Date: Dec 8/14

UNIT 7: WORKING WITH FUNCTIONS
COMPOSITE FUNCTIONS

Definition of a Composite Function

For two functions f and g, the composite function $f(g(x))$ is formed by evaluating f at g. This new ***composite function*** $f \circ g$, is called the **composition of f and g**, and is defined by $(f \circ g)(x) = f(g(x))$. In this situation, the domain of f is the range of g, provided that g is defined.

Ex. 1. Let $f = \{(3,2), (5,1), (7,4), (9,3), (11,5)\}$ and $g = \{(1,3), (2,5), (3,7), (4,9), (5,6)\}$. Determine

a) $(f \circ g)(3)$

$= f(g(3))$
$= f(7)$
$= 4$

b) $(g \circ f)(3)$

$= g(f(3))$
$= g(2)$
$= 5$

c) $(f \circ f)(9)$

$= f(f(9))$
$= f(3)$
$= 2$

d) $(g \circ g)(3)$

$= g(g(3))$
$= g(7)$
d.n.e.
or undefined.

e) $f \circ g$

all x's.

$= f(g(x))$

$= \{(1,2), (2,1), (3,4), (4,3)\}$

f) $D_{f \circ g}$: $\{1, 2, 3, 4\}$

Ex. 2. If $f(x) = \sqrt{x}$, $g(x) = x + 5$ and $h(x) = x^2 - \dfrac{1}{x}$, find each of the following:

a) $(f \circ g)(4)$

$= f(g(4))$
$= f(9)$
$= \sqrt{9}$
$= 3$

b) $(g \circ f)(4)$

$= g(f(4))$
$= g(2)$
$= 7.$

c) $(h \circ f \circ g)(-5)$

$= h(f(g(-5)))$
$= h(f(0))$
$= h(0)$
d.n.e or undefined

d) $(f \circ g)(x)$

$= f(g(x))$
$= f(x+5)$
$\therefore (f \circ g)(x) = \sqrt{x+5}$

e) $g \circ f$

$= g(f(x))$
$= g(\sqrt{x})$
$= \sqrt{x} + 5$

f) $h \circ g$

$= h(g(x))$
$= h(x+5)$
$= (x+5)^2 - \dfrac{1}{x+5}$

or $= \dfrac{(x+5)^3 - 1}{x+5}$ ***

$= \dfrac{x^3 + 15x^2 + 75x + 124}{x+5}$

***
$(x+5)^3 = (x+5)(x+5)(x+5)$
$= (x+5)(x^2 + 10x + 25)$
$= x^3 + 15x^2 + 75x + 125$

Recall: $f(x)=\sqrt{x}$, $g(x)=x+5$ and $h(x)=x^2-\frac{1}{x}$

g) x, if $(f\circ g)(x)=(g\circ f)(x)$ from d) & e)

$\sqrt{x+5}=\sqrt{x}+5$

square both sides

$x+5=(\sqrt{x}+5)(\sqrt{x}+5)$

$x+5=x+10\sqrt{x}+25$

$-20=10\sqrt{x}$

$-2=\sqrt{x}$

Square both sides

$4=x$

Check: L.S. $=\sqrt{4+5}$ $=\sqrt{9}$ $=3$ | RS $=\sqrt{4}+5$ $=2+5$ $=7$

$\because$ LS $\neq$ RS

$\therefore x=4$ is an extraneous root

$\therefore$ there is no solution

h) x, if $(h\circ g)(x)=0$ from f)

$(x+5)^2-\frac{1}{x+5}=0$

$(x+5)^2=\frac{1}{x+5}$

$\sqrt[3]{(x+5)^3}=\sqrt[3]{1}$

$x+5=1$

$x=-4$

Ex. 3. From the functions listed in the box, find two whose composite function is $h(x)$.

$p(x)=x-3$	$f(x)=x^2$
$g(x)=\sqrt{x}$	$q(x)=x+1$

a) $h(x)=(x+1)^2$

$h(x)=f(q(x))$

b) $h(x)=\sqrt{x-3}$

$h(x)=g(p(x))$

c) $h(x)=x^2-6x+9$

$h(x)=(x-3)^2$

$h(x)=f(p(x))$

Ex. 2. Express h as the composition of two functions f and g, such that $h(x)=f(g(x))$.

a) $h(x)=\sqrt{x^2-4}$

$g(x)=x^2-4$

$f(x)=\sqrt{x}$

b) $h(x)=2\sin^2 x-3\sin x+1$

$g(x)=\sin x$

$f(x)=2x^2-3x+1$

c) $h(x)=\frac{1}{1-x^2}$

$g(x)=1-x^2$

$f(x)=\frac{1}{x}$

d) $h(x)=2^{(6x+7)}$

$g(x)=6x+7$

$f(x)=2^x$

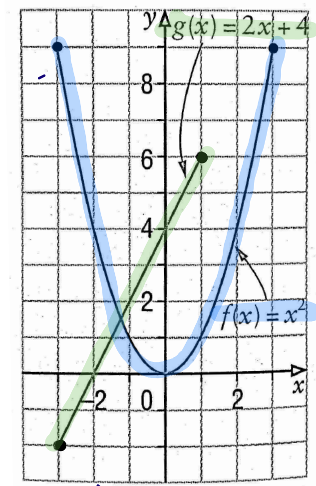
HW. pg. 152 #1, 3 odd parts, 4, 10, 11

Date: Dec 10/14 **COMPOSITION OF FUNCTIONS CONTINUED**

Definition of a Composite Function

For two functions *f* and *g*, the composite function $f(g(x))$ is formed by evaluating *f* at *g*. This new ***composite function*** $f \circ g$, is called the **composition of *f* and *g***, and is defined by $(f \circ g)(x) = f(g(x))$. In this situation, the domain of f is the range of *g*, provided that *g* is defined.

Ex. 1. State the domain and range of *f* and *g*. Use the graphs of *f* and *g* to evaluate each expression, if it exists. If it does not exist, explain why.



$D_f = \{x \in \mathbb{R} \mid -3 \leq x \leq 3\}$ $D_g = \{x \in \mathbb{R} \mid -3 \leq x \leq 1\}$

$R_f = \{y \in \mathbb{R} \mid 0 \leq y \leq 9\}$ $R_g = \{y \in \mathbb{R} \mid -2 \leq y \leq 6\}$

a) $(g \circ f)(-1)$
$= g(f(-1))$
$= g(1)$
$= 6$

b) $(g \circ g)(-2)$
$= g(g(-2))$
$= g(0)$
$= 4$

c) $(f \circ g)(0)$
$= f(g(0))$
$= f(4)$
undefined/d.n.e
(4 is not in the domain of f)

Ex. 2. If $f(x) = 2x - 1$, $h(x) = x^2 - 2x$, $g(x) = \ln(x-1)$ and $p(x) = \cos x$ determine:

a) $f^{-1}(x)$

Let $y = 2x - 1$
For inverse,
$x = 2y - 1$
to solve for y
$2y = x + 1$
$y = \frac{x+1}{2}$
$\therefore f^{-1}(x) = \frac{x+1}{2}$

b) g^{-1}

Let $y = \ln(x-1)$
For inverse:
$x = \ln(y-1)$
$e^x = y - 1$
$y = e^x + 1$
$\therefore g^{-1}(x) = e^x + 1$

c) $(f \circ f^{-1})(x)$

$= f(f^{-1}(x))$
$= f\left(\frac{x+1}{2}\right)$
$= \cancel{2}^{1}\left(\frac{x+1}{\cancel{2}_{1}}\right) - 1$
$= x + 1 - 1$
$= x$

d) $g^{-1} \circ g$

$= g^{-1}(g(x))$
$= g^{-1}(\ln(x-1))$
$= e^{(\ln(x-1))} + 1$
$= x - 1 + 1$
$= x$

Recall: $f(x) = 2x - 1$, $h(x) = x^2 - 2x$, $g(x) = \ln(x-1)$ and $p(x) = \cos x$

e) $h \circ f$

$= h(f(x))$
$= h(2x-1)$
$= (2x-1)^2 - 2(2x-1)$
$= (2x-1)(2x-1) - 2(2x-1)$
$= 4x^2 - 4x + 1 - 4x + 2$
$= 4x^2 - 8x + 3$

f) $(h \circ p)(\frac{\pi}{4})$

$= h(p(\frac{\pi}{4}))$
$= h(\cos\frac{\pi}{4})$
$= h(\frac{1}{\sqrt{2}})$
$= (\frac{1}{\sqrt{2}})^2 - 2(\frac{1}{\sqrt{2}})$
$= \frac{1}{2} - \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{1}{2} - \frac{2\sqrt{2}}{2}$
$= \frac{1 - 2\sqrt{2}}{2}$

g) x, if $(h \circ g)(x) = 0$

$h(g(x)) = 0$
$h(\ln(x-1)) = 0$
$[\ln(x-1)]^2 - 2[\ln(x-1)] = 0$
let $a = \ln(x-1)$
$a^2 - 2a = 0$
$a(a-2) = 0$

$a = 0$ or $a = 2$

$\ln(x-1) = 0$
$x - 1 = e^0$
$x - 1 = 1$
$\therefore x = 2$

$\ln(x-1) = 2$
$x - 1 = e^2$
$x = e^2 + 1$

$\therefore x = 2$ or $x = e^2 + 1$

h) x, for $0 \le x \le 2\pi$ if $(f \circ p)(x) = -2$

$f(p(x)) = -2$
$f(\cos x) = -2$
$2\cos x - 1 = -2$
$\cos x = \frac{-1}{2}$
raa $= \frac{\pi}{3}$
In QII: $x = \frac{2\pi}{3}$
In QIII: $x = \frac{4\pi}{3}$

$\therefore x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

review your special Δ's.

HW. pg. 152 #2c, 3 even, 12 and Composite Functions Worksheet

Date:_________________ **WARMUP**

1. Given $f(x)=\log_3 x$ and $g(x)=x^3-5x^2+2x+8$, determine

a) $h(x)=(f\circ g)(x)$

$h(x)=f(g(x))$

$h(x)=f(x^3-5x^2+2x+8)$

$h(x)=\log_3(x^3-5x^2+2x+8)$

$\because g(-1)=0 \therefore x+1$ is a factor

$$\begin{array}{r|l} & x^2-6x+8 \\ x+1 & x^3-5x^2+2x+8 \\ & x^3+x^2 \\ & -6x^2+2x \\ & -6x^2-6x \\ & 8x+8 \\ & 8x+8 \\ & 0 \end{array}$$

b) D_h For domain of $h(x)$

$x^3-5x^2+2x+8>0$

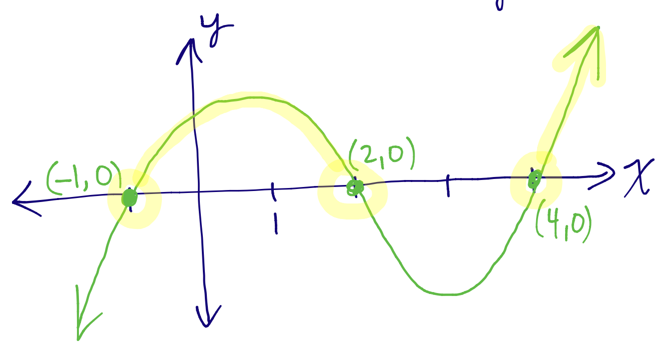
Unit 2: Day 4

Let $g(x)=x^3-5x^2+2x+8$

$g(x)=(x+1)(x^2-6x+8)$

$g(x)=(x+1)(x-4)(x-2)$

$\therefore$ x-ints are $-1, 2, 4$ (all single)



$\therefore D_h=\{x\in\mathbb{R}\mid -1<x<2,\ x>4\}$

2. Given $f(x)=\dfrac{1}{x}$ and $g(x)=\sqrt{x+2}$, determine

a) $h(x)=(g\circ f)(x)$

$h(x)=g(f(x))$

$h(x)=g\left(\frac{1}{x}\right)$

$h(x)=\sqrt{\frac{1}{x}+\frac{2}{1}}$

$h(x)=\sqrt{\frac{1+2x}{x}}$

$\therefore D_h=\{x\in\mathbb{R}\mid x\le -\frac{1}{2} \text{ or } x>0\}$

b) D_h For the domain:

Unit 2: Day 4.5

$\frac{1+2x}{x}\ge 0,\ x\ne 0$

Let $p(x)=\frac{1+2x}{x}$

x-int is $-\frac{1}{2}$,

restriction: $x\ne 0$

$p(-1) \Rightarrow +$ ✓ $\quad p(-\frac{1}{4}) \Rightarrow -$ ✗ $\quad p(1) \Rightarrow +$ ✓

Date: Dec 11/14

APPLICATIONS OF COMPOSITE FUNCTIONS

Ex. 1. Refrigeration slows down the growth of bacteria in food. The number of bacteria in a certain food is approximated by $B(T) = 15T^2 - 70T + 600$, where T represents the temperature in degrees Celsius and $3 \le T \le 12$. Once the food is removed from refrigeration, the temperature, $T(t)$, is given by $T(t) = 3.5t + 3$, where t is the time in hours and $0 \le t \le 3$.

a) Write the expression for the number of bacteria in the food, t hours after it is removed from refrigeration.

$$B(t) = (B \circ T)(t)$$
$$= B(T(t))$$
$$= B(3.5t+3)$$
$$= 15(3.5t+3)^2 - 70(3.5t+3) + 600$$
$$= 15(3.5t+3)(3.5t+3) - 70(3.5t+3) + 600$$
$$\vdots$$
$$\therefore B(t) = 183.75t^2 + 70t + 525$$

b) At 1.5 h, about how many bacteria are in the food?

$$B(1.5) = 183.75(1.5)^2 + 70(1.5) + 525$$
$$\doteq 1043$$

$\therefore$ there are approximately 1043 bacteria.

c) When will the bacteria count reach about 1200?

Find t when $B(t) = 1200$

$$183.75t^2 + 70t + 525 = 1200$$
$$183.75t^2 + 70t - 675 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-70 \pm \sqrt{(70)^2 - 4(183.75)(-675)}}{2(183.75)}$$

$$t = \frac{-70 \pm \sqrt{501025}}{367.5}$$

$$t = \frac{-70 - \sqrt{501025}}{367.5} \quad \text{or} \quad t = \frac{-70 + \sqrt{501025}}{367.5}$$

inadmissable $\because 0 \le t \le 3$

$t \doteq 1.74$

$\therefore$ the bacteria count will reach about 1200 after approximately 1 hour & 45 minutes

Ex. 2. A circle has radius *r*.

a) Write a function for the circle's area in terms of *r*.

Formula: $A = \pi r^2$ Function: $A(r) = \pi r^2$

b) Write a function for the radius in terms of the circumference, *C*.

$C = 2\pi r$

$r = \frac{C}{2\pi}$

Function: $r(C) = \frac{C}{2\pi}$

c) Determine $(A \circ r)(C)$.

$$\begin{aligned} &A(r(C)) \\ &= A\left(\frac{C}{2\pi}\right) \\ &= \pi\left(\frac{C}{2\pi}\right)^2 \\ &= \frac{\cancel{\pi}^{1}}{1} \cdot \frac{C^2}{4\cancel{\pi^2}^{1}} \end{aligned}$$

$\therefore A(C) = \frac{C^2}{4\pi}$

d) A tree's circumference is 3.6 m. What is the area of the cross section?

$$\begin{aligned} A(3.6) &= \frac{(3.6)^2}{4\pi} \\ &\doteq 1.03 \end{aligned}$$

$\therefore$ the area is approximately $1.03\ m^2$.

Ex. 3. A spherical hailstone grows in a cloud. The hailstone maintains a spherical shape while its radius increases at a rate of 0.2 mm/min.

a) Express the radius, *r*, in millimetres, of the hailstone, as a function of the time, *t*, in minutes.

$r(t) = 0.2t$

b) Express the volume, *V*, in cubic millimetres, of the hailstone, in terms of *r*.

$V = \frac{4}{3}\pi r^3$ or $V(r) = \frac{4}{3}\pi r^3$

c) Determine $(V \circ r)(t)$ and explain what it means.

$$\begin{aligned} V(t) &= V(r(t)) \\ &= V(0.2t) \\ &= \frac{4}{3}\pi\left(\frac{1}{5}t\right)^3 \\ &= \frac{4}{3}\pi \cdot \frac{1}{125}t^3 \end{aligned}$$

$\therefore V(t) = \frac{4\pi}{375}t^3$

gives the volume of the hailstone after *t* minutes.

d) What is the volume of the hailstone 1 h after it begins to form?

$$\begin{aligned} V(60) &= \frac{4\pi}{375}(60)^3 \\ &\doteq 7238\ mm^3 \end{aligned}$$

$\therefore$ the hailstone is approx. $7238\ mm^3$.

$7238\ mm^3 \div 10^3$
$= 7.238\ cm^3$

HW. Worksheet on Applications of Composite Functions

Date:______________ **WARMUP**

Ex. 1. Given $f(x)=\dfrac{1}{x}$ and $g(x)=\sqrt{x+2}$, find

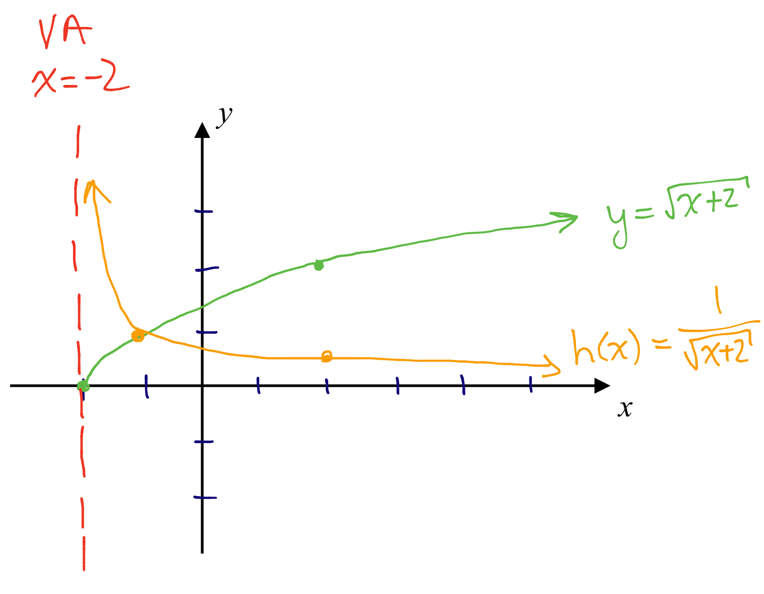
a) $h(x)=(f\circ g)(x)$, graph and state the domain and range.

$h(x)=f(g(x))$

$h(x)=f(\sqrt{x+2})$

$h(x)=\dfrac{1}{\sqrt{x+2}}$

① Graph $y=\sqrt{x+2}$



$D_h: \{x\in\mathbb{R} \mid x>-2\}$

$R_h: \{y\in\mathbb{R} \mid y>0\}$

b) x, if $f(x)=g(x)$ and illustrate your solution graphically.

$\dfrac{1}{x}=\sqrt{x+2}$

Square both sides

$\dfrac{1}{x^2}=\dfrac{x+2}{1}$

$1=x^3+2x^2$

$0=x^3+2x^2-1$

$0=(x+1)(x^2+x-1)$

$x+1=0$ $\quad$ $x=\dfrac{-1\pm\sqrt{(1)^2-4(1)(-1)}}{2(1)}$

$x=-1$ $\quad$ $x=\dfrac{-1\pm\sqrt{5}}{2}$

$x=\dfrac{-1+\sqrt{5}}{2}$ or $x=\dfrac{-1-\sqrt{5}}{2}$

$\therefore x=-1,\ \dfrac{-1+\sqrt{5}}{2},\ \dfrac{-1-\sqrt{5}}{2}$

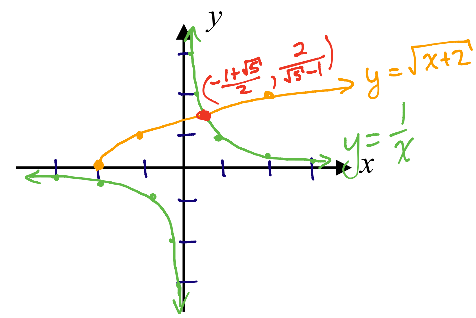
extraneous roots

Let $p(x)=x^3+2x^2-1$

$\because p(-1)=0$

$\therefore x+1$ is a factor

$$\begin{array}{r}
x^2+x-1\\
x+1\enclose{longdiv}{x^3+2x^2+0x-1}\\
\underline{x^3+x^2}\\
x^2+0x\\
\underline{x^2+x}\\
-x-1\\
\underline{-x-1}
\end{array}$$



$\therefore$ the solution for $f(x)=g(x)$ is $x=\dfrac{-1+\sqrt{5}}{2}$

Ex. 2. Given $f(x)=\dfrac{x}{x+2}$ and $g(x)=2x+1$, determine:

a) $g^{-1}(x)$

Let $y=2x+1$.
For the inverse,

$$x=2y+1$$
$$x-1=2y$$
$$y=\frac{x-1}{2}$$
$$\boxed{\therefore g^{-1}(x)=\frac{x-1}{2}}$$

b) $f^{-1}(x)$

Let $y=\dfrac{x}{x+2}$
For the inverse:

$$\frac{x}{1}=\frac{y}{y+2}$$
$$x(y+2)=y$$
$$xy+2x=y$$
$$xy-y=-2x$$
$$y(x-1)=-2x$$
$$y=\frac{-2x}{x-1}$$
$$\boxed{\therefore f^{-1}(x)=\frac{-2x}{x-1}}$$

c) $(g\circ f)(x)$

$$g(f(x))=g\left(\frac{x}{x+2}\right)$$
$$=2\left(\frac{x}{x+2}\right)+1$$
$$=\frac{2x}{x+2}+\frac{x+2}{x+2}$$
$$\boxed{\therefore (g\circ f)(x)=\frac{3x+2}{x+2}}$$

d) $(g\circ f)^{-1}(x)$

Let $y=\dfrac{3x+2}{x+2}$
For the inverse:

$$\frac{x}{1}=\frac{3y+2}{y+2}$$
$$xy+2x=3y+2$$
$$xy-3y=2-2x$$
$$y(x-3)=2-2x$$
$$y=\frac{2-2x}{x-3}$$
$$\boxed{\therefore (g\circ f)^{-1}(x)=\frac{2-2x}{x-3}}$$
or
$$\boxed{(g\circ f)^{-1}(x)=\frac{2x-2}{3-x}}$$

Ex. 3. A company produces a product for \$9.45 per unit, plus a fixed operating cost of \$52 000. The company sells the product for \$15.80 per unit.

a) Determine a function, $C(x)$, to represent the cost of producing x units, in dollars.

$$C(x)=9.45x+52000$$

b) Determine a function, $R(x)$, to represent the revenue generated by selling x units, in dollars.

$$R(x)=15.8x$$

c) Determine a function, $P(x)$, that represents profit.

$$P(x)=R(x)-C(x)$$
$$P(x)=15.8x-(9.45x+52000)$$
$$\therefore P(x)=6.35x-52000$$

HW. Using your answers to **Ex. 2.**, show that $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$.
More Applications of Composite Functions Worksheet