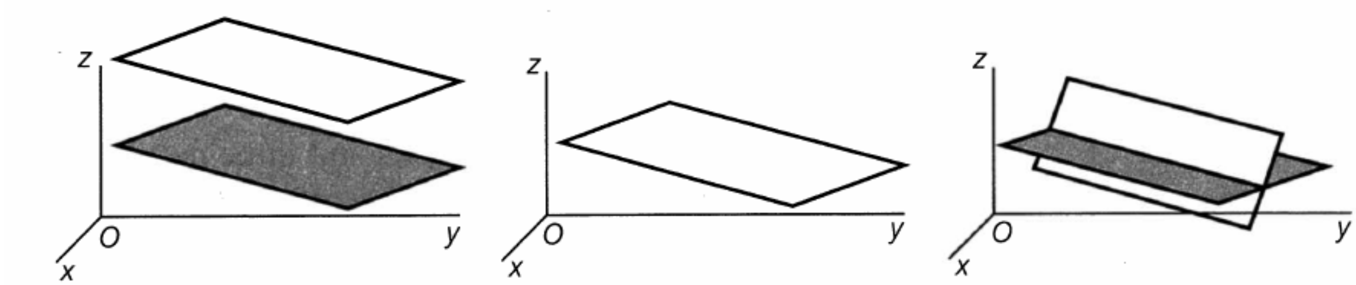
Date: June 9/14 **Section 8.4 – The Intersection of Two Planes**

There are three ways two planes can intersect.

Planes are parallel and distinct. **Planes are parallel and coincident.** **Planes intersect.**



There is no intersection.	The intersection is the plane.	The intersection is a line.
ex. $\pi_1 : 2x+5y+z-3=0$	**ex.** $\pi_1 : 2x+5y+z-3=0$	**ex.** $\pi_1 : 2x+5y+z-3=0$
$\pi_2 : 4x+10y+2z+1=0$	$\pi_2 : 4x+10y+2z-6=0$	$\pi_2 : 3x-y+5z+1=0$

Ex. 1. Find the intersection of each of the following:

a) $\pi_1 : 5x-2y+3z+1=0$
$\pi_2 : 5x-2y+3z-4=0$

$\because \vec{n}_2 = k\vec{n}_1, k=1$ but $D_2 \neq 1D_1$
The planes are parallel & distinct.
There is no intersection.

b) $\pi_1 : x+2y+7z-4=0$
$\pi_2 : 2x+4y+14z-8=0$

$\because \pi_2 = k\pi_1, k=2$ $\therefore$ The planes are coincident. The intersection is the plane $x+2y+7z-4=0$.

c) $\pi_1 : x+y-3z-4=0$ ①
$\pi_2 : x+2y-z-1=0$ ②

$\because \vec{n}_2 \neq k\vec{n}_1$, $\therefore$ the planes intersect in a line.

Solution 1: Solve algebraically.

Eliminate x

② - ① $y+2z+3=0$

Let $\boxed{z=t}$ and find y

$y+2t+3=0$

$\boxed{y=-3-2t}$

Find x, sub in ①

$x-3-2t-3t-4=0$

$x-7-5t=0$

$\boxed{x=7+5t}$

$\therefore$ the line of intersection is $x=7+5t$, $y=-3-2t$, $z=t$

or $\vec{r}=(7,-3,0)+t(5,-2,1), t\in\mathbb{R}$

Solution 2: Solve using an augmented 2×4 matrix.

$$\begin{array}{cccc} x & y & z & = \# \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 1 & 2 & -1 & 1 \end{array}\right]$$

Goal

$$\left[\begin{array}{ccc|c} \# & \# & \# & \# \\ 0 & \# & \# & \# \end{array}\right]$$

$$R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & -2 & 3 \end{array}\right]$$

$-y-2z=3$ & $x+y-3z=4$

Let $\boxed{z=t}$

Find y

$-y-2t=3$

$-y=3+2t$

$\boxed{y=-3-2t}$

Find x

$x-3-2t-3t=4$

$\boxed{x=7+5t}$

d) $\pi_1 : x+2y+7z-4=0$

$\pi_2 : x+3y-3z-1=0$

$\because \vec{n}_1 \neq k\vec{n}_2 \therefore$ the planes intersect in a line.

$$\left[\begin{array}{ccc|c} 1 & 2 & 7 & 4 \\ 1 & 3 & -3 & 1 \end{array}\right]$$

$$R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 2 & 7 & 4 \\ 0 & -1 & 10 & 3 \end{array}\right]$$

$\therefore \underbrace{-y+10z=3}$ & $x+2y+7z=4$

Let $\boxed{z=t}$

Find y

$-y+10t=3$

$-y=3-10t$

$\boxed{y=-3+10t}$

Find x

$x+2(-3+10t)+7t=4$

$x-6+20t+7t=4$

$x+27t-6=4$

$\boxed{x=10-27t}$

$\therefore$ the plane intersect in the line

$x=10-27t,\ y=-3+10t,\ z=t$ or

$\vec{r}=(10,-3,0)+t(-27,10,1),\ t\in\mathbb{R}$

e) $\pi_1 : x-4y+3z-5=0$

$\pi_2 : 2x-y+6z=0$

$\because \vec{n}_1 \neq k\vec{n}_2,\ \therefore$ the planes intersect in a line.

$$\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ 2 & -1 & 6 & 0 \end{array}\right]$$

$$2R_1 - R_2 \left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ 0 & -7 & 0 & 10 \end{array}\right]$$

$-7y=10$ & $x-4y+3z=5$

Let $\boxed{z=t}$

Find y

$\boxed{y=-\frac{10}{7}}$

Find x

$x+\frac{4}{1}\times\frac{10}{7}+3t=\frac{5}{1}$

$x+\frac{40}{7}+3t=\frac{35}{7}$

$\boxed{x=-\frac{5}{7}-3t}$

$\therefore$ the planes intersect in the line

$x=-\frac{5}{7}-3t,\ y=-\frac{10}{7},\ z=t$ or

$\vec{r}=\left(-\frac{5}{7},-\frac{10}{7},0\right)+t(-3,0,1),\ t\in\mathbb{R}$

f) $\pi_1 : x+4y-z-3=0$

$\pi_2 : 2x+8y+2z-7=0$

$\because \vec{n}_1 \neq k\vec{n}_2,\ \therefore$ the planes intersect in a line.

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 3 \\ 2 & 8 & 2 & 7 \end{array}\right]$$

$$2R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 4 & -1 & 3 \\ 0 & 0 & -4 & -1 \end{array}\right]$$

$\therefore -4z=-1$ & $x+4y-z=3$

$\boxed{z=\frac{1}{4}}$

$\boxed{\text{Let } y=t}$

Find x

$x+4t-\frac{1}{4}=\frac{3}{1}$

$x=\frac{12}{4}+\frac{1}{4}-4t$

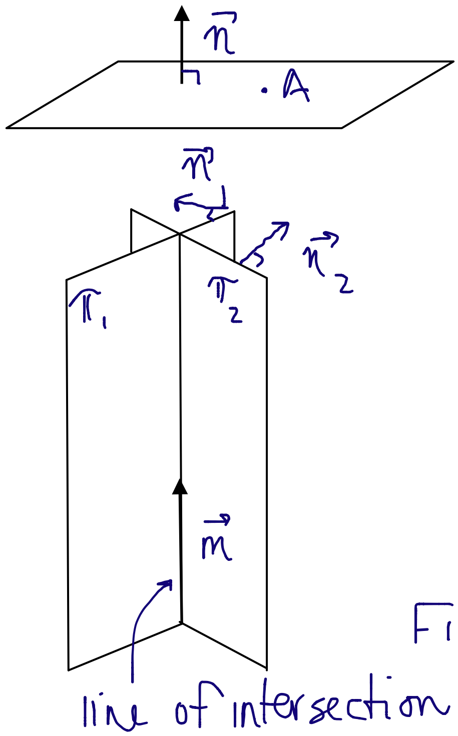
$\boxed{x=\frac{13}{4}-4t}$

$\therefore$ the planes intersect in the line $x=\frac{13}{4}-4t,\ y=t,\ z=\frac{1}{4}$ or

$\vec{r}=\left(\frac{13}{4},0,\frac{1}{4}\right)+t\underbrace{(-4,1,0)},\ t\in\mathbb{R}$

$Ax + By + Cz + D = 0, \quad \vec{n} = (A, B, C)$

Ex. 2. Find the ***scalar equation*** of the plane which passes through the point $A(2,0,-1)$ and is perpendicular to the line of intersection of $\pi_1 : 2x + y - z + 5 = 0$ and $\pi_2 : x + y + 2z + 7 = 0$.



Find $\vec{m}$ for the line of intersection of π_1 & π_2.

$\vec{m} = \vec{n}_1 \times \vec{n}_2$

$\vec{m} = (2,1,-1) \times (1,1,2)$

$= (3, -5, 1)$

For required plane

$\vec{n} = (\underset{A}{3}, \underset{B}{-5}, \underset{C}{1})$; $A(\underset{x}{2}, \underset{y}{0}, \underset{z}{-1})$

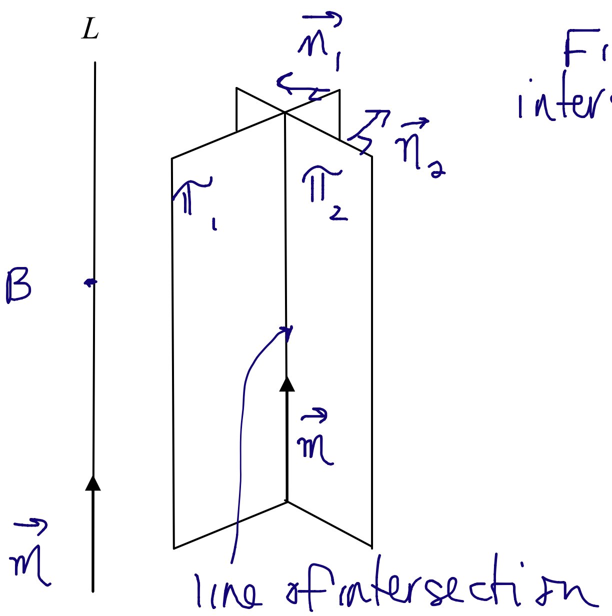
Find D: $Ax + By + Cz + D = 0$

$3(2) - 5(0) + 1(-1) + D = 0$

$D = -5$

$\therefore$ the scalar equation of the plane is $3x - 5y + z - 5 = 0$.

Ex. 3. Find the ***vector equation*** of the line that passes through the point $B(2,-1,7)$ and is parallel to the line of intersection of $\pi_1 : x + 2y - 3z + 5 = 0$ and $\pi_2 : 3x - y + 2z - 4 = 0$.



Find $\vec{m}$ for the line of intersection of π_1 and π_2.

$\vec{m} = \vec{n}_1 \times \vec{n}_2$

$= (1,2,-3) \times (3,-1,2)$

$= (1, -11, -7)$

For the required line,

$\vec{r} = \vec{r}_1 + t\vec{m}, \quad t \in \mathbb{R}$

$\therefore \vec{r} = (2,-1,7) + t(1,-11,7), \ t \in \mathbb{R}$

is the vector equation of the line.

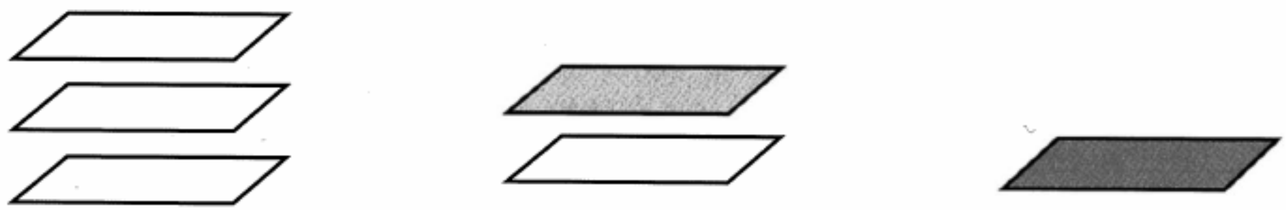
HW: pg. 300 #2-6, 8bc, 9

Date: June 10/14 **Section 8.5 – The Intersection of Three Planes**

There are eight possible ways three planes can intersect.

When the normal of all three planes are parallel, the possibilities are

Case #1 **Case #2** **Case #3**



3 planes are parallel and distinct; no intersection

2 planes are coincident, the other parallel; no intersection

3 planes are coincident; intersection: a plane

Examples of each case:

Find the intersection of the following ~~pairs of lines,~~ planes if any exist:

#1. $\pi_1: x+2y+4z-2=0 \rightarrow \pi_1: x+2y+4z-2=0$

$\pi_2: -x-2y-4z+5=0 \rightarrow \pi_2: x+2y+4z-5=0$

$\pi_3: 2x+4y+8z-12=0 \rightarrow \pi_3: x+2y+4z-6=0$

$\because \vec{n}_1 = \vec{n}_2 = \vec{n}_3$

$\&\ D_1 \neq D_2 \neq D_3$

$\therefore$ the planes are parallel and distinct. There is no intersection.

#2. $\pi_1: -5x+4y+3z-2=0 \rightarrow \pi_1: 5x-4y-3z+2=0$

$\pi_2: 10x-8y-6z+4=0 \rightarrow \pi_2: 5x-4y-3z+2=0$

$\pi_3: -15x+12y+9z-21=0 \rightarrow \pi_3: 5x-4y-3z+7=0$

$\because \vec{n}_1 = \vec{n}_2 = \vec{n}_3$

$\&\ D_1 = D_2 \neq D_3$

$\therefore$ 2 planes are coincident and the other is parallel and distinct. There is no common intersection.

#3. $\pi_1: -x+y+3z-2=0 \rightarrow \pi_1: x-y-3z+2=0$

$\pi_2: 2x-2y+6z+4=0 \rightarrow \pi_2: x-y-3z+2=0$

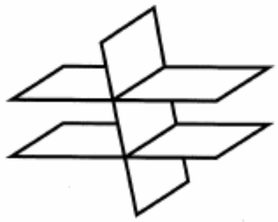
$\pi_3: -3x+3y+9z-6=0 \rightarrow \pi_3: x-y-3z+2=0$

$\because \pi_1 = \pi_2 = \pi_3$

$\therefore$ the 3 planes are coincident. The intersection is the plane $x-y-3z+2=0$.

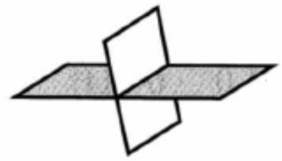
When only two of the normals of the planes are parallel, the possibilities are

Case #4



two planes are parallel and distinct, the other crossing; no common intersection

Case #5



two planes are coincident; the other crossing; intersection: a line

#4. $\pi_1: 3x+5y-2z+1=0 \rightarrow \pi_1: 3x+5y-2z+1=0$

$\pi_2: -6x-10y+4z-4=0 \rightarrow \pi_2: 3x+5y-2z+2=0$

$\pi_3: x+2y-3z-6=0 \rightarrow \pi_3: x+2y-3z-6=0$

$\because \vec{n}_1 = \vec{n}_2 \neq k\vec{n}_3$, $D_1 \neq D_2$

$\therefore$ The lanes π_1 and π_2 are paralle nd distinct with the ot r plane crossing. There is no common inl section.

#5. $\pi_1: 4x+y+7z-2=0 \rightarrow 4x+y+7z-2=0$

$\pi_2: 8x+2y+14z-4=0 \rightarrow 4x+y+7z-2=0$

$\pi_3: x-3y+5z-6=0 \rightarrow x-3y+5z-6=0$

π_1 and $_2$ are parallel and coincident with the pla π_3 crossing. The intersection is a line.

$$\left[\begin{array}{ccc|c} 1 & -3 & 5 & 6 \\ 4 & 1 & 7 & 2 \end{array}\right]$$

$$\begin{array}{c} \\ 4R_1 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 6 \\ 0 & -13 & 13 & 22 \end{array}\right]$$

$-13y+13z=22$ & $x-3y+5z=6$

Let $\boxed{y=t}$

Find z

$-13t+13z=22$

$13z=22+13t$

$$\boxed{z=\frac{22}{13}+t}$$

Find x

$x-3t+5\left(\frac{22}{13}+t\right)=6$

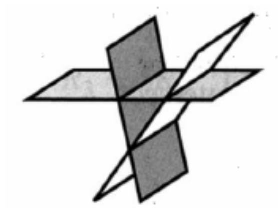
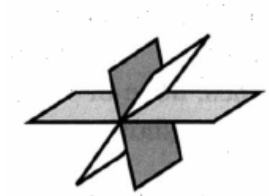
$x-3t+\frac{110}{13}+5t=\frac{78}{13}$

$$\boxed{x=-\frac{32}{13}-2t}$$

$\therefore$ the line of intersection is $x=-\frac{32}{13}-2t$, $=t$, $z=\frac{22}{13}+t$

or $\vec{r}=\left(-\frac{32}{13}, 0, \frac{22}{13}\right)+t(-2,1,1)$, $t \in \mathbb{R}$.

When none of the normals are parallel, the possibilities are

Case #6	**Case #7**	**Case #8**
		
normals coplanar; no intersection since the plane form a triangular prism.	*normals coplanar; intersection: a line* like the pages of a book	*normals are not parallel and non-coplanar; intersection: a point* like the corner of a roo

***Note*:** If all normals are coplanar, then, $(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$.

Gal: $\left[\begin{array}{ccc|c} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{array}\right]$

#6. $\pi_1: x+2y-z-1=0$

$\pi_2: x-5y+4z+4=0$

$\pi_3: 3x-y+2z-2=0$

∵ none of the normals are pa allel, the planes possibly intersect.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & -5 & 4 & -4 \\ 3 & -1 & 2 & 2 \end{array}\right]$$

$$\begin{array}{l} \\ R_1-R_2 \\ 3R_1-R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 7 & -5 & 5 \\ 0 & 7 & -5 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_2-R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 7 & -5 & 5 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

∴ $0z = 4$

$0 = 4$

no solution

∴ there is no common intr ction since the plane s form a trian ular prism.

#7. $\pi_1: x+2y+z-2=0$

$\pi_2: x+3y-z-4=0$

$\pi_3: 3x+7y+z-8=0$

∵ none of the normals are parallel, the planes possibly intersect.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & 3 & -1 & 4 \\ 3 & 7 & 1 & 8 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-R_1 \\ 3R_1-R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ $0z=0$

$0=0$

∴ they intersect in a line

$y-2z=2$; Let $\boxed{z=t}$

Find y

$y-2t=2$

$\boxed{y=2+2t}$

$x+2y+z=2$

Find x.

$x+2(2+2t)+t=2$

$x+4+4t+t=2$

$\boxed{x=-2-5t}$

∴ the planes intersect in the line $x=-2-5t$, $y=2+2t$, $z=t$ or $\vec{r}=(-2,2,0)+t(-5,2,1), t\in\mathbb{R}$ like the pages of a book.

#8. $\pi_1: x+y+z-4=0$

$\pi_2: x+2y+2z-2=0$

$\pi_3: x-y+z+2=0$

∵ none of the normals are parallel, the planes possibly intersect.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & -1 & 1 & -2 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-R_1 \\ R_2-R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 1 & 4 \end{array}\right]$$

$$\begin{array}{l} \\ \\ 3R_2-R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -10 \end{array}\right]$$

$2z=-10$; $\boxed{z=-5}$

$y+z=-2$; Find y.

$y-5=-2$

$\boxed{y=3}$

$x+y+z=4$

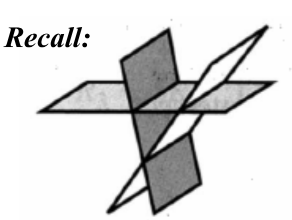
Find x.

$x+3-5=4$

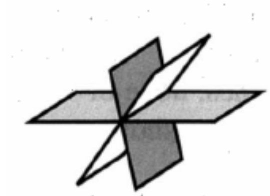
$\boxed{x=6}$

∴ the planes intersect at the point $(6,3,-5)$ like the corner of box.

Recall:



normals coplanar; no intersection



normals coplanar; intersection: a line

normals are not parallel and non-coplanar; intersection: a point

Ex. Determine the intersection, if any, of each of the following sets of planes. In each case, give a geometrical interpretation of the system of equations and the solution. Also state whether the system has no solutions, a unique solution, or an infinite number of solutions.

a) $2x+4y+z=2$
$5x+5y+3z=17$
$4x-y+3z=26$

∵ none of the normals are parallel, the planes possible intersect.

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 5 & 5 & 3 & 17 \\ 4 & -1 & 3 & 26 \end{array}\right]$$

$$\begin{array}{c} \\ 5R_1-2R_2 \\ 2R_1-R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 0 & 10 & -1 & -24 \\ 0 & 9 & -1 & -22 \end{array}\right]$$

$$\begin{array}{c} \\ \\ 9R_2-10R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 0 & 10 & -1 & -24 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

∴ $\boxed{z=4}$ & $10y-z=-24$ & $2x+4y+z=2$

Find y
$10y-4=-24$
$10y=-20$
$\boxed{y=-2}$

Find x.
$2x+4(-2)+4=2$
$2x-4=2$
$2x=6$
$\boxed{x=3}$

∴ the planes intersect at the point $(3,-2,4)$ like the corner of a room. The system has a unique solution.

b) $2x+7y+2z=3$
$6x+y-4z=-1$
$2x+9y+3z=4$

HW: pg. 308 #4 & 7 (use matrix), 8, 9, 3, 2;
Review pg. 312 #2-12 (8.1 & 8.2), 13-16 & 18 (8.3), 17 & 19-22 (8.4 & 8.5); pg. 315 #1-7ab