**Date:** Jan 5/15

# UNIT 8: SECANTS, TANGENTS & RATES OF CHANGE
## REVIEW OF PREREQUISITE SKILLS FOR UNIT 8

### I Review of Slopes and Equations of Lines

**Slope:** The **slope** is the measure of the steepness of a line.

$$slope = \frac{rise}{run}$$

$$m = \frac{\Delta y}{\Delta x}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Equation of a Line: i) slope y-intercept form or ii) standard form**

$y = mx + b$ $\qquad\qquad$ $Ax + By + C = 0,\ A,\ B,\ C \in I,\ A > 0$

**Note:**

**i)** A ***vertical line*** has a slope that is undefined and an equation of the form $x = \#$.

**ii)** A ***horizontal line*** has a slope that is 0 and an equation of the form $y = \#$.

**iii)** ***Parallel lines*** have slopes that are the same.

**iv)** ***Perpendicular lines*** have slopes that are negative reciprocals.

**Ex. 1.** Find the equation of the line determined by the given information.

**a)** slope $-2$, y-intercept 3

$m = -2,\ b = 3$

$\therefore y = -2x + 3$ is the required equation.

**b)** horizontal, through $(-2, 5)$

$\therefore y = 5$ is the required equation

**c)** perpendicular to $2x - 3y - 6 = 0$ & having an $x$-intercept of $-2$

① $2x - 3y - 6 = 0$

$-3y = -2x + 6$

$y = \frac{2}{3}x - 2$

$m = \frac{2}{3}$

$\boxed{\therefore m_\perp = -\frac{3}{2}}$

$m_{perp.} = -\frac{3}{2}$

② $m = -\frac{3}{2},\ b = \_\_$

$x = -2,\ y = 0$

$0 = -\frac{3}{\cancel{2}_1}(-\cancel{2}^1) + b$

$0 = 3 + b$

$\boxed{\therefore b = -3}$

$\therefore y = -\frac{3}{2}x - 3$ is the equation.

**d)** through $(-2, 4)$ & $(-6, 6)$

① $m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{6 - 4}{-6 - (-2)}$

$m = \frac{2}{-4}$

$\boxed{\therefore m = -\frac{1}{2}}$

② $m = -\frac{1}{2},\ b = \_\_$

$x = -2\ \ y = 4$

Find $b$:

$4 = -\frac{1}{2}(-2) + b$

$4 = 1 + b$

$\boxed{3 = b}$

$\therefore y = -\frac{1}{2}x + 3$ is the equation.

**II Rationalizing the Denominator or Numerator**

A ***rational number*** either repeats or terminates in its decimal form.
An ***irrational number*** neither repeats nor terminates in its decimal form.

**Ex. 2.** Rationalize each ***denominator***.

**a)** $\dfrac{1+2\sqrt{2}}{3\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{\sqrt{2}+4}{6}$

**b)** $\dfrac{\sqrt{3}}{1-2\sqrt{3}} \cdot \dfrac{1+2\sqrt{3}}{1+2\sqrt{3}}$

$(a-b)(a+b) = a^2-b^2$

$= \dfrac{\sqrt{3}+6}{1+\cancel{2\sqrt{3}}-\cancel{2\sqrt{3}}-12}$

$= \dfrac{\sqrt{3}+6}{1-12}$

$= -\dfrac{\sqrt{3}+6}{11}$ or $\dfrac{-\sqrt{3}-6}{11}$

**Ex. 3.** Rationalize each ***numerator***.

**a)** $\dfrac{\sqrt{3}}{1-2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{3}{\sqrt{3}-6}$

**b)** $\dfrac{1+2\sqrt{2}}{3\sqrt{2}} \cdot \dfrac{1-2\sqrt{2}}{1-2\sqrt{2}}$

$= \dfrac{1-8}{3\sqrt{2}-12}$

$= \dfrac{-7}{3\sqrt{2}-12}$ or $\dfrac{7}{12-3\sqrt{2}}$

**Ex. 4.** Write an equivalent expression for **a)** by rationalizing the ***numerator*** and for **b)** by rationalizing the ***denominator***.

**a)** $\dfrac{\sqrt{9+h}-3}{h} \cdot \dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$

$= \dfrac{(9+h)-9}{h(\sqrt{9+h}+3)}$

$= \dfrac{\cancel{h}^{\,1}}{\cancel{h}_{1}(\sqrt{9+h}+3)}$

$= \dfrac{1}{\sqrt{9+h}+3}$

**b)** $\dfrac{x-3}{2-\sqrt{x+1}} \cdot \dfrac{2+\sqrt{x+1}}{2+\sqrt{x+1}}$

$= \dfrac{(x-3)(2+\sqrt{x+1})}{4-(x+1)}$

$= \dfrac{(x-3)(2+\sqrt{x+1})}{4-x-1}$

$= \dfrac{(x-3)(2+\sqrt{x+1})}{-x+3}$

$= \dfrac{\cancel{(x-3)}^{\,1}(2+\sqrt{x+1})}{-\cancel{(x-3)}_{1}}$

$= -2-\sqrt{x+1}$

## III Simplifying Rational Expressions

**Ex. 5.** Simplify each of the following.

**a)** $\dfrac{x^2-16}{x^3+64}$

$= \dfrac{(x-4)\cancel{(x+4)}^{1}}{\cancel{(x+4)}_{1}(x^2-4x+16)}$

$= \dfrac{x-4}{x^2-4x+16}$

**b)** $\left(\dfrac{1}{x-1}\right)\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$

$= \left(\dfrac{1}{x-1}\right)\left[\dfrac{1(3x+5)-2(x+3)}{(x+3)(3x+5)}\right]$

$= \left(\dfrac{1}{x-1}\right)\left[\dfrac{3x+5-2x-6}{(x+3)(3x+5)}\right]$

$= \left(\dfrac{1}{\cancel{x-1}_{1}}\right)\left[\dfrac{\cancel{x-1}^{1}}{(x+3)(3x+5)}\right]$

$= \dfrac{1}{(x+3)(3x+5)}$

or

$\boxed{= \dfrac{1}{3x^2+14x+15}}$

**c)** $\dfrac{\dfrac{1}{2+h}-\dfrac{1}{2}}{h} \cdot \dfrac{2(2+h)}{2(2+h)}$

$= \dfrac{2-(2+h)}{2h(2+h)}$

$= \dfrac{-\cancel{h}^{1}}{2\cancel{h}_{1}(2+h)}$

$= \dfrac{-1}{2(2+h)}$

**d)** $\dfrac{(3+h)^3-27}{h}$

expand $(3+h)(3+h)(3+h)$

$= \dfrac{[(3+h)-3][(3+h)^2+3(3+h)+9]}{h}$

$= \dfrac{\cancel{h}^{1}(h^2+6h+9+9+3h+9)}{\cancel{h}_{1}}$

$= h^2+9h+27$ ✓

**HW. Day 1 pg. 83 #1 to 5 all and #6 below.**

**6.** Rationalize each of the following denominators to obtain an equivalent expression.

**a)** $\dfrac{x}{2-\sqrt{4+x}}$ **b)** $\dfrac{x}{\sqrt{x+5}-\sqrt{5-x}}$ **c)** $\dfrac{2x-4}{\sqrt{7x+2}-\sqrt{6x+4}}$

***Answers***

**6. a)** $-2-\sqrt{4+x}$ **b)** $\dfrac{\sqrt{x+5}+\sqrt{5-x}}{2}$ **c)** $2\left(\sqrt{7x+2}+\sqrt{6x+4}\right)$

**Date:** Jan 6/15

# SLOPES OF LINES GIVEN TWO POINTS

**Recall**

| *slope formula* | *slope, y-intercept form* | *standard form* |
|---|---|---|
| $m=\dfrac{y_2-y_1}{x_2-x_1}$ | $y=mx+b$ | $Ax+By+C=0$ |

**Ex. 1.** Find the slope $m$, in simplified form, of each pair of points.

**a)** $(-3,6)$ and $(6,0)$

$m=\dfrac{y_2-y_1}{x_2-x_1}$

$m=\dfrac{0-6}{6-(-3)}$

$m=\dfrac{-6}{9}$

$\boxed{\therefore m=-\dfrac{2}{3}}$

**b)** $\left(\dfrac{3}{4},\dfrac{1}{4}\right)$ and $\left(\dfrac{7}{4},-\dfrac{3}{4}\right)$

$m=\dfrac{-\frac{3}{4}-\frac{1}{4}}{\frac{7}{4}-\frac{3}{4}}$

$m=\dfrac{-1}{1}$

$\boxed{\therefore m=-1}$

**Ex. 2.** Find the equation of the line determined by the given information.

**a)** parallel to $x+4y-2=0$ & through $(-2,5)$

① $x+4y-2=0$

$4y=-x+2$

$y=-\frac{1}{4}x+\frac{1}{2}$

$m=-\frac{1}{4}$

$\therefore m_{parallel}=-\frac{1}{4}$

$(m_{\parallel}=-\frac{1}{4})$

② $m=-\frac{1}{4}$, $b=$ ___

$x=-2$, $y=5$

Find $b$: $5=-\frac{1}{4}(-2)+b$

$5=\frac{1}{2}+b$

$b=\frac{9}{2}$

$\therefore y=-\frac{1}{4}x+\frac{9}{2}$ is the equation

$(x+4y-18=0)$

**b)** through $(-5, 3)$ & $(-1, 2)$

① Find $m$:

$m=\dfrac{2-3}{-1-(-5)}$

$m=\dfrac{-1}{4}$

② $m=-\frac{1}{4}$, $b=$ ___

$x=-1$, $y=2$

Find $b$:

$2=-\frac{1}{4}(-1)+b$

$2=\frac{1}{4}+b$

$b=\frac{7}{4}$

$\therefore y=-\frac{1}{4}x+\frac{7}{4}$ is the equation. $(x+4y-7=0)$

**Ex. 3.** Find the slope $m$, in simplified form, of each pair of points.

**a)** $P(\underset{x_1}{1},\ \underset{y_1}{3})$, $Q\left(\underbrace{1+h}_{x_2},\ \underbrace{3(1+h)^2}_{y_2}\right)$

$m=\dfrac{3(1+h)^2-3}{1+h-1}$

$m=\dfrac{3(1+h)(1+h)-3}{h}$

$m=\dfrac{3(1+2h+h^2)-3}{h}$

$m=\dfrac{3h^2+6h+3-3}{h}$

$m=\dfrac{3h^2+6h}{h}$

$\boxed{\therefore m=3h+6}$

**b)** $P(9, 3)$, $Q\left(9+h,\ \sqrt{9+h}\right)$

$m=\dfrac{\sqrt{9+h}-3}{9+h-9}$

$m=\dfrac{\sqrt{9+h}-3}{h}\cdot\dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$

$m=\dfrac{(9+h)-9}{h(\sqrt{9+h}+3)}$

$m=\dfrac{\cancel{h}^{\,1}}{\cancel{h}_{\,1}(\sqrt{9+h}+3)}$

$m=\dfrac{1}{\sqrt{9+h}+3}$

**c)** $P(-2, 2),\ Q\left(-2+h,\ \dfrac{4}{(-2+h)+4}\right)$

$$m = \frac{\frac{4}{h+2} - 2}{-2+h-(-2)} \cdot \frac{h+2}{h+2}$$

$$m = \frac{4-2(h+2)}{h(h+2)}$$

$$m = \frac{4-2h-4}{h(h+2)}$$

$$m = \frac{-2\cancel{h}^{1}}{{}^{1}\cancel{h}(h+2)}$$

$$\therefore m = \frac{-2}{h+2}$$

**d)** $P(2,5),\ Q\left(2+h,\ (2+h)^3-(2+h)^2+1\right)$

$$m = \frac{(2+h)^3-(2+h)^2+1 \ -5}{2+h \ -2}$$

$$m = \frac{^{**}(2+h)(2+h)(2+h) - (2+h)(2+h) - 4}{h}$$

$$m = \frac{(h^3+6h^2+12h+8)-(h^2+4h+4)-4}{h}$$

$$m = \frac{h^3+5h^2+8h}{h}$$

$$m = h^2+5h+8$$

$$** \quad (2+h)(2+h)(2+h)$$
$$= (2+h)(4+4h+h^2)$$
$$= 8+8h+2h^2+4h+4h^2+h^3$$
$$= h^3+6h^2+12h+8$$

**H.W. Day 2 pg. 72 #1 odd parts, 2 odd parts and #3 below.**

**3.** Find the slope $m$, in simplified form, of each pair of points.

**a)** $P(-2,0),\ Q\left(-2+h, 4-(-2+h)^2\right)$

**b)** $P(1,4),\ Q\left(1+h, (1+h)^2-6(1+h)+9\right)$

**c)** $P(-1,2),\ Q\left(-1+h, \sqrt{3-(-1+h)}\right)$

**d)** $P\left(-3,-\dfrac{1}{3}\right),\ Q\left(-3+h, \dfrac{1}{-3+h}\right)$

**e)** $P(2,5),\ Q\left(2+h, \dfrac{2(2+h)+1}{(2+h)-1}\right)$

**f)** $P(1,1),\ Q\left(1+h, \dfrac{1}{\sqrt{1+h}}\right)$

***Answers***

**3. a)** $4-h$ **b)** $h-4$ **c)** $\dfrac{-1}{\sqrt{4-h}+2}$ **d)** $\dfrac{1}{3(h-3)}$ **e)** $\dfrac{-3}{1+h}$ **f)** $\dfrac{-1}{\left(\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)}$

**Date:** Jan 7/15

**SLOPES OF SECANTS AND TANGENTS**
**AVERAGE AND INSTANTANEOUS RATES OF CHANGE**

**Definitions:**

**Secants and Average Rates of Change**

A ***secant*** is a line that passes through two points on the graph of a function $y = f(x)$.
The ***average rate of change*** of $y$ with respect to $x$ is the ***slope of the secant*** between those points.

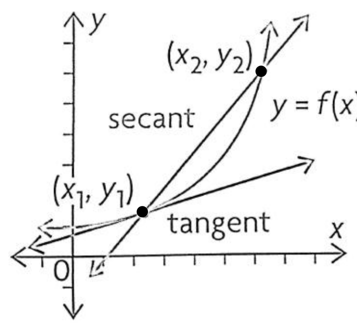
$$\textit{average rate of change} = m_{\text{secant}}$$

**Tangents and Instantaneous Rates of Change**

A ***tangent*** is a line that touches the graph of a function $y = f(x)$ at exactly one point.
The tangent is the straight line that most resembles the graph near that point.
The ***instantaneous rate of change*** of $y$ with respect to $x$ is the ***slope of the tangent*** at that point.

$$\textit{instantaneous rate of change} = m_{\text{tangent}}$$

$$= \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

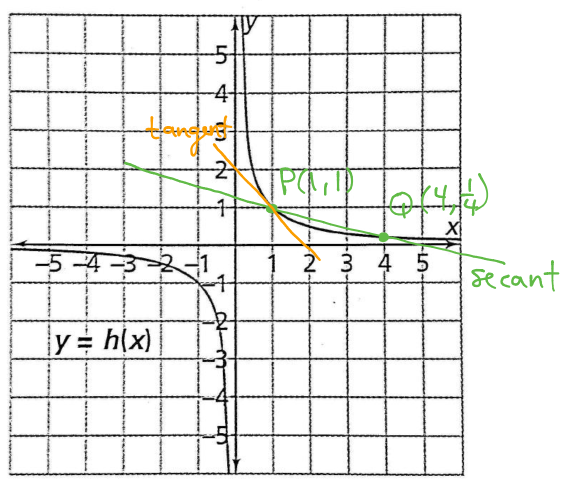



**Ex. 1.** Given the graph of $h(x) = \frac{1}{x}$,

**a)** draw the secant line that passes through $P(1,1)$ and $Q\left(4, \frac{1}{4}\right)$ and calculate its slope.

$$m = \frac{\frac{1}{4} - 1}{4 - 1} = \frac{-\frac{3}{4}}{3} = -\frac{\cancel{3}^1}{4} \cdot \frac{1}{\cancel{3}_1} = -\frac{1}{4}$$

$$\therefore m_{PQ} = -\frac{1}{4}$$




**b)** draw the tangent line to the curve at $P(1,1)$ and use the graph to estimate its slope.

$$m_t = \frac{-1}{1}$$

$$\therefore m_t = -1$$

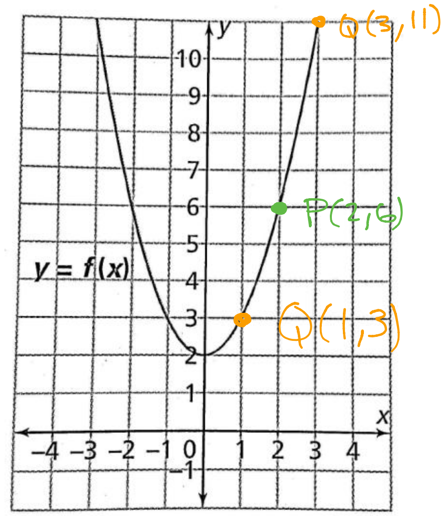
**c)** determine the equation of the tangent line to the curve at $P(1,1)$.

$m_t = -1$, $b = \underline{2}$, $(x = 1, y = 1)$

$\therefore y = -x + 2$ is the equation of the tangent.

**Ex. 2.** Consider the function $f(x) = x^2 + 2$.

**a)** Complete the following tables to estimate the slope of the tangent to $f(x) = x^2 + 2$ at point $P(2, 6)$.




**i)** *Q approaches P from the right, ie.* $Q \to P^+$

| $P$ | $Q$ | Slope of Line $PQ$ |
|---|---|---|
| (2, 6) | (3, 11) | 5 |
| (2, 6) | (2.5, 8.25) | 4.5 |
| (2, 6) | (2.1, 6.41) | 4.1 |
| (2, 6) | (2.01, 6.0401 | 4.01 |

**ii)** *Q approaches P from the left, ie.* $Q \to P^-$

| $P$ | $Q$ | Slope of Line $PQ$ |
|---|---|---|
| (2, 6) | (1, 3) | 3 |
| (2, 6) | (1.5, 4.25) | 3.5 |
| (2, 6) | (1.9, 5.61) | 3.9 |
| (2, 6) | (1.99, 5.9601) | 3.99 |

***Conclusions:*** The slope of the tangent at P(2, 6) is 4.

**b)** Let $Q$ be a point on the curve $h$ units to the right of $P$ and then calculate the slope of the secant $PQ$.

$f(x) = x^2 + 2$

$f(2+h) = (2+h)^2 + 2$

$= h^2 + 4h + 6$

$P(2,6)$, $Q(2+h, h^2+4h+6)$

$$m_{PQ} = \frac{h^2+4h+6-6}{2+h-2}$$

$$m_{PQ} = \frac{h^2+4h}{h}$$

$$\boxed{\therefore m_{PQ} = h+4} \leftarrow \text{slope of the secant.}$$

**c)** Use the result of part **b)** to calculate the slope of the tangent to the graph of $f(x)$ at point $P$.

For tangent : $Q \to P^+$, so $h \to 0$

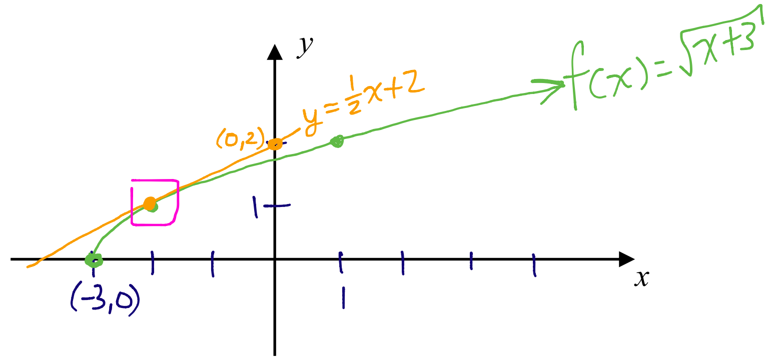
as $h \to 0$, $m_{PQ} \to 4$

$\therefore m_t = 4$

**Ex. 3.** For each curve,

**i)** find the slope of the tangent at the given point
**ii)** find the equation of the tangent at the given point
**iii)** graph the curve and the tangent

**a)** $f(x)=\sqrt{x+3}$ at $P(-2,1)$




i) Let Q be a point on the curve h units to the right of P.

$P(-2,1)$ ; $Q(-2+h, \sqrt{h+1})$

For secant:

$$m_{PQ}=\frac{\sqrt{h+1}-1}{-2+h-(-2)}\cdot\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$

$$m_{PQ}=\frac{(h+1)-1}{h(\sqrt{h+1}+1)}$$

$$m_{PQ}=\frac{\cancel{h}\,1}{\cancel{h}(\sqrt{h+1}+1)}$$

$$\therefore m_{PQ}=\frac{1}{\sqrt{h+1}+1}$$

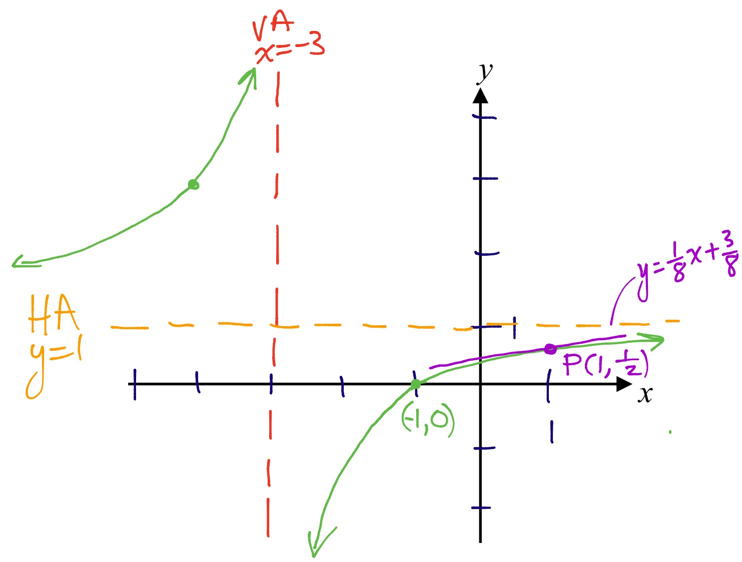
For tangent:
$Q\to P^+$, so $h\to 0$
as $h\to 0$, $m_{PQ}\to\frac{1}{2}$
$\therefore m_t=\frac{1}{2}$

ii) $m_t=\frac{1}{2}$, $b=$___, $x=-2$, $y=1$

Find b: $1=\frac{1}{2}(-2)+b$
$1=-1+b$
$\therefore b=2$

$\therefore$ the equation of the tangent at $P(-2,1)$ is $y=\frac{1}{2}x+2$

**b)** $f(x)=\dfrac{x+1}{x+3}$ at $P\left(1,\dfrac{1}{2}\right)$




Let Q be a point on the curve h units to the right of P.

$P(1,\frac{1}{2})$ ; $Q(1+h, \frac{h+2}{h+4})$

For secant:

$$m_{PQ}=\frac{\frac{h+2}{h+4}-\frac{1}{2}}{h}\cdot\frac{2(h+4)}{2(h+4)}$$

$$m_{PQ}=\frac{2(h+2)-(h+4)}{2h(h+4)}$$

$$m_{PQ}=\frac{2h+4-h+4}{2h(h+4)}$$

$$m_{PQ}=\frac{\cancel{h}\,1}{2\cancel{h}(h+4)}$$

$$\boxed{\therefore m_{PQ}=\frac{1}{2(h+4)}}$$

For tangent:
$Q\to P^+$, so $h\to 0$
as $h\to 0$, $m_{PQ}\to\frac{1}{8}$ $\boxed{\therefore m_t=\frac{1}{8}}$

ii) $m_t=\frac{1}{8}$, $b=$___
$x=1$, $y=\frac{1}{2}$

Find b:
$\frac{1}{2}=\frac{1}{8}(1)+b$
$\frac{4}{8}-\frac{1}{8}=b$
$\boxed{\therefore b=\frac{3}{8}}$

$\therefore y=\frac{1}{8}x+\frac{3}{8}$ is the equation of the tangent at $(1,\frac{1}{2})$.

**HW. Worksheet on Secants and Tangents #1 to 7**