** $f(x) = \ln(4 - x^2)$ **c)** $f(x) = \ln\left(\frac{4 - x^2}{x+1}\right)$ **d)** $f(x) = \ln(e^x - 3)$

4. Use the properties of logarithms to evaluate each of the following.

a) $\ln e^{-2}$ **b)** $\ln\left(\frac{1}{e}\right)$ **c)** $e^{\ln\sqrt{3}}$ **d)** $e^{-2\ln 5}$

e) $e^{-2\ln 3 + 3\ln 2}$ **f)** $\ln\left(\ln e^{e^{10}}\right)$ **g)** $\frac{\ln 128}{\ln 0.5}$ **h)** $\frac{\ln\sqrt{27}}{\ln 9}$

5. Use the properties of logarithms to write each of the following as a sum and/ or a difference and/or a multiple of logarithms.

a) $\ln\left[\frac{(x+1)(x-3)^3}{(x+2)^4}\right]$ **b)** $\ln\left[\frac{\sqrt{x^2+1}}{\sqrt[3]{x^4}(3x+2)^5}\right]$ **c)** $\ln(\tan^2 x)$ **d)** $\ln\left[\frac{e^{-x}\cdot\cos x}{x^2+2x+1}\right]$

6. Use the properties of logarithms to write each of the following as a single logarithm of a ***simplified*** expression.

a) $\ln 2 + \ln\sin x + \ln\cos x$ **b)** $\frac{1}{2}\ln(4x^{-4}) + \frac{1}{5}\ln(32x^{10})$

c) $\ln(x^3 - y^3) - \ln(x-y) - 2\ln x$ **d)** $\ln(4x^3 - 3x + 1) - \ln(4x^3 + 4x^2 - x - 1)$

7. Solve each equation for *x*.

a) $e^{-2x} = 4$ **b)** $\ln\left(\frac{x+1}{x}\right) = 1$ **c)** $e^x - 2e^{-x} = 1$ **d)** $\ln(x^2 + e) - \ln(x+1) = 1$

e) $2^{x-5} = 3$ **f)** $8\ln x - 4 = 0$ **g)** $(\ln x)^2 - 4 = 0$ **h)** $(\ln x)^2 + (\ln x) - 2 = 0$

i) $2\ln x = \ln(4x+5)$ **j)** $\ln x + \ln(x+8) = 3$ **k)** $2e^{-2x} - 2e^{-2x}(2x) - [4x\cdot e^{-2x} - 2e^{-2x}(2x^2)] = 0$

8. Solve the equation $x = \ln\left(y + \sqrt{y^2 - 1}\right)$ for *y*, where $y > 1$.

9. The world's population was 2.5 billion in 1950 and reached 5 billion in 1987. Assume that the world's population grows exponentially, so that at time *t* (measured in years) the population is

$$N(t) = 2.5e^{kt}$$

where $t = 0$ corresponds to the year 1950, and *N* is measured in billions of people.

a) Find the constant *k*.

b) According to this model, when will the population reach 10 billion?

10. Assembly-line operations tend to have a high turnover of employees, forcing companies to spend much time and effort in training new workers. It has been found that a worker new to a task on the line will produce items according to the function defined by $P(x) = 25 - 25e^{-0.3x}$, where $P(x)$ items are produced by the worker on day x.

a) How many days will it take for a new worker to produce at least 20 items in a day?

b) What will be the limit on the number of items a worker can produce in a day if x gets larger and larger?

11. If a deposit of P dollars is invested at a rate of interest r ***compounded continuously*** for t years, the compound amount is $A = Pe^{rt}$.

a) Find the value of $2500 after five years if the money is invested at 10% /a,

i) compounded quarterly **ii)** compounded continuously

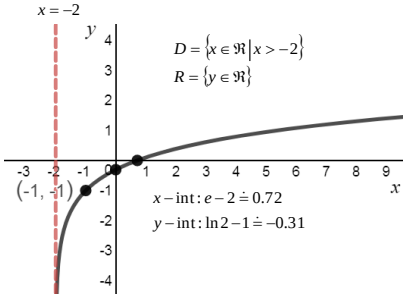
b) Find the interest rate that will cause $5000 to grow to $7250 in 6 years if the money is compounded continuously.

Answers

1. *Transformation on* $y = \ln x$ *are:*

i) *horizontal translation left* 2 *units*

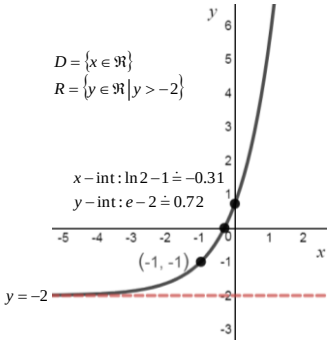
ii) *vertical translation down* 1 *unit*



2. $y = e^{x+1} - 2$, *Transformations on* $y = e^x$ *are:*

i) *horizontal translation left 1 unit*

ii) *vertical translation down 2 units*



3. a) $D = \{x \in R \mid x > 1\}$ **b)** $D = \{x \in R \mid -2 < x < 2\}$ **c)** $D = \{x \in R \mid x < -2 \text{ or } -1 < x < 2\}$ **d)** $D = \{x \in R \mid x > \ln 3\}$

4. a) -2 **b)** -1 **c)** $\sqrt{3}$ **d)** $\frac{1}{25}$ **e)** $\frac{8}{9}$ **f)** 10 **g)** -7 **h)** $\frac{3}{4}$

5. a) $\ln(x+1) + 3\ln(x-3) - 4\ln(x+2)$ **b)** $\frac{1}{2}\ln(x^2+1) - \frac{4}{3}\ln x - 5\ln(3x+2)$ **c)** $2\ln(\sin x) - 2\ln(\cos x)$ **d)** $-x + \ln(\cos x) - 2\ln(x+1)$

6. a) $\ln(\sin 2x)$ **b)** $\ln 4$ **c)** $\ln\left(\frac{x^2+xy+y^2}{x^2}\right)$ **d)** $\ln\left(\frac{2x-1}{2x+1}\right)$

7. a) $-\ln 2$ **b)** $\frac{1}{e-1}$ **c)** $\ln 2$ **d)** $0, e$ **e)** $\frac{\ln 3}{\ln 2} + 5 \text{ or } \log_2 3 + 5$ **f)** $\sqrt{e}$ **g)** $\frac{1}{e^2}, e^2$ **h)** $\frac{1}{e^2}, e$ **i)** 5 **j)** $4 + \sqrt{16 + e^3}$

k) $\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}$

8. $y = \frac{1}{2}(e^x + e^{-x}) \text{ or } \frac{e^{2x}+1}{2e^x}$

9. a) $\frac{\ln 2}{37}$ **b)** 2024 **10. a)** 6 days **b)** 25 items **11. a) i)** $4096.54 **ii)** $4121.80 **b)** 6.19%

Date:_____ **6.6 Logarithmic Scales & Their Applications**

1. If one earthquake has a magnitude of 5 on the Richter scale and a second earthquake has a magnitude of 6, compare the intensities of the two earthquakes.

2. A sound is 1 000 000 times more intense than a sound you can just hear. What is the loudness of the sound?

3. Find the pH of a liquid with a hydrogen ion concentration of 8.7×10^{-6} mol/L.

4. An earthquake of magnitude 2 cannot be felt. An earthquake of magnitude 4 will be noticed but usually causes no damage. Compare the intensities of two such earthquakes.

5. An earthquake in Gansu, China, on December 16, 1920, measured 8.6 on the Richter scale and killed 100 000 people. An earthquake that usually causes no damage measures 4 on the Richter scale. Compare the intensities of the two earthquakes.

6. An earthquake in the Quetta area of Pakistan on May 31, 1935 measured 6.8 on the Richter scale. This quake killed 50 000 people. On October 2, 1987, an earthquake of magnitude 6.1 shook Los Angeles, California, and killed six people.
a) Compare the magnitude of the two earthquakes.
b) Why do you think the death toll was so much higher with the earthquake in Pakistan?

7. On January 24, 1939, an earthquake measuring 8.3 occurred in Chillan, Chile, killing 28 000 people. On September 21, 1999, an earthquake in Taiwan measured 7.6 on the Richter scale and killed 2100 people. Compare the intensities of these two earthquakes.

8. Sasha needs a new muffler on her car. She has been told that the sound from her car was measured at 120 dB. After installing the new muffler, the loudness of her car is 75 dB. How many times more intense was the sound from her defective muffler?

9. Tania's infant daughter has colic and cries during the night. The noise level in the house at these times is 75 dB. When the baby finally falls asleep, the noise level is 35 dB. How many times more intense is the noise level in the house when the baby is crying?

10. How many times more intense is the sound of a space-shuttle launch (180 dB) than the sound of a jet engine (140 dB)?

11. Jonathan lives near a busy street. He has all the windows in his home open and measures the noise level inside as 79 dB. He closes the windows and finds the noise level is 68 dB. By what factor did the intensity of the noise decrease when Jonathan closed the windows?

12. Find the hydrogen ion concentration of milk, which has a pH of 6.50.

13. Find the hydrogen ion concentration of milk of magnesia which has a pH of 10.50.

Answers

1. 10 times **2.** 60 dB **3.** 5.06 **4.** 100 times **5.** 40 000 times **6. a)** 5 times **7.** 5 times **8.** 32 000 times
9. 10 times **10.** 10 000 times **11.** 13 **12.** 3.2×10^{-7} mol/L **13.** 3.2×10^{-11} mol/L

Date:_____ **Unit 6 Test Review**

PART A: ***No calculator is allowed.***

1. In each of the following, change to the equivalent logarithmic form.

a) $5^4 = 625$ **b)** $4^{-2} = \frac{1}{16}$ **c)** $e^t = 3$ **d)** $10^y = x$

2. In each of the following, change to the equivalent exponential form.

a) $\log_{\frac{1}{11}}(121) = -2$ **b)** $\log_{125}(x) = \frac{1}{3}$ **c)** $\ln 1296 = 4$ **d)** $\log A = W$

3. Evaluate each of the following:

a) -3^{-2} **b)** $(-32)^{\frac{3}{5}}$ **c)** $64^{-\frac{2}{3}}$ **d)** $\left(\frac{2}{3}\right)^{-4}$

e) $\log_2 32$ **f)** $\log_3 27$ **g)** $\log_{125} 5$ **h)** $\log_2 \frac{1}{16}$

i) $\log 0.0001$ **j)** $\log_2 8 + \log_3 9$ **k)** $7^{\log_7 5}$ **l)** $\ln e^4$

4. Evaluate each of the following by applying appropriate logarithm properties.

a) $\log_6 9 + \log_6 4$ **b)** $\log_4 20 - \log_4 5$ **c)** $2\log 2 + \log 5 - \log 20$

d) $\log_6 3 - \log_6 108$ **e)** $\log_2 1 - \log_2 \frac{8}{5} - \log_2 10$ **f)** $\log_2 3.2 + \log_2 100 - \log_2 5$

g) $\log_{\frac{1}{3}}\left(5^3 9^{-3} 25^{-\frac{3}{2}}\right)$ **h)** $3^{-2\log_3 5}$ **i)** $\log\left(\frac{\sqrt[3]{100}}{\sqrt[5]{1000}}\right)$

j) $\log_{\sqrt{3}} 9^{\frac{1}{3}}$ **k)** $\log_5 \sqrt[4]{25}$ **l)** $\log_5 \sqrt[3]{25} + \log_3 \sqrt{27}$

m) $\ln e^{2\ln 4 - 3\ln 2}$ **n)** $\log_2 \sqrt{36} - \log_2 \sqrt{72}$ **o)** $\log_8 2 + 3\log_8 2 - \frac{1}{2}\log_8 16$

p) $\frac{\log 625}{\log 5}$ **q)** $\frac{\ln \sqrt{216}}{\ln 36}$ **r)** $\frac{2}{\log_{128} 2}$

5. a) Write the following logarithm as a sum and/ or a difference and/or a multiple of logarithms.

$$\ln\left[\frac{x^6 \cdot \sqrt[3]{x+1}}{(x+1)^3}\right]$$

b) Write the following as a single logarithm of a ***simplified*** expression.

$$3\log(\cos x) - \log(\sin x) - \log(1 - \cos^2 x)$$

6. Use the properties of logarithms to write each of the second logarithmic functions as a sum and/or a difference and/or a multiple of logarithms. Then, graph, by naming and applying transformations on the first function.

a) $y = \log_3 x$ and $y = \log_3(9x^{-2})$ **b)** $y = \log_2 x$ and $y = \log_2\left(\frac{x}{4}\right)^{\frac{3}{2}}$

7. Graph each function below using transformations. Find and label any asymptotes and intercepts and state the domain and range.

a) $y = \log_{\frac{1}{3}}(0.5x + 2) - 1.5$ **b)** $f(x) = 3\ln(3 - x)$

8. Solve the following equations, if possible. State any restrictions on *x*.

a) $3 = \log_2\left(\frac{1}{x}\right)$ **b)** $\log x^2 = 3\log 4$ **c)** $\ln 3 - \ln x = \ln 12$

d) $\log_2(x+2) + \log_2 x = 3$ **e)** $\log_2(x-2) + \log_2(x+1) = 2$ **f)** $\frac{4}{3}e^{2x} + 2e^x - \frac{4}{3} = 0$

g) $\log_x\left(\frac{1}{64}\right) = -\frac{6}{5}$ **h)** $\log_5(x+2) - \log_5(x-1) = 2\log_5 3$ **i)** $\log_x(19x-30) = 3$

j) $\log_2(\log_8 x) + 3 = 4$ **k)** $\log\sqrt[3]{x^2 - 10x} = \frac{2}{3}$ **l)** $\frac{\log(35 - x^3)}{\log(5-x)} = 3$

9. **a)** If $\log_b x = 0.4$, determine the value of $\log_b x\sqrt{x}$.

b) If $\log_b a^2 = 3$, determine the value of $\log_a b^2$.

c) If $\left(\frac{1}{2}\right)^{x+y} = 16$ and $\log_{x-y} 8 = -3$, calculate the values of *x* and *y*.

d) If $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$, determine the value(s) of *x*.

PART B: ***A calculator is required.***

10. In each of the following, use the change of base formula to express the given logarithm in terms of the specified base *b*, and then evaluate to three decimal places.

a) $\log_2 80,\ b = 10$ **b)** $\log_5 34.62,\ b = e$ **c)** $3\log_5 22 - 2\log_5 15,\ b = 10$

11. Emilio invests $1000. How long, to the nearest month, will it take for the investment to grow to $1500 at 7.5%/a compounded

a) semi-annually? **b)** continuously?

12. A radioactive substance decays from 20 g to 15 g in 7 h. Determine the half-life of the substance to two decimal places.

13. The population of a town is decreasing at a rate of 1.6%/a. If the population today is 20 000, how long will it take for the population to decline to 15 000 to one decimal place?

14. In a particular circuit, the current *I*, in amperes, can be found using the formula $I = 0.8\left(1 - 10^{-0.0434t}\right)$, after *t* seconds.

a) What is the current in the circuit after 5 s to two decimal places?

b) Determine the time it takes for the current to reach 0.5 A to the nearest hundredth of a second

15. An earthquake of magnitude 8.3 on the Richter scale killed 200 000 in Tokyo, Japan, on September 1, 1923. On February 4, 1976, an earthquake of magnitude 7.5 killed 23 000 in Guatemala. Compare the intensity of these earthquakes.

16. A liquid has a pH of 8.31. Find the hydrogen ion concentration $[H^+]$.

17. a) Determine the pH level of a cheese for which $[H^+] = 3.3 \times 10^{-4}$ mol/L.

b) Is the cheese acidic or basic?

18. Kendra is talking to her friend on the subway platform, where the noise level is 60 dB. When a subway train enters the station, Kendra can no longer hear her friend. The noise level from the train is 90 dB. How many times more intense is the noise level in the station when the train enters, to the nearest thousand?

Answers:

1. a) $4=\log_5 625$ **b)** $-2=\log_4\left(\frac{1}{16}\right)$ **c)** $t=\ln 3$ **d)** $y=\log x$

2. a) $\left(\frac{1}{11}\right)^{-2}=121$ **b)** $125^{\frac{1}{3}}=x$ **c)** $e^4=1296$ **d)** $10^W=A$

3. a) $-\frac{1}{9}$ **b)** -8 **c)** $\frac{1}{16}$ **d)** $\frac{81}{16}$ **e)** 5 **f)** 3 **g)** $\frac{1}{3}$ **h)** -4 **i)** -4 **j)** 5 **k)** 5 **l)** 4

4. a) 2 **b)** 1 **c)** 0 **d)** -2 **e)** -4 **f)** 6 **g)** 6 **h)** $\frac{1}{25}$ **i)** $\frac{1}{15}$ **j)** $\frac{4}{3}$ **k)** $\frac{1}{2}$ **l)** $\frac{13}{6}$ **m)** $\ln 2$ **n)** $-\frac{1}{2}$ **o)** $\frac{2}{3}$ **p)** 4 **q)** $\frac{3}{4}$ **r)** 14

5. a) $6\ln x-\frac{8}{3}\ln(x+1)$ **b)** $\log(\cot^3 x)$

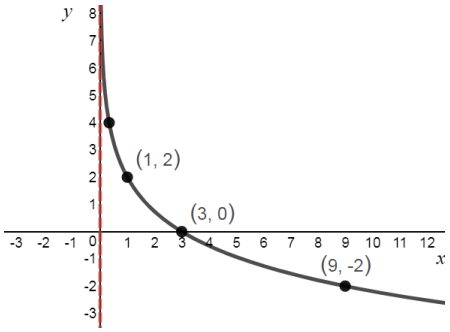
6. a) $y=-2\log_3 x+2$

Transformations on $y=\log_3 x$ *are:*

i) *vertical reflection in the x-axis*

ii) *vertical expansion by a factor of* 2

iii) *vertical translation up* 2 *units*

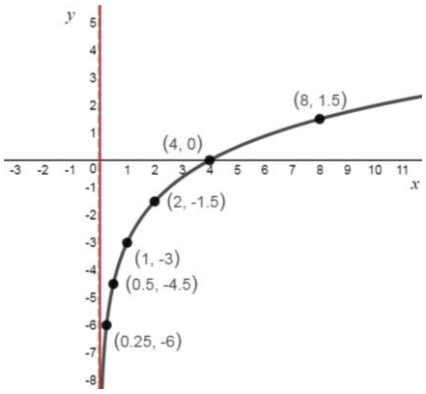


b) $y=\frac{3}{2}\log_2 x-3$

Transformations on $y=\log_2 x$ *are:*

i) *vertical expansion by a factor of* $\frac{3}{2}$

ii) *vertical translation down* 3 *units*



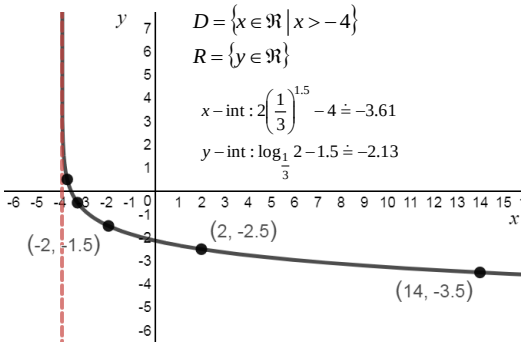
7. a) *Transformatiosn on* $y=\log_{\frac{1}{3}} x$ *are:*

i) *horizontal expansion by a factor of* 2

ii) *horizontal translation left* 4 *units*

iii) *vertical translation down* 1.5 *units*

$x=-4$



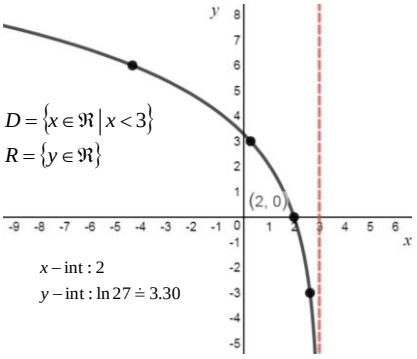
b) *Transformations on* $y=\ln x$ *are:*

i) *vertical expansion by a factor of* 3

ii) *horizontal reflection in the y-axis*

iii) *horizontal translation right* 3 *units*

$x=3$



8. a) $\frac{1}{8}$, $x>0$ **b)** $-8, 8$, $x\neq 0$ **c)** $\frac{1}{4}$, $x>0$ **d)** 2, $x>0$ **e)** 3, $x>2$ **f)** $-\ln 2$ **g)** 32, $x>0$, $x\neq 1$ **h)** $\frac{11}{8}$, $x>1$ **i)** 2, 3, $x>\frac{30}{19}$

j) 64, $x>1$ **k)** $5-5\sqrt{5}$, $5+5\sqrt{5}$, $x<0$ *or* $x>10$ **l)** 2, 3, $x<\sqrt[3]{35}$

9. a) $\frac{3}{5}$ **b)** $\frac{4}{3}$ **c)** $x=-1\frac{3}{4}$, $y=-2\frac{1}{4}$ **d)** $\frac{1}{\sqrt[3]{4}}, 8$ **10. a)** 6.322 **b)** 2.202 **c)** 2.397

11. a) 5 years and 6 months **b)** 5 years and 5 months **12.** 16.87 h **13.** 17.8 years **14. a)** 0.31 A **b)** 9.81 s

15. 6.3 times **16.** 4.9×10^{-9} mol/L **17. a)** 3.48 **b)** acidic **18.** 1000 times