**Date:**_____________

# UNIT 8: SECANTS, TANGENTS & RATES OF CHANGE

## 8.1 Review of Prerequisite Skills For Unit 8

### I Review of Slopes and Equations of Lines

**Slope:** The **slope** is the measure of the steepness of a line.

$$slope = \frac{rise}{run}$$

$$m = \frac{\quad}{\quad}$$

**Equation of a Line: i) slope y-intercept form or ii) standard form**

$$y = mx + b \qquad\qquad Ax + By + C = 0,\ A,\ B,\ C \in I,\ \ A > 0$$

**Note:**

**i)** A ***vertical line*** has a slope that is ________________ and an equation of the form ____________.

**ii)** A ***horizontal line*** has a slope that is ______________ and an equation of the form ____________.

**iii)** ***Parallel lines*** have slopes that are the _________________.

**iv)** ***Perpendicular lines*** have slopes that are __________________________________________________.

**Ex. 1.** Find the equation of the line determined by the given information.

**a)** slope $-2$, y-intercept 3

**b)** horizontal, through $(-2,5)$

**c)** perpendicular to $2x - 3y - 6 = 0$ & having an $x$-intercept of $-2$

**d)** through $(-2,4)$ & $(-6,6)$

## II Rationalizing the Denominator or Numerator

A ***rational number*** either repeats or terminates in its decimal form.
An ***irrational number*** neither repeats nor terminates in its decimal form.

**Ex. 2.** Rationalize each ***denominator***.

**a)** $\dfrac{1+2\sqrt{2}}{3\sqrt{2}}$

**b)** $\dfrac{\sqrt{3}}{1-2\sqrt{3}}$

**Ex. 3.** Rationalize each ***numerator***.

**a)** $\dfrac{\sqrt{3}}{1-2\sqrt{3}}$

**b)** $\dfrac{1+2\sqrt{2}}{3\sqrt{2}}$

**Ex. 4.** Write an equivalent expression for **a)** by rationalizing the ***numerator*** and for **b)** by rationalizing the ***denominator***.

**a)** $\dfrac{\sqrt{9+h}-3}{h}$

**b)** $\dfrac{x-3}{2-\sqrt{x+1}}$

**III Simplifying Rational Expressions**

**Ex. 5.** Simplify each of the following.

**a)** $\dfrac{x^2-16}{x^3+64}$

**b)** $\left(\dfrac{1}{x-1}\right)\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$

**c)** $\dfrac{\dfrac{1}{2+h}-\dfrac{1}{2}}{h}$

**d)** $\dfrac{(3+h)^3-27}{h}$

**HW. Exercise 8.1**

**Date:**_______________ 

# 8.2 Slopes of Lines Given Two Points

***Recall:***

| ***slope formula*** | ***slope, y-intercept form*** | ***standard form*** |
|---|---|---|
| $m=\dfrac{y_2-y_1}{x_2-x_1}$ | $y=mx+b$ | $Ax+By+C=0$ |

**Ex. 1.** Find the slope *m*, in simplified form, of each pair of points.

**a)** $(-3,6)$ and $(6,0)$

**b)** $\left(\frac{3}{4},\frac{1}{4}\right)$ and $\left(\frac{7}{4},-\frac{3}{4}\right)$

**Ex. 2.** Find the equation of the line determined by the given information.

**a)** parallel to $x+4y-2=0$ & through $(-2, 5)$

**b)** through $(-5, 3)$ & $(-1, 2)$

**Ex. 3.** Find the slope *m*, in simplified form, of each pair of points.

**a)** $P(1, 3),\ Q\left(1+h,\ 3(1+h)^2\right)$

**b)** $P(9, 3),\ Q\left(9+h,\ \sqrt{9+h}\right)$

**c)** $P(-2, 2),\ Q\left(-2+h,\ \dfrac{4}{(-2+h)+4}\right)$

**d)** $P(2,5),\ Q\left(2+h,\ (2+h)^3-(2+h)^2+1\right)$

**HW. Exercise 8.2**

**Date:______________**

# 8.3 Slopes of Secants and Tangents
# Average and Instantaneous Rates of Change

**Definitions:**

**Secants and Average Rates of Change**

A ***secant*** is a line that passes through two points on the graph of a function $y = f(x)$.
The ***average rate of change*** of *y* with respect to *x* is the ***slope of the secant*** between those points.

$$average\ rate\ of\ change = m_{\text{secant}}$$

$$= \frac{change\ in\ y}{change\ in\ x}$$

$$= \frac{\Delta y}{\Delta x}$$

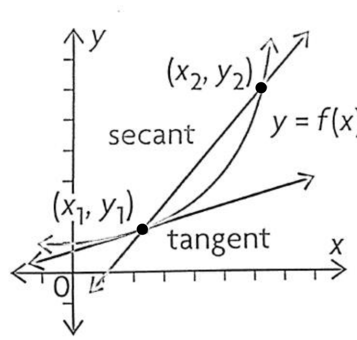
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

**Tangents and Instantaneous Rates of Change**

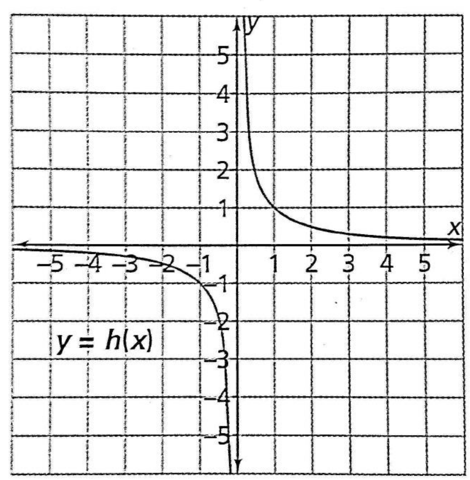
A ***tangent*** is a line that touches the graph of a function $y = f(x)$ at exactly one point.
The tangent is the straight line that most resembles the graph near that point.
The ***instantaneous rate of change*** of *y* with respect to *x* is the ***slope of the tangent*** at that point.

$$instantaneous\ rate\ of\ change = m_{\text{tangent}}$$




**Ex. 1.** Given the graph of $h(x) = \frac{1}{x}$,

**a)** draw the secant line that passes through $P(1,1)$ and $Q\left(4, \frac{1}{4}\right)$ and calculate its slope.

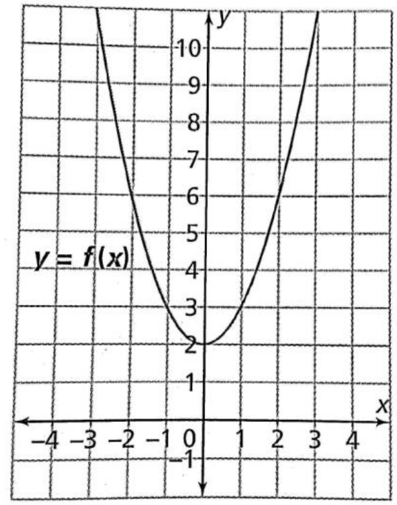



**b)** draw the tangent line to the curve at $P(1,1)$ and use the graph to estimate its slope.

**c)** determine the equation of the tangent line to the curve at $P(1,1)$.

**Ex. 2.** Consider the function $f(x) = x^2 + 2$.

**a)** Complete the following tables to estimate the slope of the tangent to $f(x) = x^2 + 2$ at point $P(2,6)$.




**i)** *Q approaches P from the right, ie.* $Q \to P^+$

| $P$ | $Q$ | Slope of Line $PQ$ |
|---|---|---|
| (2, 6) | (3, | |
| (2, 6) | (2.5, | |
| (2, 6) | (2.1, | |
| (2, 6) | (2.01, | |

**ii)** *Q approaches P from the left, ie.* $Q \to P^-$

| $P$ | $Q$ | Slope of Line $PQ$ |
|---|---|---|
| (2, 6) | (1, | |
| (2, 6) | (1.5, | |
| (2, 6) | (1.9, | |
| (2, 6) | (1.99, | |

***Conclusions:***

**b)** Let $Q$ be a point on the curve $h$ units to the right of $P$ and then calculate the slope of the secant $PQ$.

**c)** Use the result of part **b)** to calculate the slope of the tangent to the graph of $f(x)$ at point $P$.

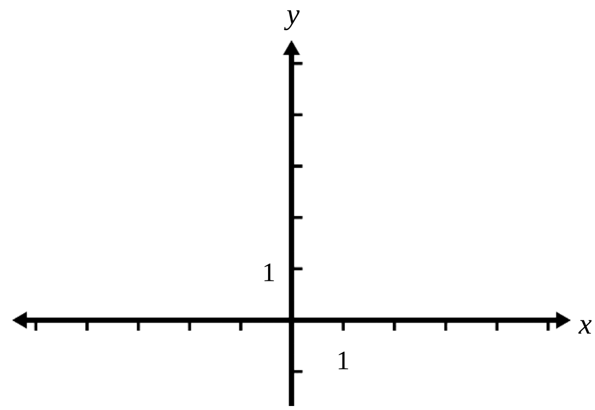
**Ex. 3.** For each curve,

**i)** find the slope of the tangent at the given point

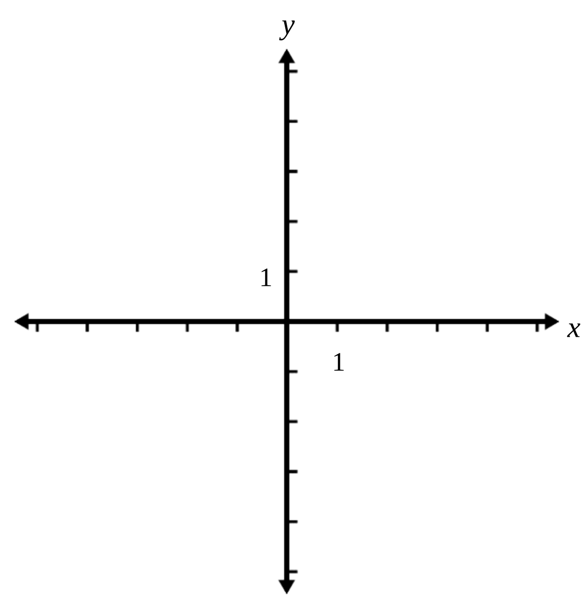
**ii)** find the equation of the tangent at the given point

**iii)** graph the curve and the tangent

**a)** $f(x)=\sqrt{x+3}$ at $P(-2,1)$




**b)** $f(x)=\dfrac{x+1}{x+3}$ at $P\left(1,\dfrac{1}{2}\right)$




**HW. Exercise 8.3**

**Date:_______________** 

# 8.4 Applications of Rates of Change

**Recall:**

**Secants and Average Rates of Change**

A ***secant*** is a line that passes through two points on the graph of a function $y = f(x)$.
The ***average rate of change*** of *y* with respect to *x* is the ***slope of the secant*** between those points.

$$average\ rate\ of\ change = m_{\text{secant}}$$

$$= \frac{change\ in\ y}{change\ in\ x}$$

$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

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**Tangents and Instantaneous Rates of Change**

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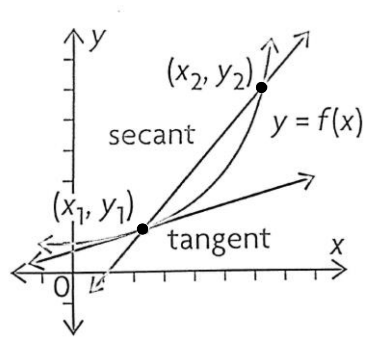
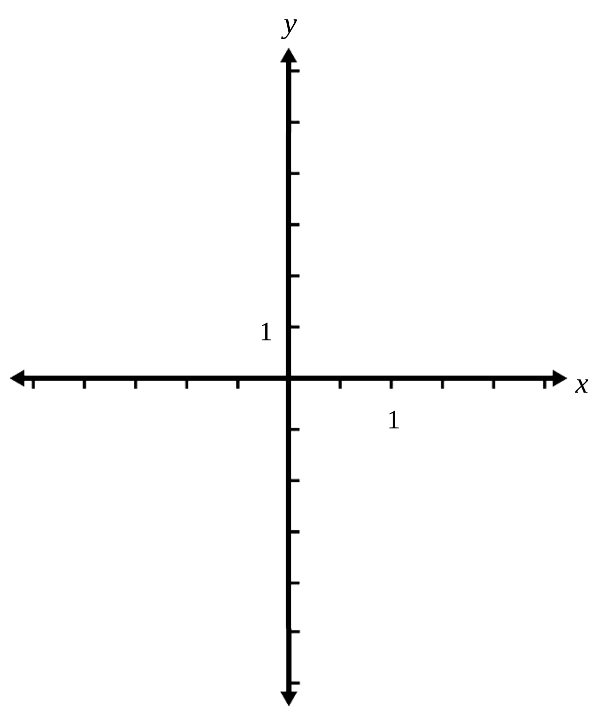
$$instantaneous\ rate\ of\ change = m_{\text{tangent}}$$

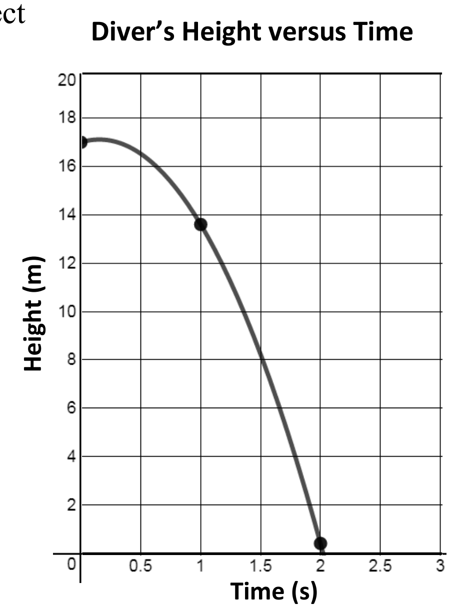



**Ex. 1.** Find the slope and equation of the tangent to $f(x) = x - x^3$ at $x = -1$. Illustrate graphically.



**Ex. 2.** A cliff diver in Acapulco, Mexico, dives from about 17 m above the water. The function $s(t) = -4.9t^2 + 1.5t + 17$ models the diver's height above the water, in metres, at $t$ seconds.

**a)** Determine the diver's average rate of descent with respect to time during each of the first two seconds.




**b)** Determine the diver's instantaneous rate of descent with respect to time at 1 second.

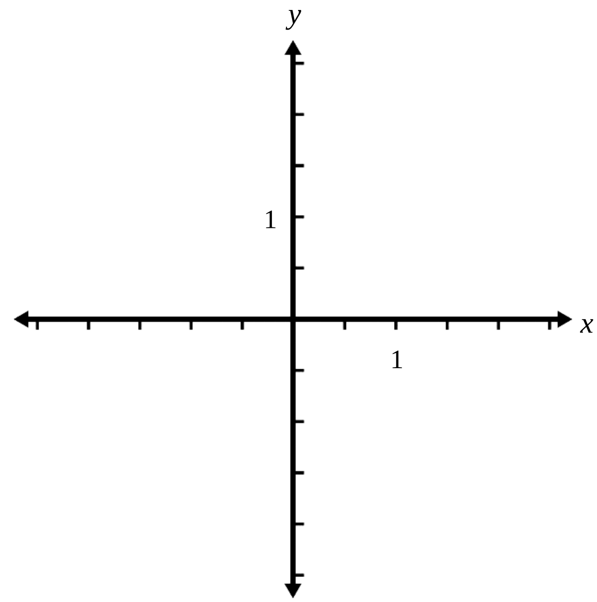
**HW. Exercise 8.4**

**Date:________________** 

# 8.5 Applications of Rates of Change continued

**Warmup**

**1.** Find the equation of the tangent to $f(x)=\dfrac{1}{x^2-1}$ at $x=2$. Illustrate your solution graphically.



**2.** A motorboat coasts toward a dock with its engine off. Its distance *s*, in metres, from the dock *t* seconds after the engine is turned off is $s(t)=\dfrac{10(6-t)}{t+3}$ for $0 \le t \le 6$.

**a)** Find the average velocity for $t \in [0, 6]$.

**b)** Find the boat's velocity when it bumps into the dock..

**3.** It can be shown that from a height of *s* metres above ground level, a person can see a distance, *d* kilometres to the horizon, where $d(s) = 3.53\sqrt{s}$ .

**a)** When the elevator of the CN Tower passes the 225 m height, how far can the passengers see across Lake Ontario?

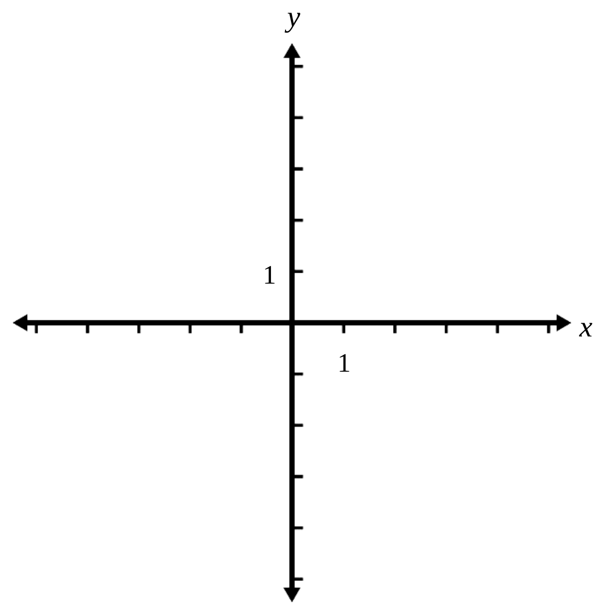
**b)** Find the rate of change of this distance to the horizon with respect to height, when the height of the elevator is 225 m

**HW. Exercise 8.5**

**Date:____________** **8.6 Expressions For Slopes of Tangents & Rates of Change**

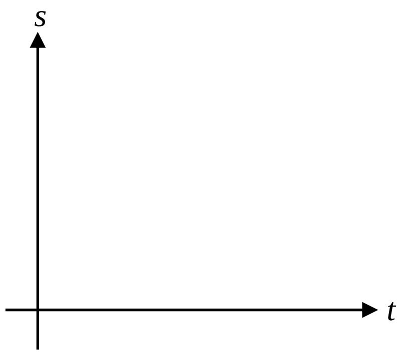
**Ex. 1.** Given the function $y = -x^2 - 2x + 3$, find:

**a)** an expression for the slope of the tangent to the curve at any point (*x*, *y*) and illustrate graphically.
**b)** the slope of the tangent at $x = -3$ by using the formula obtained from part **a**.
**c)** the point on the curve where the tangent is horizontal.
**d)** the equation of the ***normal*** at $x = -3$.
**Note:** The ***normal*** is the line perpendicular to the tangent at the point of tangency.



**Ex. 2.** An unoccupied test rocket meant for orbit is fired straight up from a launch pad. Something goes wrong and the rocket comes back down. The height of the rocket above the ground is modelled by $s(t) = 5t^2 - \frac{1}{6}t^3$, $t \geq 0$, where *s* is measured in metres and *t*, in seconds.

**a)** Sketch the graph of the function. (Scale the horizontal axis only.)
**b)** Find an expression for the **velocity** (slope of the tangent) at any time, *t*.
**c)** When is the velocity (slope of the tangent) zero?
**d)** Find the maximum height of the rocket.
**e)** At what velocity does the rocket hit the ground?




**HW. Exercise 8.6**

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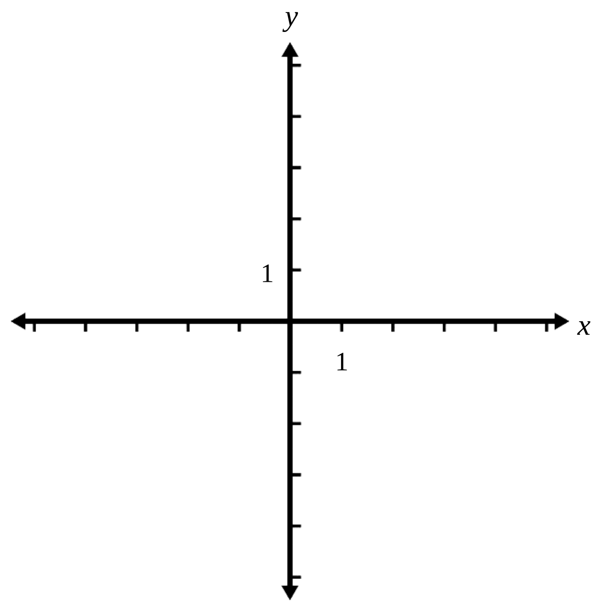
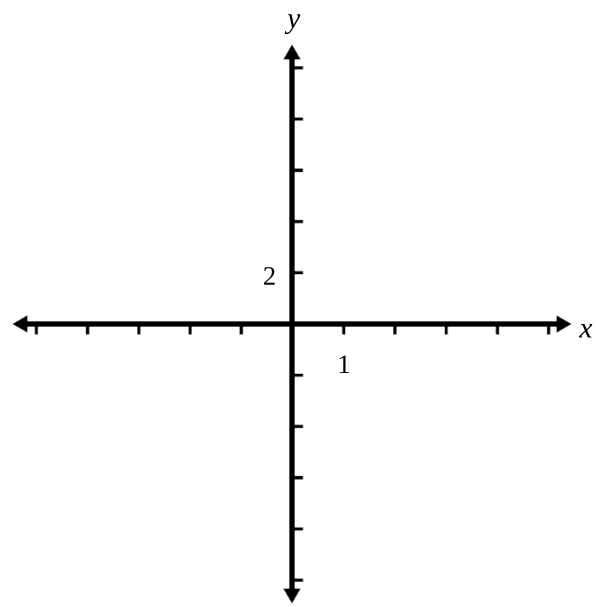
# The Ultimate Warmup

**1.** Find the equation of the tangent to $f(x)=\dfrac{x}{2-\sqrt{4+x}}$ at $x=5$. Illustrate your solution graphically.

***Hint:*** *Simplify the function by rationalizing the denominator and identify any holes.*

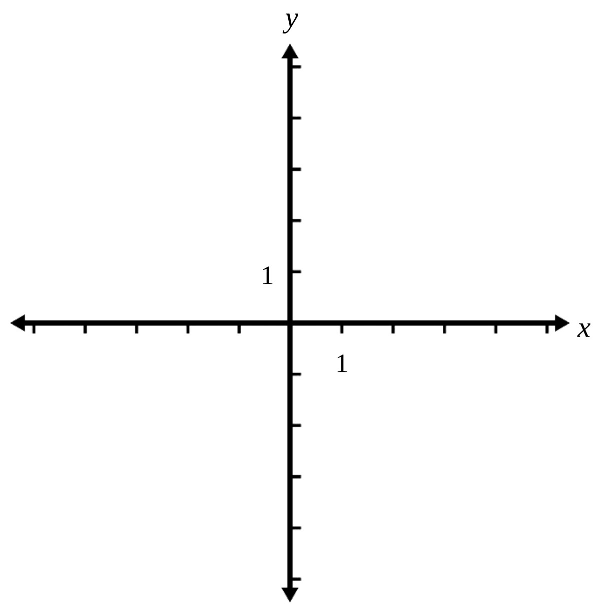


**2.** Find the equation of the tangent to $f(x) = \dfrac{x^3 - 6x^2 + 12x - 8}{|x-2|}$ at $x = -1$. Illustrate your solution graphically. ***Hint:*** *Rewrite the function as a piecewise function first.*



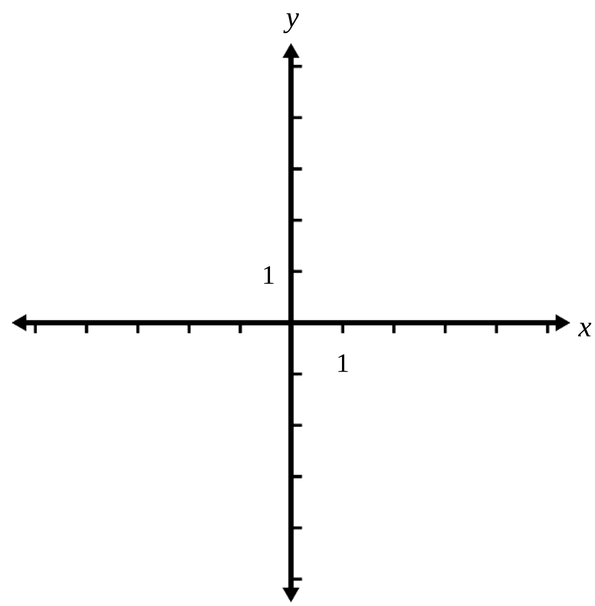
**3.** Find the equations of the tangent and normal to $g(x) = \dfrac{2x^2 - 3x + 1}{x^2 - 1}$ at $x = -2$.

Illustrate your solution graphically. ***Hint:*** *Simplify the function first and identify any holes.*



**4.** Given the function $f(x)=\dfrac{x^2}{x-1}$, find:

**a)** an expression for the slope of the tangent to the curve at any point $(x, y)$ and illustrate graphically.
**b)** the slope of the tangent at $x=-3$ by using the formula obtained from part **a**.
**c)** the point(s) on the curve where the tangent is horizontal.
**d)** where on the curve the normal has a slope of $\dfrac{3}{2}$.




**HW. For Unit 8 Test do: Unit 8 Review Exercise**