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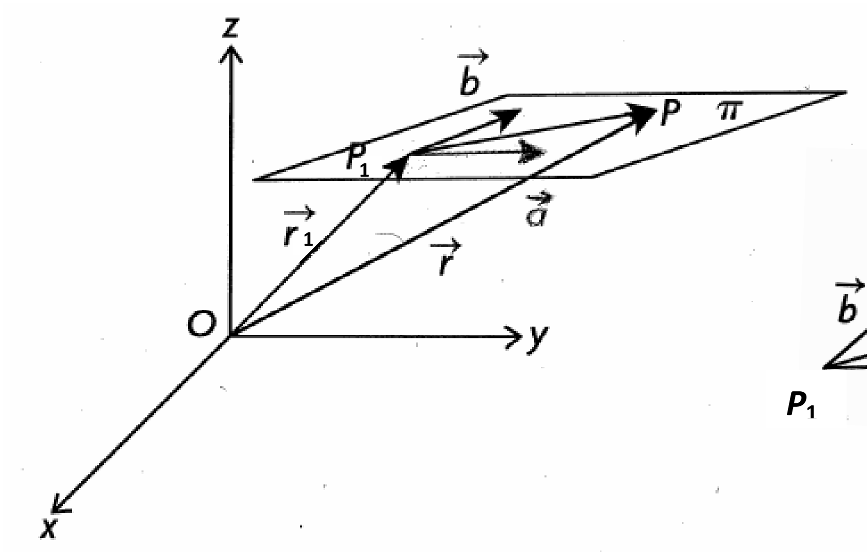
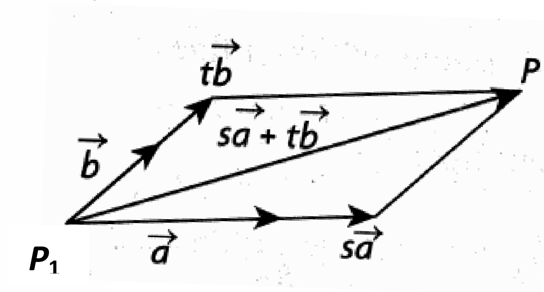
UNIT 10 – EQUATIONS OF PLANES

Section 8.1 – The Vector Equation of a Line in Space





The ***vector equation*** of a plane can be determined by one point and two non-parallel vectors.

Let $P(x, y, z)$ be any point in the plane π, $P_1(x_1, y_1\ z_1)$ be a particular point in the plane and vectors $\vec{a}$ and $\vec{b}$ be two non-parallel direction vectors in the plane.

SUMMARY OF EQUATIONS OF PLANES IN 3-SPACE:

Vector Equation

$\vec{r} = \vec{r}_1 + s\vec{a} + t\vec{b}$

or

$(x, y, z) = (x_1, y_1, z_1) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$

Parametric Equations

$x = x_1 + a_1 s + b_1 t$

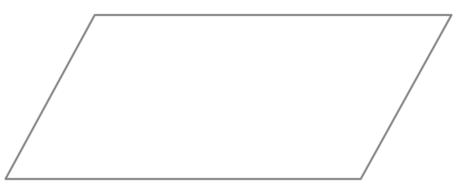
$y = y_1 + a_2 s + b_2 t$

$z = z_1 + a_3 s + b_3 t$

where (x, y, z) is the position vector of any point in the plane
(x_1, y_1, z_1) is the position vector of some particular point in the plane
$\vec{a}$ and $\vec{b}$ are two non-parallel *direction vectors* for the plane
and $s, t \in \Re$ are the parameters.

Ex. 1. Determine the ***vector*** and ***parametric equations*** of the plane through the points $F(-3,7,1)$, $G(-1,2,-3)$ and $H(5,-1,-2)$.

Ex. 2. Determine the ***vector equation*** of the plane containing the point $P(-2,3,5)$ and the line $\vec{r}=(-1,4,3)+t(-1,-3,6),\ t\in\Re$.



Ex. 3. Find the ***vector equation*** of the plane that contains the two lines

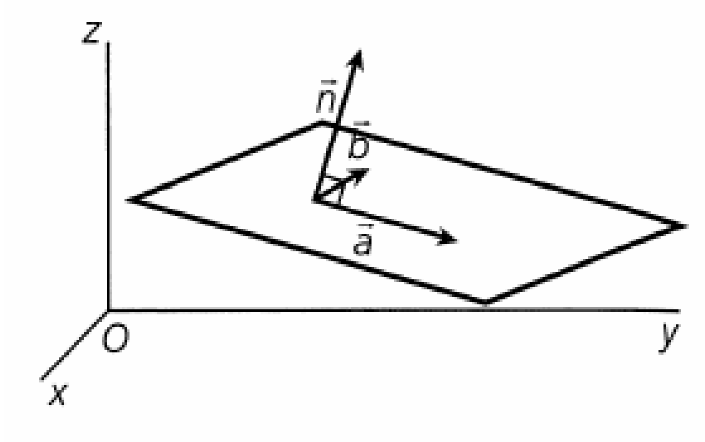
$\vec{r} = (1,2,3) + s(2,4,-1),\ s \in \Re$ and $\vec{r} = (4,5,1) + t(-2,-4,1),\ t \in \Re$.

Ex. 4. Does the point $(1,5,3)$ lie in the plane $\vec{r} = (4,-3,1) + s(3,1,-2) + t(6,5,-1),\ s,t \in \Re$?

HW: p. 279 #1-12

Date:_____________ **Section 8.2 – The Scalar Equation of a Plane in Space**

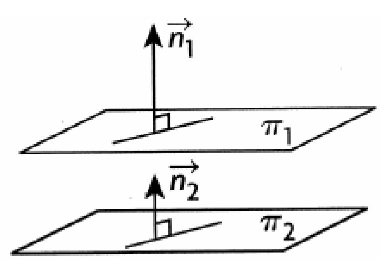
A ***normal vector*** to a plane is a vector that is perpendicular to every vector in the plane.



Find the general equation of a plane containing the point $P_1(x_1, y_1, z_1)$ with normal vector $\vec{n} = (A, B, C)$.

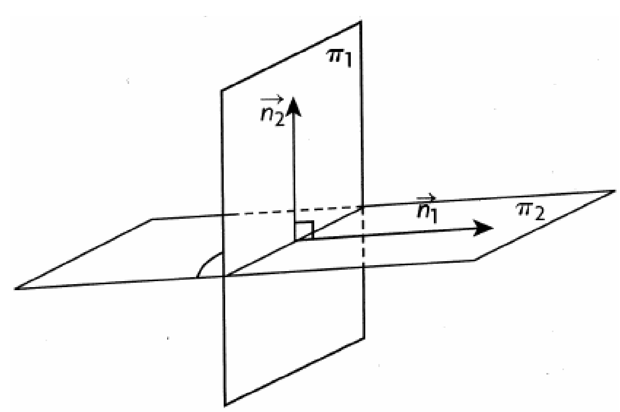
Let $P(x, y, z)$ be any point in the plane.

Parallel Planes



If $\pi_1 \parallel \pi_2$ then $\vec{n}_1 \parallel \vec{n}_2$ where $\vec{n}_1 = k\vec{n}_2$.

Perpendicular Planes



If $\pi_1 \perp \pi_2$ then $\vec{n}_1 \perp \vec{n}_2$ where $\vec{n}_1 \cdot \vec{n}_2 = 0$.

SUMMARY: The **scalar** or **Cartesian equation** of a plane in space has the form

$$Ax + By + Cz + D = 0$$

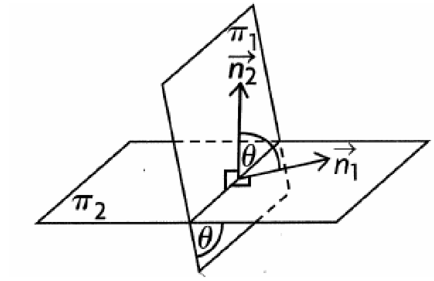
where (A, B, C) is a vector *normal* to the plane.

Ex. 1. Find the ***scalar equation*** of the plane through the point $A(-2,1,3)$ having a normal vector $\vec{n} = (4,-2,5)$.

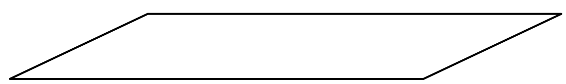
Ex. 2. Find the ***Cartesian equation*** of the plane that passes through the points $A(-2,3,1)$, $B(-1,2,-1)$ and $C(1,-2,3)$.

Ex. 3. Find the ***scalar equation*** of the plane containing the point $Q(1,2,-1)$ and the line $\vec{r} = (2,1,-1) + t(1,0,-3),\ t \in \Re$.

Ex. 4. Find the acute angle between the planes
$\pi_1 : x + 2y - 3z - 4 = 0$ and $\pi_2 : x - 3y + 5z + 7 = 0$.



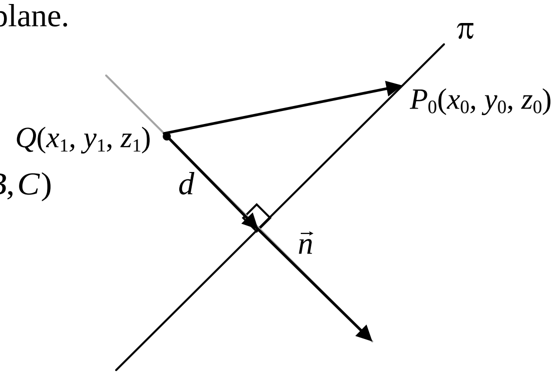
Ex. 5. Use vector projections to find the exact distance from the point $Q(1,2,3)$ to the plane
$4x - 5y + 7z - 10 = 0$.



Ex. 6. a) Show that the ***shortest distance*** from a point $Q(x_1, y_1, z_1)$ to a plane with a scalar equation $Ax + By + Cz + D = 0$ is given by the formula $d = \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

b) Use the formula to find the distance from the point $Q(1,2,3)$ to the plane $4x - 5y + 7z - 10 = 0$.

a) Let $P_0(x_0, y_0, z_0)$ be a specific point in the plane.



$$\overrightarrow{QP_0} = \overrightarrow{OP_0} - \overrightarrow{OQ}$$

$$\overrightarrow{QP_0} = (x_0, y_0, z_0) - (x_1, y_1, z_1)$$

$$\therefore \overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1) \text{ and } \vec{n} = (A, B, C)$$

$$d = \left| \underset{\overrightarrow{QP_0}\, on\, \vec{n}}{VP} \right|$$

$$= \frac{|\overrightarrow{QP_0} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (A, B, C)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_0 - Ax_1 + By_0 - By_1 + Cz_0 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}}$$

Note: Since the point $P_0(x_0, y_0, z_0)$ is in the plane $Ax + By + Cz + D = 0$, it satisfies the equation.
ie. $Ax_0 + By_0 + Cz_0 + D = 0$, so $Ax_0 + By_0 + Cz_0 = -D$

**** see Note above***

$$= \frac{|-Ax_1 - By_1 - Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|-(Ax_1 + By_1 + Cz_1 + D)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$\therefore d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

HW: p. 285 #1-13 & Additional Questions

1. What is the exact distance from the point $Q(1, 3, -2)$ to the plane $4x - y - z + 6 = 0$?

2. What is the exact distance between the planes $2x - y - 2z + 3 = 0$ and $4x - 2y - 4z - 9 = 0$?

Answers: **1.** $\dfrac{3\sqrt{2}}{2}$ **2.** $\dfrac{5}{2}$

Date:_____________ **Section 8.3 – The Intersection of a Line and a Plane**

Warm-up:

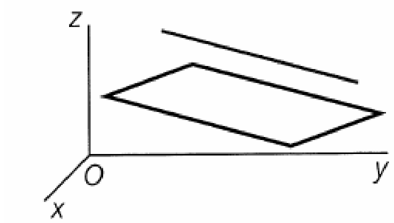
Given $A(1,5,9)$, $B(-2,6,8)$ and $C(12,7,16)$ are points in a plane, find:

a) the vector, parametric and scalar equations of the plane.

b) the exact distance the point $Q(1,-2,1)$ is from the plane.

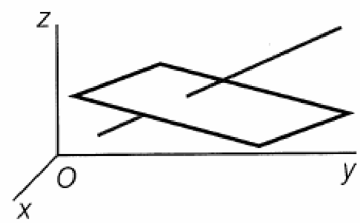
There are three possible ways that a line and a plane in three dimensions can intersect.

The line is parallel and distinct from the plane.



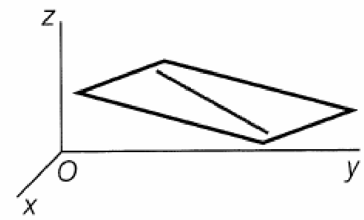
There is no intersection.

The line intersects the plane.



The intersection is a point.

The line lies in the plane.



The intersection is the line.

Ex. 1. For each of the following, find the intersection of the line and the plane.

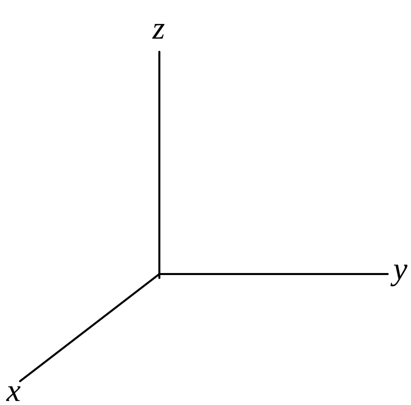
a) $L: \vec{r} = (1,-3,-2) + t(2,1,3),\ t \in \Re$ and $\pi: 2x + 4y - z + 3 = 0$

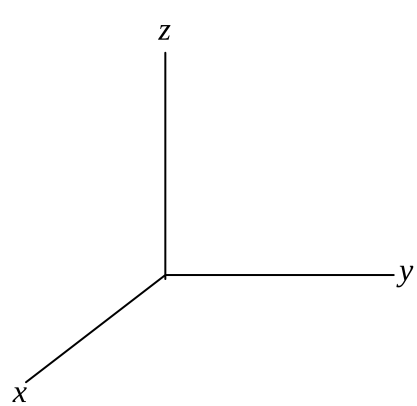
b) $L: \frac{x-2}{5} = \frac{y+1}{3} = z$ and $\pi: 8x - 13y - z - 29 = 0$

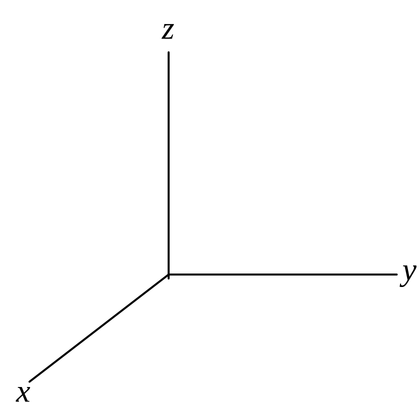
c) $L: \vec{r} = (2,1,5) + t(11,9,1),\ t \in \Re$ and $\pi: 2x - 3y + 5z - 8 = 0$

Ex. 2. For each of the following planes, find the *x*, *y* and *z*-intercepts and sketch the plane.

a) $2x + 3y + 4z - 12 = 0$ **b)** $3y + 2z - 6 = 0$ **c)** $y - 6 = 0$







HW: p. 292 #1-11

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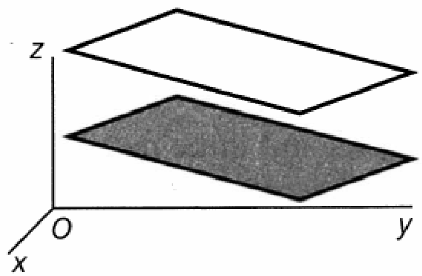
Section 8.4 – The Intersection of Two Planes

There are three ways two planes can intersect.

Planes are parallel and distinct.

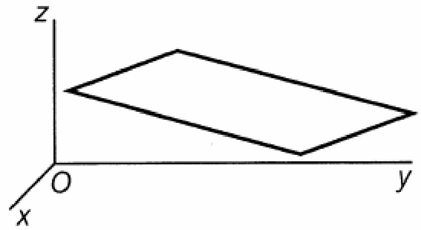


There is no intersection.

ex. $\pi_1 : 2x+5y+z-3=0$

$\pi_2 : 4x+10y+2z+1=0$

Planes are parallel and coincident.

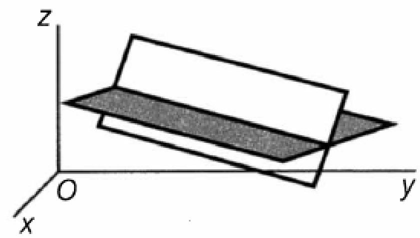


The intersection is the plane.

ex. $\pi_1 : 2x+5y+z-3=0$

$\pi_2 : 4x+10y+2z-6=0$

Planes intersect.



The intersection is a line.

ex. $\pi_1 : 2x+5y+z-3=0$

$\pi_2 : 3x-y+5z+1=0$

Ex. 1. Find the intersection of each of the following:

a) $\pi_1 : 5x-2y+3z+1=0$

$\pi_2 : 5x-2y+3z-4=0$

b) $\pi_1 : x+2y+7z-4=0$

$\pi_2 : 2x+4y+14z-8=0$

c) $\pi_1 : x+y-3z-4=0$

$\pi_2 : x+2y-z-1=0$

Solution 1: Solve algebraically.

Solution 2: Solve using an augmented 2×4 matrix.

d) $\pi_1 : x + 2y + 7z - 4 = 0$

$\pi_2 : x + 3y - 3z - 1 = 0$

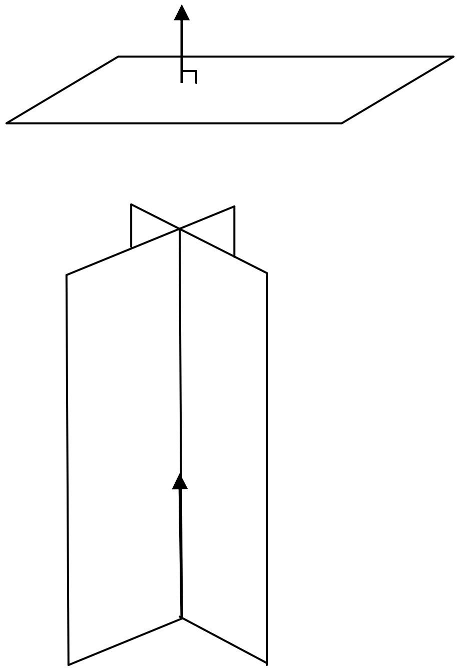
e) $\pi_1 : x - 4y + 3z - 5 = 0$

$\pi_2 : 2x - y + 6z = 0$

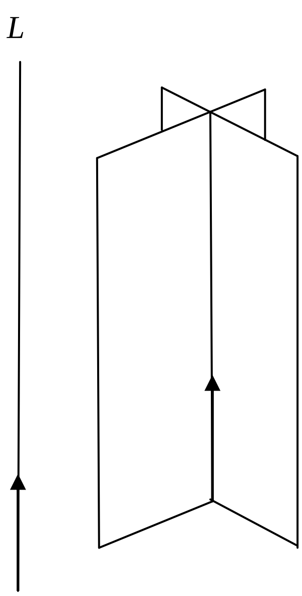
f) $\pi_1 : x + 4y - z - 3 = 0$

$\pi_2 : 2x + 8y + 2z - 7 = 0$

Ex. 2. Find the ***scalar equation*** of the plane which passes through the point $A(2,0,-1)$ and is perpendicular to the line of intersection of $\pi_1 : 2x + y - z + 5 = 0$ and $\pi_2 : x + y + 2z + 7 = 0$.



Ex. 3. Find the ***vector equation*** of the line that passes through the point $B(2,-1,7)$ and is parallel to the line of intersection of $\pi_1 : x + 2y - 3z + 5 = 0$ and $\pi_2 : 3x - y + 2z - 4 = 0$.



HW: p. 300 #2-6, 8bc, 9

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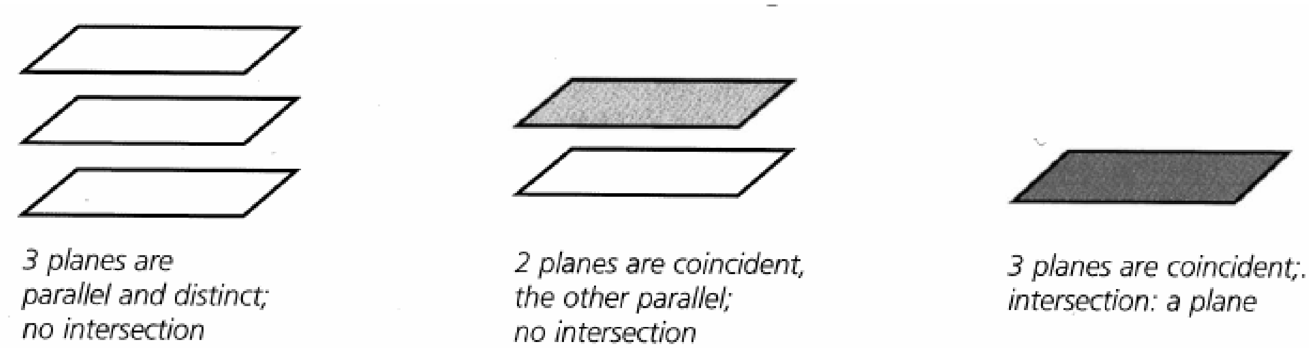
Section 8.5 – The Intersection of Three Planes

There are eight possible ways three planes can intersect.

When the normal of all three planes are parallel, the possibilities are

Case #1 **Case #2** **Case #3**



Examples of each case:

Find the intersection of the following pairs of lines, if any exist:

#1. $\pi_1 : x+2y+4z-2=0 \rightarrow \pi_1 : x+2y+4z-2=0$

$\pi_2 : -x-2y-4z+5=0 \rightarrow \pi_2 : x+2y+4z-5=0$

$\pi_3 : 2x+4y+8z-12=0 \rightarrow \pi_3 : x+2y+4z-6=0$

#2. $\pi_1 : -5x+4y+3z-2=0 \rightarrow \pi_1 : 5x-4y-3z+2=0$

$\pi_2 : 10x-8y-6z+4=0 \rightarrow \pi_2 : 5x-4y-3z+2=0$

$\pi_3 : -15x+12y+9z-21=0 \rightarrow \pi_3 : 5x-4y-3z+7=0$

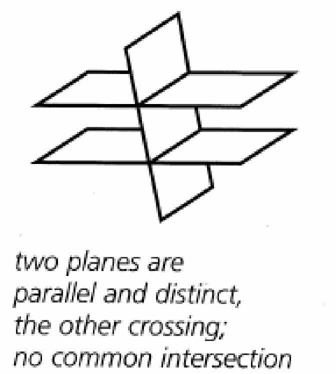
#3. $\pi_1 : -x+y+3z-2=0 \rightarrow \pi_1 : x-y-3z+2=0$

$\pi_2 : 2x-2y+6z+4=0 \rightarrow \pi_2 : x-y-3z+2=0$

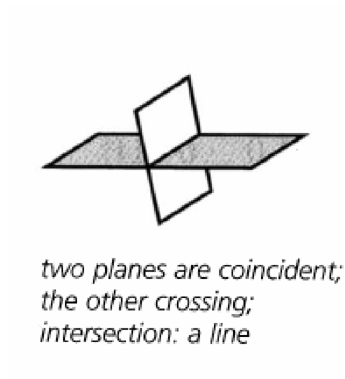
$\pi_3 : -3x+3y+9z-6=0 \rightarrow \pi_3 : x-y-3z+2=0$

When only two of the normals of the planes are parallel, the possibilities are

Case #4



Case #5

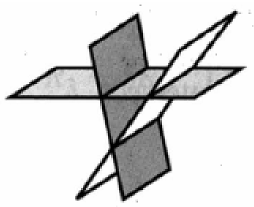


#4. $\pi_1 : 3x + 5y - 2z + 1 = 0$
$\pi_2 : -6x - 10y + 4z - 4 = 0$
$\pi_3 : x + 2y - 3z - 6 = 0$

#5. $\pi_1 : 4x + y + 7z - 2 = 0$
$\pi_2 : 8x + 2y + 14z - 4 = 0$
$\pi_3 : x - 3y + 5z - 6 = 0$

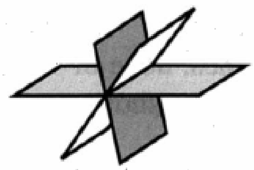
When none of the normals are parallel, the possibilities are

Case #6



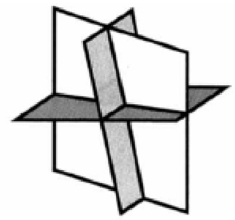
normals coplanar;
no intersection
since the planes form
a triangular prism

Case #7



normals coplanar;
intersection: a line
like the pages of
a book

Case #8



normals are not parallel
and non-coplanar;
intersection: a point
like the corner of
a room

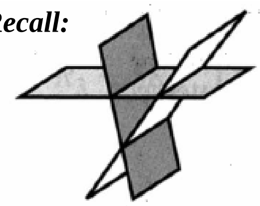
***Note*:** If all normals are coplanar, then, $(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$.

#6. $\pi_1 : x + 2y - z - 1 = 0$
$\pi_2 : x - 5y + 4z + 4 = 0$
$\pi_3 : 3x - y + 2z - 2 = 0$

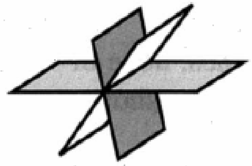
#7. $\pi_1 : x + 2y + z - 2 = 0$
$\pi_2 : x + 3y - z - 4 = 0$
$\pi_3 : 3x + 7y + z - 8 = 0$

#8. $\pi_1 : x + y + z - 4 = 0$
$\pi_2 : x + 2y + 2z - 2 = 0$
$\pi_3 : x - y + z + 2 = 0$

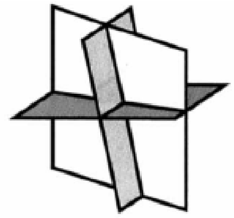
Recall:



normals coplanar; no intersection



normals coplanar; intersection: a line



normals are not parallel and non-coplanar; intersection: a point

Ex. Determine the intersection, if any, of each of the following sets of planes. In each case, give a geometrical interpretation of the system of equations and the solution. Also state whether the system has no solutions, a unique solution, or an infinite number of solutions.

a) $2x+4y+z=2$
$5x+5y+3z=17$
$4x-y+3z=26$

b) $2x+7y+2z=3$
$6x+y-4z=-1$
$2x+9y+3z=4$

HW: p. 308 #4 & 7 (use matrix), 8, 9, 3, 2;
REVIEW for TEST: p. 312 #2-12 (8.1 & 8.2), 13-16 & 18 (8.3), 17 & 19-22 (8.4 & 8.5); p. 315 #1-7ab