Date: Jan 8/15

APPLICATIONS OF RATES OF CHANGE

Recall:

Secants and Average Rates of Change

A ***secant*** is a line that passes through two points on the graph of a function $y = f(x)$.
The ***average rate of change*** of y with respect to x is the ***slope of the secant*** between those points.

$$average\ rate\ of\ change = m_{\text{secant}}$$

$$= \frac{change\ in\ y}{change\ in\ x}$$

$$= \frac{\Delta y}{\Delta x}$$

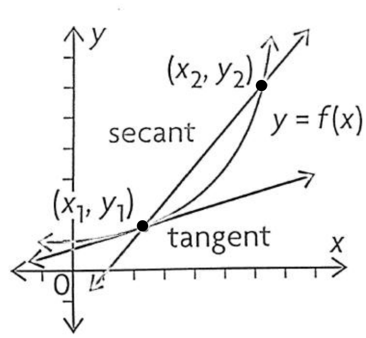
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Tangents and Instantaneous Rates of Change

A ***tangent*** is a line that touches the graph of a function $y = f(x)$ at exactly one point.
The tangent is the straight line that most resembles the graph near that point.
The ***instantaneous rate of change*** of y with respect to x is the ***slope of the tangent*** at that point.

$$instantaneous\ rate\ of\ change = m_{\text{tangent}}$$



Ex. 1. Find the slope and equation of the tangent to $f(x) = x - x^3$ at $x = -1$. Illustrate graphically.

** $f(-1+h) = (-1+h) - (-1+h)^3$
$= (h-1) - (h-1)(h^2-2h+1)$
$= (h-1) - (h^3-3h^2+3h-1)$
$= -h^3 + 3h - 2h$

i) Let Q be a point on the curve h units to the right of P.

$P(-1,0)$; $Q(-1+h, f(-1+h))$ **

$P(-1,0)$; $Q(-1+h, -h^3+3h^2-2h)$

For the secant:

$$m_{PQ} = \frac{-h^3+3h^2-2h}{h}$$

$$\boxed{\therefore m_{PQ} = -h^2+3h-2}$$

For the tangent:

$Q \to P^+$, so $h \to 0$

as $h \to 0$, $m_{PQ} \to -2$

$\therefore m_t = -2$

ii) $m_t = -2$, $b =$ ____

$x = -1$, $y = 0$

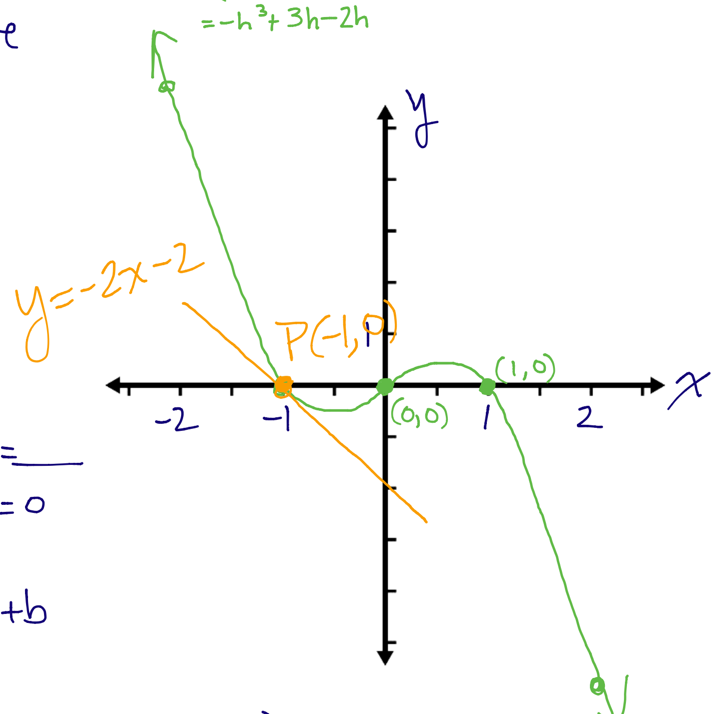
Find b:

$0 = -2(-1) + b$

$0 = 2 + b$

$\therefore b = -2$

$\therefore$ the equation of the tangent at $(-1,0)$ is $y = -2x - 2$.



iii) Graph f(x):

$f(x) = x(1-x^2)$
$= x(1-x)(1+x)$

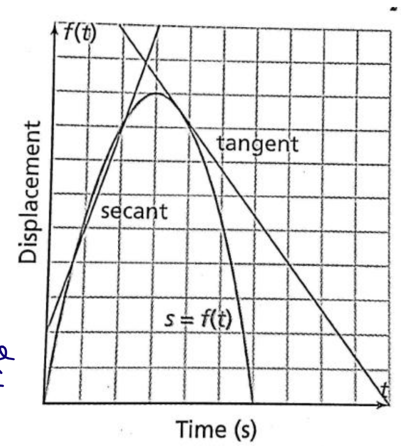
$\therefore$ x-ints are 0, 1, -1.

x	$f(x)$
-2	6
2	-6

Ex. 2. A cliff diver in Acapulco, Mexico, dives from about 17 m above the water. The function $s(t) = -4.9t^2 + 1.5t + 17$ models the diver's height above the water, in metres, at t seconds.

a) Determine the diver's average rate of descent with respect to time during each of the first two seconds.

slope of secant.



Displacement versus time

t	$s(t)$
0	17
1	13.6
2	0.4

1st second (0 to 1); 2nd second (1 to 2)

i) $m_{secant} = \dfrac{s(1) - s(0)}{1 - 0}$

$V_{avg} = \dfrac{13.6 - 17}{1 - 0}$

$V_{avg} = -3.4$

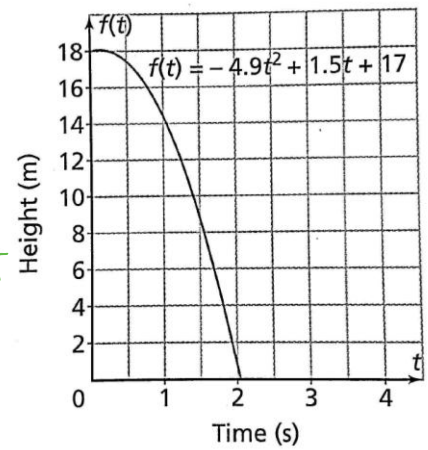
$\therefore$ during the 1st second, the average rate of descent is -3.4 m/s.

ii) $m_{secant} = \dfrac{s(2) - s(1)}{2 - 1}$

$V_{avg} = \dfrac{0.4 - 13.6}{1}$

$V_{avg} = -13.2$

$\therefore$ during the 2nd second the average rate of descent is -13.2 m/s.



Diver's height versus time

b) Determine the diver's instantaneous rate of descent with respect to time at 1 second.

slope of tangent

Find m_t at $t = 1$:

Let Q be a point on the curve h units to the right of P.

$P(1, 13.6)$; $Q(1+h, s(1+h))$ **

$P(1, 13.6)$; $Q(1+h, -4.9h^2 - 8.3h + 13.6)$

** $s(1+h) = -4.9(1+h)^2 + 1.5(1+h) + 17$

$= -4.9(h^2 + 2h + 1) + 1.5(1+h) + 17$

$= -4.9h^2 - 9.8h - 4.9 + 1.5 + 1.5h + 17$

$= -4.9h^2 - 8.3h + 13.6$

For secant:

$m_{PQ} = \dfrac{-4.9h^2 - 8.3h + 13.6 - 13.6}{h}$

$m_{PQ} = \dfrac{-4.9h^2 - 8.3h}{h}$

$\therefore m_{PQ} = -4.9h - 8.3$

For tangent:

$Q \to P^+$, so $h \to 0$

as $h \to 0$, $m_{PQ} \to -8.3$

$\therefore m_t = -8.3$

$\therefore$ at 1 second the divers instantaneous rate descent is -8.3 m/s.

HW. Worksheet on Tangents and Applications of Rates of Change #1 ace, 2, 5

Date: Jan 9/15 **WARMUP**

1. Find the equation of the tangent to $f(x)=\frac{1}{x^2-1}$ at $x=2$. Illustrate your solution graphically.

To Graph $f(x)=\frac{1}{x^2-1}$

VA: $x^2-1=0$

$(x-1)(x+1)=0$

$\therefore x=1$ & $x=-1$ are the V.A.

x-int: none

y-int: $f(0)=-1$

$\therefore$ y-int is -1.

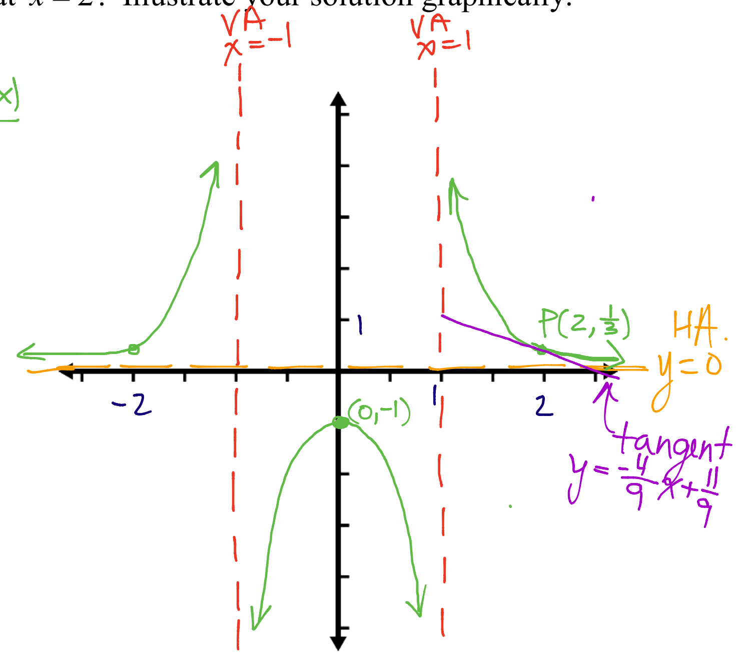
HA: $f(x)=\frac{1}{x^2-1} \div \frac{x^2}{x^2}$

$=\frac{\frac{1}{x^2}}{1-\frac{1}{x^2}}$

As $x \to \pm\infty$, $f(x) \to 0$

$\therefore y=0$ is the H.A.

x	$f(x)$
-2	$\frac{1}{3}$
2	$\frac{1}{3}$



Let Q be a point on the curve, h units to the right of P.

$P(2, f(2))$; $Q(2+h, f(2+h))$

$P(2, \frac{1}{3})$; $Q(2+h, \frac{1}{h^2+4h+3})$

$*f(2+h)=\frac{1}{(2+h)^2-1}$

$=\frac{1}{h^2+4h+4-1}$

For secant:

$$m_{PQ}=\frac{\frac{1}{h^2+4h+3}-\frac{1}{3}}{h}\cdot\frac{3(h^2+4h+3)}{3(h^2+4h+3)}$$

$$m_{PQ}=\frac{3-(h^2+4h+3)}{3h(h^2+4h+3)}$$

$$m_{PQ}=\frac{-h^2-4h}{3h(h^2+4h+3)}$$

$$\boxed{\therefore m_{PQ}=\frac{-h-4}{3(h^2+4h+3)}}$$

For tangent:

$Q \to P^+$, so $h \to 0$

as $h \to 0$, $m_{PQ} \to -\frac{4}{9}$

$$\boxed{\therefore m_t=-\frac{4}{9}}$$

$m_t=-\frac{4}{9}$, $b=$ ___

$x=2$, $y=\frac{1}{3}$

Find b:

$\frac{1}{3}=-\frac{4}{9}(2)+b$

$\frac{3}{9}=-\frac{8}{9}+b$

$\frac{11}{9}=b$

$\boxed{\therefore y=-\frac{4}{9}x+\frac{11}{9} \text{ is the equation of the tangent at } x=2.}$

2. A motorboat coasts toward a dock with its engine off. Its distance s, in metres, from the dock t seconds after the engine is turned off is $s(t) = \dfrac{10(6-t)}{t+3}$ for $0 \le t \le 6$.

a) Find the average velocity for $t \in [0,6]$. ← interval notation

b) Find the boat's velocity when it bumps into the dock..

a) $m_{secant} = \dfrac{s(6)-s(0)}{6-0}$

$= \dfrac{0-20}{6}$

$= -\dfrac{10}{3}$

∴ the average velocity is $-\frac{10}{3}$ m/s (or $-3\frac{1}{3}$ m/s)

b) Find m_t at $t=6$:

Let Q be a point on the curve, h units to the right of P.

$P(6,0)$, $Q(6+h, s(6+h))$

or $P(6,0)$; $Q\left(6+h, \frac{-10h}{h+9}\right)$

For secant:

$m_{PQ} = \dfrac{\frac{-10h}{h+9} - 0}{h} \cdot \dfrac{h+9}{h+9}$

$m_{PQ} = \dfrac{-10h}{h(h+9)}$

$\boxed{\therefore m_{PQ} = \dfrac{-10}{h+9}}$

For tangent:

$Q \to P^+$, so $h \to 0$

as $h \to 0$, $m_{PQ} \to \frac{-10}{9}$

$\therefore m_t = -\frac{10}{9}$

∴ the boat's velocity when it bumps the dock is $-1\frac{1}{9}$ m/s.

3. It can be shown that from a height of s metres above ground level, a person can see a distance, d kilometres to the horizon, where $d(s) = 3.53\sqrt{s}$.

a) When the elevator of the CN Tower passes the 225 m height, how far can the passengers see across Lake Ontario?

b) Find the rate of change of this distance to the horizon with respect to height, when the height of the elevator is 225 m

a) $d(225) = 3.53\sqrt{225}$

$\doteq 52.95$

∴ at a height of 225m passengers can see 52.95 km across the Lake.

b) Find m_t at $s = 225$

Let Q be a point on the curve, h units to the right of P.

$P(225, 52.95)$, $Q(225+h, 3.53\sqrt{225+h})$

For secant:

$m_{PQ} = \dfrac{3.53\sqrt{225+h} - 52.95}{h} \cdot \dfrac{3.53\sqrt{225+h} + 52.95}{3.53\sqrt{225+h} + 52.95}$

$m_{PQ} = \dfrac{12.4609(225+h) - 2803.7025}{h(3.53\sqrt{225+h} + 52.95)}$

$m_{PQ} = \dfrac{2803.7025 + 12.4609h - 2803.7025}{h(3.53\sqrt{225+h} + 52.95)}$

$\therefore m_{PQ} = \dfrac{12.4609}{3.53\sqrt{225h} + 52.95}$

For tangent:

$Q \to P^+$, so $h \to 0$

as $h \to 0$, $m_{PQ} \to \frac{353}{3000}$

$\therefore m_t \doteq 0.118$

∴ the distance is changing at a rate of 0.118 km/m at a height of 225m

HW. Worksheet on Tangents and Applications of Rates of Change #1bdf, 3, 4, 6

Date:___________ **EXPRESSIONS FOR SLOPES OF TANGENTS & RATES OF CHANGE**

Ex. 1. Given the function $y = -x^2 - 2x + 3$, find:

a) an expression for the slope of the tangent to the curve at any point (x, y) and illustrate graphically.
b) the slope of the tangent at $x = -3$ by using the formula obtained from part **a**.
c) the point on the curve where the tangent is horizontal.
d) the equation of the ***normal*** at $x = -3$.
Note: The ***normal*** is the line perpendicular to the tangent at the point of tangency.

a) Let $f(x) = -x^2 - 2x + 3$
Find m_t at $P(x, f(x))$
Let Q be a point on the curve, h units to the right of P.

$f(x+h) = -(x+h)^2 - 2(x+h) + 3$
$= -x^2 - 2xh - h^2 - 2x - 2h + 3$

$P(x, -x^2 - 2x + 3)$; $Q(x+h, -x^2 - 2xh - h^2 - 2x - 2h + 3)$

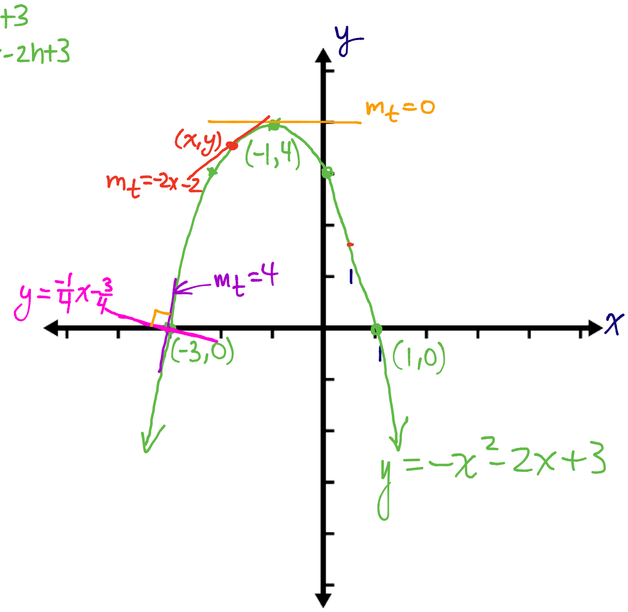
For secant:
$$m_{PQ} = \frac{(\cancel{-x^2} - 2xh - h^2 \cancel{- 2x} - 2h \cancel{+3}) - (\cancel{-x^2} \cancel{-2x} \cancel{+3})}{h}$$

$$m_{PQ} = \frac{-2xh - h^2 - 2h}{h}$$

$$\boxed{\therefore m_{PQ} = -2x - h - 2}$$

For tangent: $Q \to P^+$, so $h \to 0$
as $h \to 0$, $m_{PQ} \to -2x - 2$

$$\boxed{\therefore m_t = -2x - 2 \text{ for any point } (x, y) \text{ on the curve}}$$



b) at $x = -3$: $m_t = -2(-3) - 2$
$$\boxed{\therefore m_t = 4 \text{ at } x = -3}$$

c) If tangent is horizontal
$m_t = 0$
Find x if $m_t = 0$

$0 = -2x - 2$
$2x = -2$
$\therefore x = -1$

$\therefore$ the tangent is horizontal at the point $(-1, 4)$
vertex

d) at $x = -3$, $m_t = 4$ (from (b))

$m_n = -\frac{1}{4}$, $b =$ ___ , $x = -3$, $y = 0$

Find b:
$0 = -\frac{1}{4}(-3) + b$
$0 = \frac{3}{4} + b$
$\therefore b = -\frac{3}{4}$

$\therefore$ the equation of the normal at $x = -3$ is
$y = -\frac{1}{4}x - \frac{3}{4}$.

Ex. 2. An unoccupied test rocket meant for orbit is fired straight up from a launch pad. Something goes wrong and the rocket comes back down. The height of the rocket above the ground is modelled by $s(t) = 5t^2 - \frac{1}{6}t^3$, $t \geq 0$, where s is measured in metres and t, in seconds.

a) Sketch the graph of the function. (Scale the horizontal axis only.)
b) Find an expression for the **velocity** (slope of the tangent) at any time, t.
c) When is the velocity (slope of the tangent) zero?
d) Find the maximum height of the rocket.
e) At what velocity does the rocket hit the ground?

a) $s(t) = 5t^2 - \frac{1}{6}t^3$

For the t-intercepts, $s(t) = 0$

$5t^2 - \frac{1}{6}t^3 = 0$

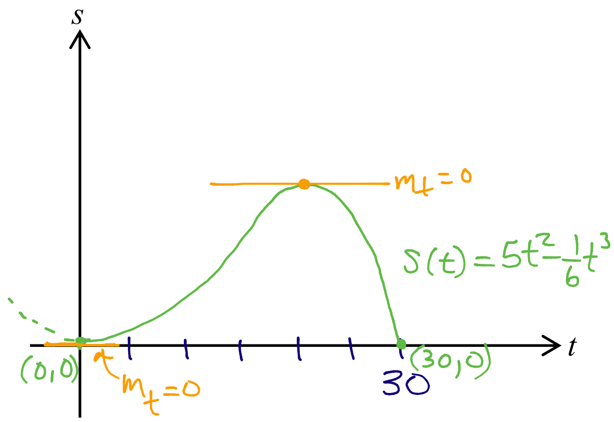
$\times 6)$ $30t^2 - t^3 = 0$

$\div -1)$ $t^3 - 30t^2 = 0$

$t^2(t-30) = 0$

$\therefore t = 0$ or $t = 30$

double single



$S(t+h) = 5(t+h)^2 - \frac{1}{6}(t+h)^3$

$= 5(t^2 + 2th + h^2) - \frac{1}{6}(t+h)(t^2 + 2th + h^2)$

$= 5t^2 + 10th + 5h^2 - \frac{1}{6}(t^3 + 3th^2 + 3t^2h + h^3)$

$= 5t^2 + 10th + 5h^2 - \frac{1}{6}t^3 - \frac{1}{2}th^2 - \frac{1}{2}t^2h - \frac{1}{6}h^3$

b) Find m_t at $P(t, s(t))$

Let Q be a point on the curve, h units to the right of P.

$P(t, 5t^2 - \frac{1}{6}t^3)$; $Q(t+h, 5t^2 + 10th + 5h^2 - \frac{1}{6}t^3 - \frac{1}{2}th^2 - \frac{1}{2}t^2h - \frac{1}{6}h^3)$

For Secant:

$$m_{PQ} = \frac{(\cancel{5t^2} + 10th + 5h^2 \cancel{- \frac{1}{6}t^3} - \frac{1}{2}th^2 - \frac{1}{2}t^2h - \frac{1}{6}h^3) - (\cancel{5t^2} \cancel{- \frac{1}{6}t^3})}{h}$$

$$m_{PQ} = \frac{10th + 5h^2 - \frac{1}{2}th^2 - \frac{1}{2}t^2h - \frac{1}{6}h^3}{h}$$

$$\boxed{\therefore m_{PQ} = 10t + 5h - \frac{1}{2}th - \frac{1}{2}t^2 - \frac{1}{6}h^2 \text{ at any time, } t \geq 0.}$$

For tangent: $Q \to P^+$, so $h \to 0$

as $h \to 0$, $m_{PQ} \to 10t - \frac{1}{2}t^2$

$\boxed{\therefore m_t = 10t - \frac{1}{2}t^2} \to \boxed{\therefore v(t) = 10t - \frac{1}{2}t^2 \text{ m/s.}}$

c) Find t when $m_t = 0$

$10t - \frac{1}{2}t^2 = 0$

$20t - t^2 = 0$

$t^2 - 20t = 0$

$t(t-20) = 0$

$\therefore t = 0$ or $t = 20$

$\therefore$ the velocity is zero at $t = 0$ sec or $t = 20$ sec.

d) $S(20) = 5(20)^2 - \frac{1}{6}(20)^3$

$= 666\frac{2}{3}$

$\therefore$ the rocket reaches a maximum height of $666\frac{2}{3}$ m.

e) using $v(t) = 10t - \frac{1}{2}t^2$

$V(30) = 10(30) - \frac{1}{2}(30)^2$

$= -150$

$\therefore$ the rocket hits the ground with a velocity of -150 m/s.

HW. Worksheet on Expressions for Slopes of Tangents & Rates of Change #1 to 5