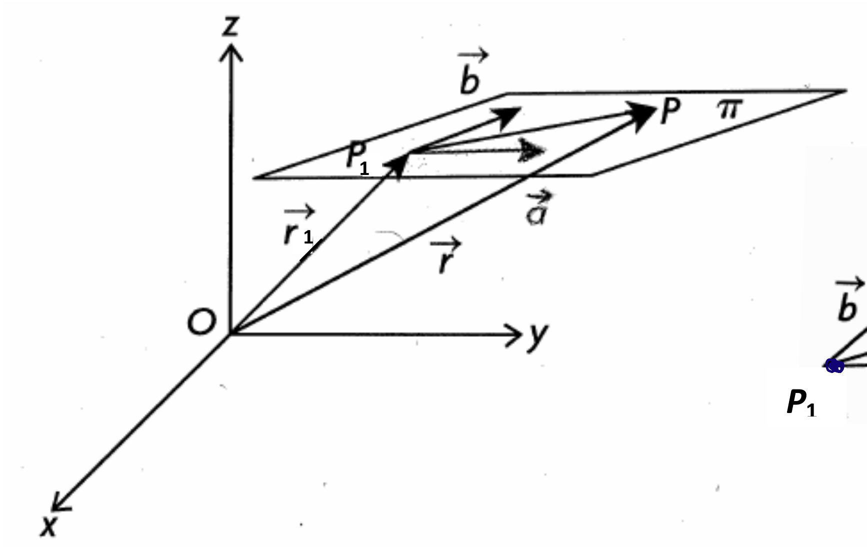
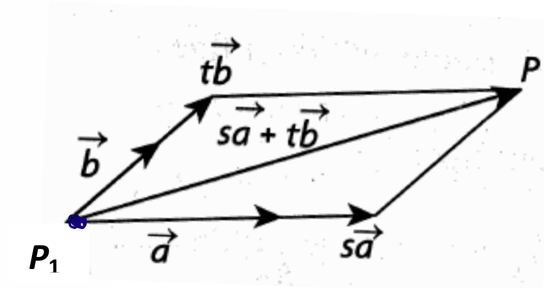
Date: June 2/14

UNIT 10 – EQUATIONS OF PLANES

Section 8.1 – The Vector Equation of a Line in Space





The ***vector equation*** of a plane can be determined by one point and two non-parallel vectors.

Let $P(x, y, z)$ be any point in the plane π, $P_1(x_1, y_1\, z_1)$ be a particular point in the plane and vectors $\vec{a}$ and $\vec{b}$ be two non-parallel direction vectors in the plane.

$$\vec{OP} = \vec{OP_1} + s\vec{a} + t\vec{b}, \quad s, t \in R$$
$$\vec{r} = \vec{r_1} + s\vec{a} + t\vec{b}$$

SUMMARY OF EQUATIONS OF PLANES IN 3-SPACE:

Vector Equation

$\vec{r} = \vec{r}_1 + s\vec{a} + t\vec{b}$

or

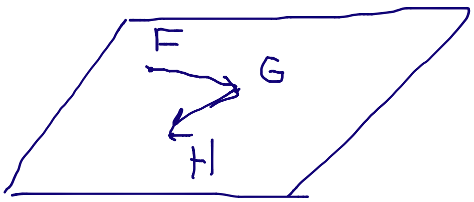
$(x, y, z) = (x_1, y_1, z_1) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$

Parametric Equations

$x = x_1 + a_1 s + b_1 t$
$y = y_1 + a_2 s + b_2 t$
$z = z_1 + a_3 s + b_3 t$

where (x, y, z) is the position vector of any point in the plane
(x_1, y_1, z_1) is the position vector of some particular point in the plane
$\vec{a}$ and $\vec{b}$ are two non-parallel *direction vectors* for the plane
and $s, t \in \Re$ are the parameters.

Ex. 1. Determine the ***vector*** and ***parametric equations*** of the plane through the points $F(-3,7,1)$, $G(-1,2,-3)$ and $H(5,-1,-2)$.

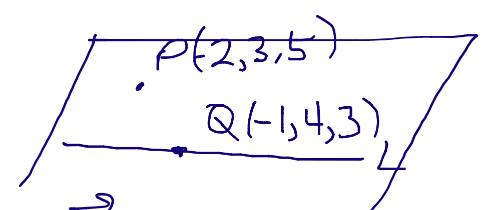


$\vec{r_1} = \vec{OF}$
$= (-3,7,1)$
$\vec{a} = \vec{FG}$
$= \vec{OG} - \vec{OF}$
$= (-1,2,-3) - (-3,7,1)$
$= (2,-5,-4)$
$\vec{b} = \vec{GH}$
$= \vec{OH} - \vec{OG}$
$= (5,-1,-2) - (-1,2,-3)$
$= (6,-3,1)$

vector equation:
$\vec{r} = \vec{r_1} + s\vec{a} + t\vec{b}$
$\therefore \vec{r} = (-3,7,1) + s(2,-5,-4) + t(6,-3,1)$, $s,t \in \mathbb{R}$

parametric equations:
$x = -3 + 2s + 6t$
$y = 7 - 5s - 3t$
$z = 1 - 4s + t$

Ex. 2. Determine the ***vector equation*** of the plane containing the point $P(-2,3,5)$ and the line $\vec{r} = (-1,4,3) + t(-1,-3,6)$, $t \in \Re$.



$\vec{r} = (-1,4,3) + t(-1,-3,6)$
$\vec{r_1} = \vec{OP}$
$= (-2,3,5)$
$\vec{a} = (-1,-3,6)$
$\vec{b} = \vec{PQ}$
$= \vec{OQ} - \vec{OP}$
$= (-1,4,3) - (-2,3,5)$
$= (1,1,-2)$

vector equation:
$\vec{r} = \vec{r_1} + s\vec{a} + t\vec{b}$, $s,t \in \mathbb{R}$
$\therefore \vec{r} = (-2,3,5) + s(-1,-3,6) + t(1,1,-2)$, $s,t \in \mathbb{R}$.

Ex. 3. Find the ***vector equation*** of the plane that contains the two lines

$\vec{r} = (1,2,3) + s(2,4,-1),\ s \in \Re$ and $\vec{r} = (4,5,1) + t(-2,-4,1),\ t \in \Re$.

L_1 $\qquad$ L_2

Note: $L_1 \parallel L_2$

Let points $P(1,2,3)$ & $Q(4,5,1)$ be points in the plane.

$\vec{r}_1 = \vec{OP}$
$= (1,2,3)$

$\vec{a} = (2,4,-1)$

$\vec{b} = \vec{PQ}$
$= \vec{OQ} - \vec{OP}$
$= (4,5,1) - (1,2,3)$
$= (3,3,-2)$

Vector Equation:

$\vec{r} = \vec{r}_1 + s\vec{a} + t\vec{b},\ s,t \in \mathbb{R}$

$\therefore \vec{r} = (1,2,3) + s(2,4,-1) + t(3,3,-2),$
$s,t \in \mathbb{R}$.

Ex. 4. Does the point $(1,5,3)$ lie in the plane $\vec{r} = (4,-3,1) + s(3,1,-2) + t(6,5,-1),\ s,t \in \Re$?

Sub $(1,5,3)$ for $\vec{r}$ & solve for s & t

$(1,5,3) = (4,-3,1) + s(3,1,-2) + t(6,5,-1)$
$(1,5,3) - (4,-3,1) = s(3,1,-2) + t(6,5,-1)$
$(-3,8,2) = s(3,1,-2) + t(6,5,-1)$

$-3 = 3s + 6t$ ①
$8 = s + 5t$ ②
$2 = -2s - t$ ③

Solve ② & ③
Eliminate "t"

②×1 $\quad 8 = s + 5t$
③×5 $\quad 10 = -10s - 5t$
Add $\quad 18 = -9s$
$\boxed{-2 = s}$

Sub in ③
$2 = -2(-2) - t$
$t = 4 - 2$
$\boxed{t = 2}$

Check in ①
$s = -2$ & $t = 2$

L.S. $= -3$

R.S. $= 3s + 6t$
$= 3(-2) + 6(2)$
$= -6 + 12$
$= 6$

$\therefore$ L.S $\neq$ R.S.

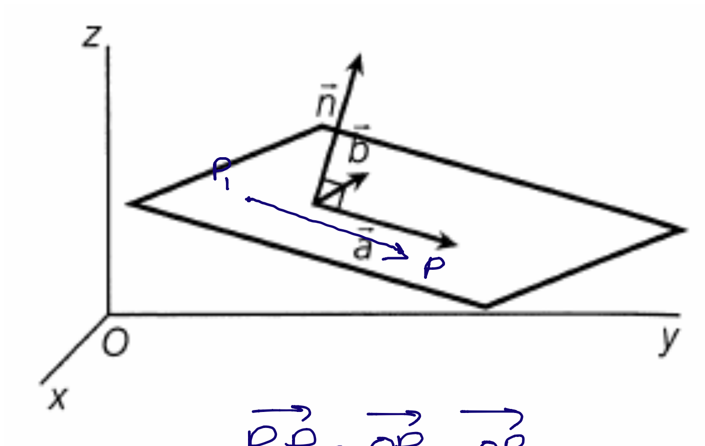
$\therefore$ the point $(1,5,3)$ does not lie in the plane.

HW: pg. 279 #1-12

Date: June 4/14 **Section 8.2 – The Scalar Equation of a Plane in Space**

A ***normal vector*** to a plane is a vector that is perpendicular to every vector in the plane.

Find the general equation of a plane containing the point $P_1(x_1, y_1, z_1)$ with normal vector $\vec{n} = (A, B, C)$.



Let $P(x, y, z)$ be any point in the plane.

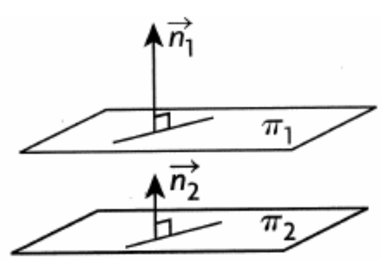
$$\overrightarrow{P_1P} = \overrightarrow{OP} - \overrightarrow{OP_1}$$
$$= (x,y,z) - (x_1,y_1,z_1)$$
$$= (x-x_1, y-y_1, z-z_1)$$

$$\overrightarrow{P_1P} \cdot \vec{n} = 0$$
$$(x-x_1, y-y_1, z-z_1)\cdot(A,B,C) = 0$$
$$Ax - Ax_1 + By - By_1 + Cz - Cz_1 = 0$$
$$Ax + By + Cz \underbrace{-Ax_1 - By_1 - Cz_1}=0, \quad \text{Let } D = -Ax_1 - By_1 - Cz_1$$

$\therefore$ $Ax + By + Cz + D = 0$ is the scalar or Cartesian equation of a plane.

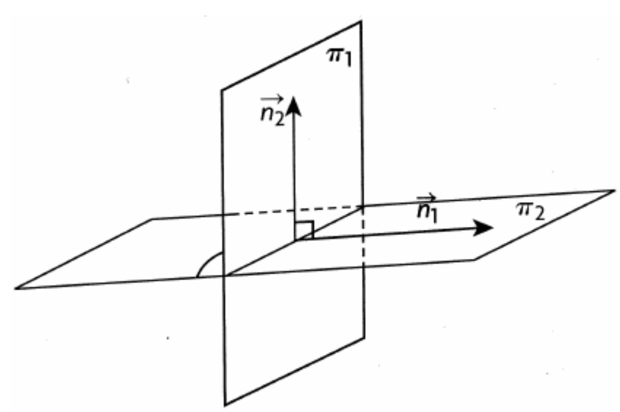
Note: $\vec{n} = (A, B, C)$

Parallel Planes



If $\pi_1 \parallel \pi_2$ then $\vec{n}_1 \parallel \vec{n}_2$ where $\vec{n}_1 = k\vec{n}_2$.

Perpendicular Planes



If $\pi_1 \perp \pi_2$ then $\vec{n}_1 \perp \vec{n}_2$ where $\vec{n}_1 \cdot \vec{n}_2 = 0$.

> ***SUMMARY:*** The **scalar** or **Cartesian equation** of a plane in space has the form
> $$Ax + By + Cz + D = 0$$
> where (A, B, C) is a vector *normal* to the plane.

Ex. 1. Find the ***scalar equation*** of the plane through the point $A(-2,1,3)$ having a normal vector $\vec{n} = (4,-2,5)$.

$x \; y \; z$

$A \; B \; C$

$$Ax + By + Cz + D = 0$$
$$4x - 2y + 5z + D = 0$$

Find if $x = -2$, $y = 1$, $z = 3$

$$4(-2) - 2(1) + 5(3) + D = 0$$
$$D = -5$$

$\therefore$ the scalar equation of the plane is
$$4x - 2y + 5z - 5 = 0$$

Ex. 2. Find the ***Cartesian equation*** of the plane that passes through the points $A(-2,3,1)$, $B(-1,2,-1)$ and $C(1,-2,3)$.

$$\vec{a} = \overrightarrow{AB}$$
$$= \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (-1,2,-1) - (-2,3,1)$$
$$\therefore \vec{a} = (1,-1,-2)$$

$$\vec{b} = \overrightarrow{AC}$$
$$= \overrightarrow{OC} - \overrightarrow{OA}$$
$$= (1,-2,3) - (-2,3,1)$$
$$\therefore \vec{b} = (3,-5,2)$$

$$\vec{n} = \vec{a} \times \vec{b}$$
$$= (-12,-8,-2)$$

$$\begin{matrix} -1 & -2 & 1 & -1 \\ -5 & 2 & 3 & -5 \end{matrix}$$

Use $\vec{n} = (6,4,1)$ & $B(-1,2,-1)$

$A \; B \; C \quad x \; y \; z$

$$Ax + By + Cz + D = 0$$

Find D

$$6(-1) + 4(2) + 1(-1) + D = 0$$
$$-6 + 8 - 1 + D = 0$$
$$D = -1$$

$\therefore$ the Cartesian equation of the plane is
$$6x + 4y + z - 1 = 0.$$

Ex. 3. Find the ***scalar equation*** of the plane containing the point $Q(1,2,-1)$ and the line $\vec{r} = (2,1,-1) + t(1,0,-3),\ t \in \Re$. , $P(2,1,-1)$

$$\vec{a} = (1,0,-3)$$

$$\vec{b} = \overrightarrow{PQ}$$
$$= \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (1,2,-1) - (2,1,-1)$$
$$\vec{b} = (-1,1,0)$$

$$\vec{n} = \vec{a} \times \vec{b}$$
$$= (3,3,1)$$

$$\begin{matrix} 0 & -3 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{matrix}$$

Use $\vec{n} = (3,3,1)$ & $Q(1,2,-1)$

$A \; B \; C \quad x \; y \; z$

$$Ax + By + Cz + D = 0$$

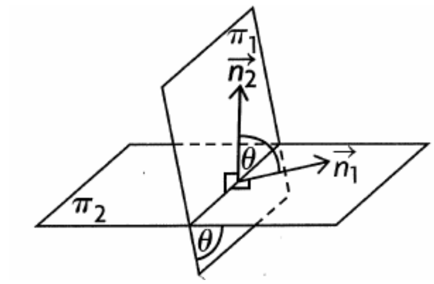
Find D

$$3(1) + 3(2) + 1(-1) + D = 0$$
$$8 + D = 0$$
$$D = -8$$

$\therefore$ $3x + 3y + z - 8 = 0$ is the scalar equation of the plane.

Ex. 4. Find the acute angle between the planes

$\pi_1 : x + 2y - 3z - 4 = 0$ and $\pi_2 : x - 3y + 5z + 7 = 0$.



For π_1, $\vec{n}_1 = (1,2,-3)$, for π_2: $\vec{n}_2 = (1,-3,5)$

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2|\cos\theta$$

$$(1,2,-3)\cdot(1,-3,5) = \sqrt{14} \cdot \sqrt{35} \cdot \cos\theta$$

$$1 - 6 - 15 = \sqrt{14} \cdot \sqrt{35} \cdot \cos\theta$$

$$\frac{-20}{\sqrt{14}\cdot\sqrt{35}} = \cos\theta$$

$$\theta \doteq 155^\circ$$

$\therefore$ the acute angle between the planes is approximately 25°.

Ex. 5. Use vector projections to find the exact distance from the point $Q(1,2,3)$ to the plane $4x - 5y + 7z - 10 = 0$.

$\vec{n} = (4,-5,7)$

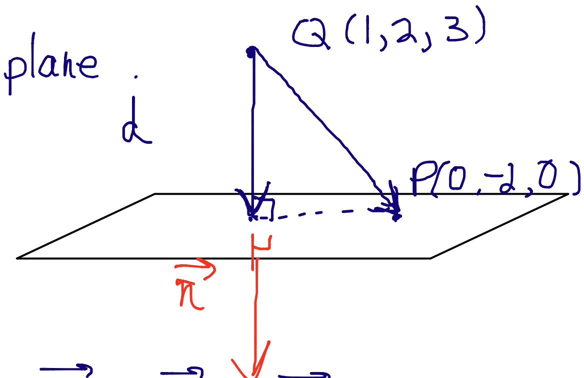
Find a point, P, in the plane.

$x = 0$, $z = 0$, find y

$-5y = 10$

$y = -2$

$P(0,-2,0)$



$\vec{QP} = \vec{OP} - \vec{OQ}$

$= (0,-2,0) - (1,2,3)$

$= (-1,-4,-3)$

$$d = |\text{VP}_{\vec{QP}\text{ on }\vec{n}}|$$

$$= \frac{|\vec{QP}\cdot\vec{n}|}{|\vec{n}|}$$

$$= \frac{|(-1,-4,-3)\cdot(4,-5,7)|}{\sqrt{16+25+49}}$$

$$= \frac{|-4+20-21|}{\sqrt{90}}$$

$$= \frac{|-5|}{3\sqrt{10}}$$

$$= \frac{5}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$= \frac{\cancel{5}\sqrt{10}}{3(\cancel{10})2}$$

$$= \frac{\sqrt{10}}{6}$$

$\therefore$ Q (1,2,3) is exactly $\frac{\sqrt{10}}{6}$ units from the plane.

Ex. 6. a) Show that the ***shortest distance*** from a point $Q(x_1, y_1, z_1)$ to a plane with a scalar equation $Ax + By + Cz + D = 0$ is given by the formula $d = \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

b) Use the formula to find the distance from the point $Q(1, 2, 3)$ to the plane $4x - 5y + 7z - 10 = 0$.

a) Let $P_0(x_0, y_0, z_0)$ be a specific point in the plane.

$\overrightarrow{QP_0} = \overrightarrow{OP_0} - \overrightarrow{OQ}$

$\overrightarrow{QP_0} = (x_0, y_0, z_0) - (x_1, y_1, z_1)$

$\therefore \overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ and $\vec{n} = (A, B, C)$

$$d = \left| \underset{QP_0\, on\, \vec{n}}{VP} \right|$$

$$= \frac{|\overrightarrow{QP_0} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (A, B, C)|}{\sqrt{A^2 + B^2 + C^2}}$$

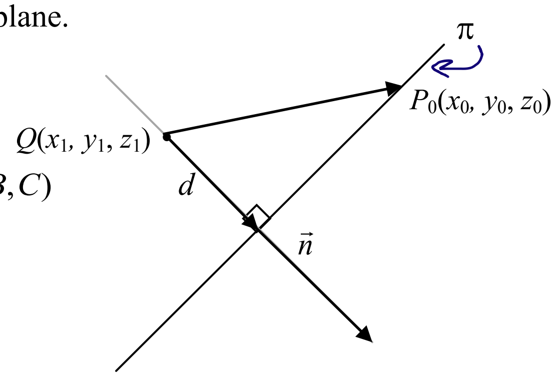
$$= \frac{|Ax_0 - Ax_1 + By_0 - By_1 + Cz_0 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}}$$

*** see Note above**

$$= \frac{|-Ax_1 - By_1 - Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|-(Ax_1 + By_1 + Cz_1 + D)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$\therefore d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$



Note: Since the point $P_0(x_0, y_0, z_0)$ is in the plane $Ax + By + Cz + D = 0$, it satisfies the equation.
ie. $Ax_0 + By_0 + Cz_0 + D = 0$, so $Ax_0 + By_0 + Cz_0 = -D$

b) $Q(1, 2, 3)$ $\vec{n} = (4, -5, 7)$ $D = -10$
$x_1\ y_1\ z_1$ $A\ B\ C$

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|4(1) - 5(2) + 7(3) - 10|}{\sqrt{(4)^2 + (-5)^2 + (7)^2}}$$

$$= \frac{|5|}{\sqrt{90}}$$

$$= \frac{5}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$= \frac{\sqrt{10}}{6}$$

$\therefore Q(1, 2, 3)$ is $\frac{\sqrt{10}}{6}$ units from the plane.

HW: pg. 285 #1-13 & Additional Questions

1. What is the exact distance from the point $Q(1, 3, -2)$ to the plane $4x - y - z + 6 = 0$?

2. What is the exact distance between the planes $2x - y - 2z + 3 = 0$ and $4x - 2y - 4z - 9 = 0$?

Answers: **1.** $\frac{3\sqrt{2}}{2}$ **2.** $\frac{5}{2}$

Date:______________ **Section 8.3 – The Intersection of a Line and a Plane**

Warm-up:

Given $A(1,5,9)$, $B(-2,6,8)$ and $C(12,7,16)$ are points in a plane, find:

a) the vector, parametric and scalar equations of the plane.

b) the exact distance the point $Q(1,-2,1)$ is from the plane.

a) vector equation:

$\vec{r} = \vec{r}_1 + s\vec{a} + t\vec{b},\ s,t \in \mathbb{R}$

$\vec{r}_1 = \vec{OA} = (1,5,9)$

$\vec{a} = \vec{AB} = \vec{OB} - \vec{OA} = (-2,6,8) - (1,5,9) = (-3,1,-1)$

$\vec{b} = \vec{BC} = \vec{OC} - \vec{OB} = (12,7,16) - (-2,6,8) = (14,1,8)$

$\therefore \vec{r} = (1,5,9) + s(-3,1,-1) + t(14,1,8),\ s,t \in \mathbb{R}$

parametric equations:

$x = 1 - 3s + 14t$

$y = 5 + s + t$

$z = 9 - s + 8t$

Scalar equation:

$Ax + By + Cz + D = 0$

$\vec{n} = \vec{a} \times \vec{b} = (9, 10, -17)$

$$\begin{array}{cccc} 1 & -1 & -3 & 1 \\ 1 & 8 & 14 & 1 \end{array}$$

$\vec{n} = (\underset{A}{9}, \underset{B}{10}, \underset{C}{-17})$ $\quad A(\underset{x}{1}, \underset{y}{5}, \underset{z}{9})$

Find D

$9(1) + 10(5) - 17(9) + D = 0$

$9 + 50 - 153 + D = 0$

$D - 94 = 0$

$D = 94$

$\therefore$ the scalar equation is $9x + 10y - 17z + 94 = 0$

b) $Q(\overset{x_1}{1}, \overset{y_1}{-2}, \overset{z_1}{+1})$; $\overset{A}{9}x + \overset{B}{10}y - \overset{C}{17}z + \overset{D}{94} = 0$

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|9(1) + 10(-2) - 17(+1) + 94|}{\sqrt{(9)^2 + (10)^2 + (-17)^2}}$$

$$= \frac{|66|}{\sqrt{470}}$$

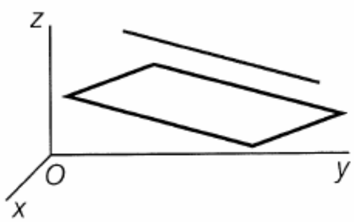
$$d = \frac{66}{\sqrt{470}} \times \frac{\sqrt{470}}{\sqrt{470}}$$

$$= \frac{66\sqrt{470}}{470}$$

$$= \frac{33\sqrt{470}}{235}$$

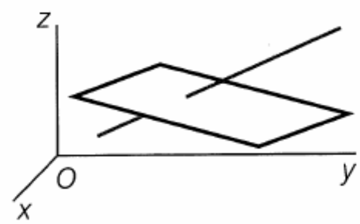
There are three possible ways that a line and a plane in three dimensions can intersect.

The line is parallel and distinct from the plane.



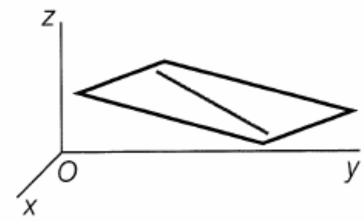
There is no intersection.

The line intersects the plane.



The intersection is a point.

The line lies in the plane.



The intersection is the line.

Ex. 1. For each of the following, find the intersection of the line and the plane.

a) $L: \vec{r} = (1,-3,-2) + t(2,1,3),\ t \in \Re$ and $\pi: 2x + 4y - z + 3 = 0$

For L,

$x = 1 + 2t$

$y = -3 + t$

$z = -2 + 3t$

Sub parametric equations of line into equation of π.

$2(1+2t) + 4(-3+t) - (-2+3t) + 3 = 0$

$2 + 4t - 12 + 4t + 2 - 3t + 3 = 0$

$5t - 5 = 0$

$5t = 5$

$t = 1$

If $t = 1$

$x = 3$

$y = -2$

$z = 1$

$\therefore$ the line intersects the plane at the point $(3,-2,1)$.

b) $L: \dfrac{x-2}{5} = \dfrac{y+1}{3} = z$ and $\pi: 8x - 13y - z - 29 = 0$

$\dfrac{x-2}{5} = \dfrac{y-(-1)}{3} = \dfrac{z-0}{1}$

For L,

$x = 2 + 5t$

$y = -1 + 3t$

$z = t$

Sub parametric equations into π:

$8(2+5t) - 13(-1+3t) - t - 29 = 0$

$16 + 40t + 13 - 39t - t - 29 = 0$

$0t = 0$

$t \in \mathbb{R}$

$\therefore$ the line lies in the plane so the intersection is the line $\dfrac{x-2}{5} = \dfrac{y+1}{3} = z$.

c) $L: \vec{r} = (2,1,5) + t(11,9,1),\ t \in \Re$ and $\pi: 2x - 3y + 5z - 8 = 0$

For L,
$x = 2 + 11t$
$y = 1 + 9t$
$z = 5 + t$

sub in π
$2(2+11t) - 3(1+9t) + 5(5+t) - 8 = 0$
$4 + 22t - 3 - 27t + 25 + 5t - 8 = 0$
$0t + 18 = 0$
$0t = -18$
no solution

$\therefore$ no intersection since the line is parallel and distinct from the plane.

Ex. 2. For each of the following planes, find the *x*, *y* and *z*-intercepts and sketch the plane.

a) $2x + 3y + 4z - 12 = 0$

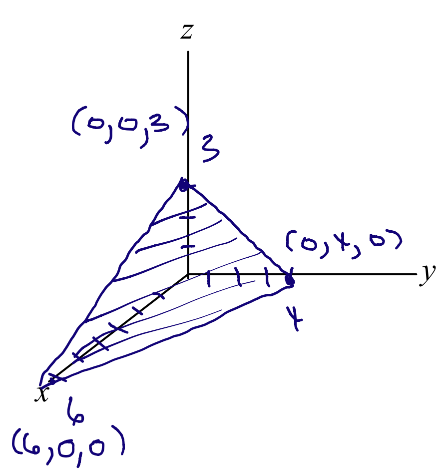
For x-int, $y=0, z=0$
$2x - 12 = 0$
$x = 6$
$\therefore$ x-int is 6
For y-int, $x=0, z=0$
$3y - 12 = 0$
$y = 4$
$\therefore$ y-int is 4
For z-int, $x=0, y=0$
$4z - 12 = 0$
$z = 3$ $\therefore$ z-int is 3

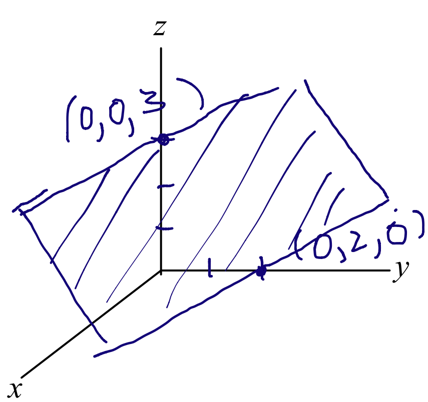
b) $3y + 2z - 6 = 0$

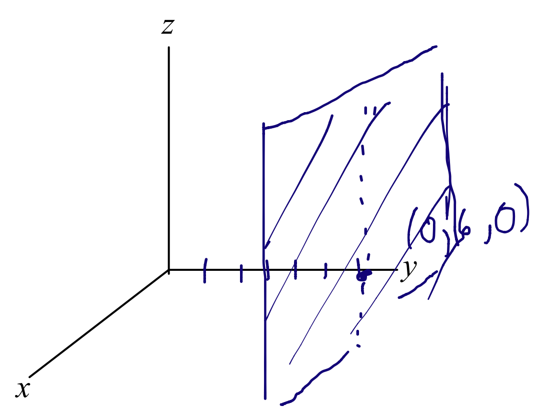
No x-int,
For y-int, $z=0$
$3y - 6 = 0$
$y = 2$
$\therefore$ y-int is 2
For z-int, $y=0$
$2z - 6 = 0$
$z = 3$
$\therefore$ z-int is 3

c) $y - 6 = 0$

No x-int,
y-int is 6
No z-int,







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