Date:_____________ **UNIT 3 – APPLICATIONS OF DERIVATIVES – Part B**

Section 5.4 – MAXIMUM AND MINIMUM ON AN INTERVAL

Algorithm for Maximum or Minimum (Extreme Values)

If a function $f(x)$ has a derivative at every point in the interval $a \le x \le b$, calculate $f(x)$ at
- all points in the interval $a \le x \le b$ where $f'(x) = 0$;
- the end points $x = a$ and $x = b$.

The ***maximum value*** of $f(x)$ on the interval $a \le x \le b$ is the *largest* of these values, and the ***minimum value*** of $f(x)$ on the interval is the *smallest* of these values.

Ex. 1. Find the maximum and minimum values(s) of $f(x)$ for $f(x) = -2x^2 - 12x - 10$ on the interval $-4 \le x \le 0$. Illustrate graphically.

$f(x) = -2x^2 - 12x - 10$

$f'(x) = -4x - 12$

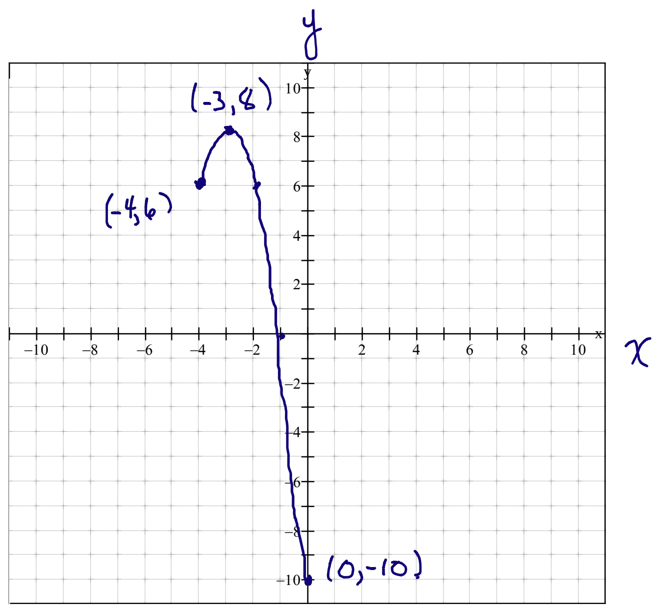
set $f'(x) = 0$

$-4x - 12 = 0$

$-4x = 12$

$* \; x = -3$

x	$f(x)$	
-4	6	
-3	8	max
0	-10	min.



∴ the maximum value of $f(x)$ is 8 and the minimum value of $f(x)$ is -10.

Ex. 2. Find the maximum and minimum values of the function $f(x) = x^3 - 6x^2 - 4$ for $-1 \le x \le 7$. Illustrate your results graphically.

$f(x) = x^3 - 6x^2 - 4$

$f'(x) = 3x^2 - 12x$

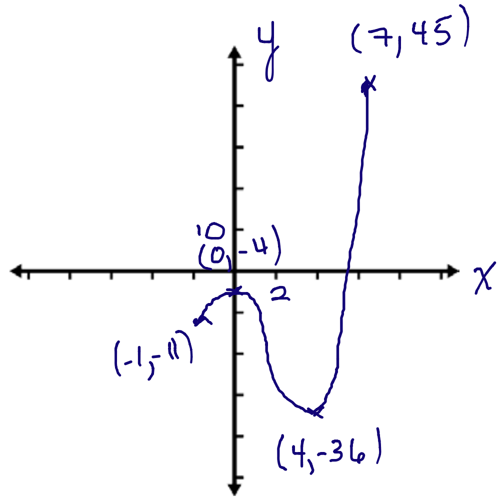
set $f'(x) = 0$

$3x^2 - 12x = 0$

$3x(x-4) = 0$

$x = 0$ or $x = 4$

x	$f(x)$
-1	-11
* 0	-4
* 4	-36 min
7	45 max



$\therefore$ the maximum value of $f(x)$ is 45 and the minimum value is -36.

Ex. 3. A flu epidemic breaks out in Waterloo. The fraction of the population that is infected at time, t, in weeks, is given by the function $f(t) = \dfrac{64t}{(8+t)^3}$.

What is the largest fraction of the population that is infected during the first 10 weeks? Assume that $t = 0$ is when the epidemic starts.

$0 \le t \le 10$

$f(t) = \dfrac{64t}{(8+t)^3}$

$f(t) = \underbrace{64t}\cdot\underbrace{(8+t)^{-3}}$

$f'(t) = 64\cdot(8+t)^{-3} - 3(8+t)^{-4}(1)(64t)$

$f'(t) = 64(8+t)^{-4}[(8+t) - 3t]$

$f'(t) = \dfrac{64(8-2t)}{(8+t)^4}$

set $f'(t) = 0$

$\dfrac{64(8-2t)}{(8+t)^4} = \dfrac{0}{1}$

$64(8-2t) = 0$

$8 - 2t = 0$

$t = 4$

$f(4) = \dfrac{\overset{4}{\cancel{\overset{16}{\cancel{64}}}}\times\cancel{4}}{\underset{3}{\cancel{12}}\times\underset{3}{\cancel{12}}\times\cancel{12}\,3}$

$f(10) = \dfrac{80}{729}$

t	$f(t)$
0	0
4	$\frac{4}{27} \doteq 0.15$ max.
10	$\frac{80}{729} \doteq 0.11$

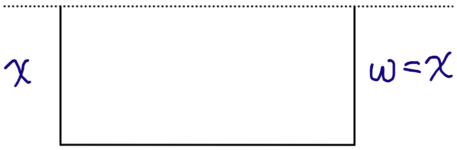
$\therefore$ the largest fraction of the population that is infected is $\frac{4}{27}$ during the first 10 weeks.

HW: p. 200 #2 to 10

Date: Mar. 25/14

Section 5.5 – OPTIMIZATION PROBLEMS

Ex. 1. A man wishes to enclose a rectangular area against the wall of a chicken house to form a chicken run. The wall is 50 m long. He has 60 m of fencing. What is the largest area he can enclose and what are its dimensions?



Find l in terms of x

60 m of fencing

$l + 2x = 60$

$l = 60 - 2x$

x	A
0	0
15	450 max.
30	0

Maximize area, A in m^2

$A = l \cdot w$

$A = (60-2x)(x)$

$A = 60x - 2x^2$

$\frac{dA}{dx} = 60 - 4x$

For max/min $\frac{dA}{dx} = 0$

$60 - 4x = 0$

$x = 15$

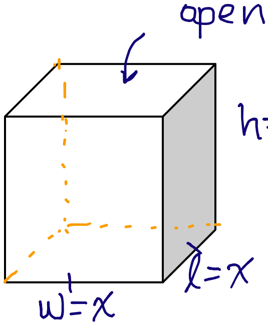
if $x = 15$

$A = 450$

$w = 15$ & $l = 30$

∴ the largerst area he can enclose is 450 m^2 when the chicken run measures 30 m by 15 m.

Ex. 2. An open-topped storage box is to have a square base and vertical sides. If 108 m^2 of sheet metal is available for its construction, find the dimensions for maximum volume.



$h = \frac{108 - x^2}{4x}$

Find h in terms of x

$S.A. = 108 \text{ m}^2$

$A_{bottom} + A_{front \,\&\, back} + A_{left \,\&\, right} = 108$

$x^2 + 2xh + 2xh = 108$

$x^2 + 4xh = 108$

$4xh = 108 - x^2$

$h = \frac{108 - x^2}{4x}$

Maximize volume, V, in m^3.

$V = lwh$

$V = x \cdot \cancel{x} \cdot \frac{108 - x^2}{4\cancel{x}}$

$V = \frac{108x - x^3}{4}$

$V = 27x - \frac{1}{4}x^3$

$\frac{dV}{dx} = 27 - \frac{3}{4}x^2$

For max/min $\frac{dV}{dx} = 0$

$27 - \frac{3}{4}x^2 = 0$

$108 - 3x^2 = 0$

$x^2 = 36$

$x = 6, \quad x \geq 0$

if $x = 6$,

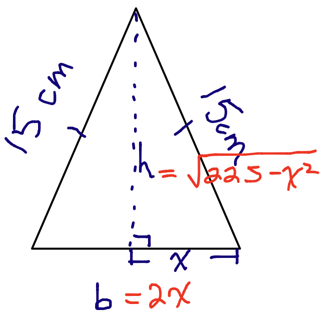
$w = 6$, $l = 6$ &

$h = 3$

x	V
0	0
6	108 max.
$\sqrt{108}$	0

∴ for maximum volume, the storage box measures 6 m × 6 m × 3 m.

Ex. 3. What is the maximum possible area of an isosceles triangle if the two equal sides are each 15 cm long?



Find h in terms of x

$$h^2 + x^2 = 15^2$$
$$h^2 = 225 - x^2$$
$$h = \sqrt{225 - x^2}$$

x	A
0	0
$15/\sqrt{2}$	$112\frac{1}{2}$ max.
15	0

Maximize the area, A, in cm^2

$$A = \frac{1}{2} bh$$
$$A = \frac{1}{\cancel{2}}(\cancel{2}x) \cdot \sqrt{225 - x^2}$$
$$A = x \cdot (225 - x^2)^{\frac{1}{2}}$$
$$\frac{dA}{dx} = 1 \cdot (225 - x^2)^{\frac{1}{2}} + \frac{1}{\cancel{2}}(225 - x^2)^{-\frac{1}{2}}(-\cancel{2}x) \cdot x$$
$$\frac{dA}{dx} = (225 - x^2)^{\frac{1}{2}} - x^2(225 - x^2)^{-\frac{1}{2}}$$
$$\frac{dA}{dx} = (225 - x^2)^{-\frac{1}{2}}\left[(225 - x^2) - x^2\right]$$
$$\frac{dA}{dx} = \frac{225 - 2x^2}{\sqrt{225 - x^2}}$$

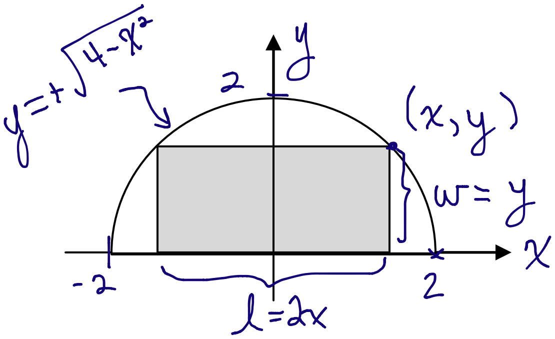
For max/min.
$$\frac{dA}{dx} = 0$$
$$225 - 2x^2 = 0$$
$$x^2 = \frac{225}{2}$$
$$x = \frac{15}{\sqrt{2}}$$

if $x = \frac{15}{\sqrt{2}}$

$$A = \sqrt{\frac{225}{2}} \cdot \sqrt{\frac{225}{1} - \frac{225}{2}}$$
$$A = \sqrt{\frac{225}{2}} \cdot \sqrt{\frac{225}{2}}$$
$$= \frac{225}{2}$$

$\therefore$ the maximum area is $112\frac{1}{2}$ cm^2

Ex. 4. Find the largest area of a rectangle inscribed in a semicircle of radius 2 cm.



For a circle with $C(0,0)$ & $r = 2$,

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$

$\therefore y = \sqrt{4 - x^2}$ for top half of circle

Maximize the area, A, in cm^2

$$A = lw$$
$$A = 2x \cdot y$$
$$A = 2x \cdot \sqrt{4 - x^2}$$
$$A = 2x \cdot (4 - x^2)^{\frac{1}{2}}$$
$$\frac{dA}{dx} = 2 \cdot (4 - x^2)^{\frac{1}{2}} + \frac{1}{\cancel{2}}(4 - x^2)^{-\frac{1}{2}}(-2x) \cdot \cancel{2}x$$
$$\frac{dA}{dx} = 2(4 - x^2)^{\frac{1}{2}} - 2x^2(4 - x^2)^{-\frac{1}{2}}$$
$$\frac{dA}{dx} = 2(4 - x^2)^{-\frac{1}{2}}\left[4 - x^2 - x^2\right]$$
$$\frac{dA}{dx} = \frac{2(4 - 2x^2)}{\sqrt{4 - x^2}}$$

For max/min
$$\frac{dA}{dx} = 0$$
$$4 - 2x^2 = 0$$
$$x^2 = 2$$
$$x = \sqrt{2}$$

x	A
0	0
$\sqrt{2}$	4 max.
2	0

if $x = \sqrt{2}$

$$A = 2 \cdot \sqrt{2} \cdot \sqrt{4 - 2}$$
$$= 2\sqrt{2} \cdot \sqrt{2}$$
$$= 4$$

$\therefore$ the maximum are is 4 cm^2.

HW: p. 206 #1 to 9

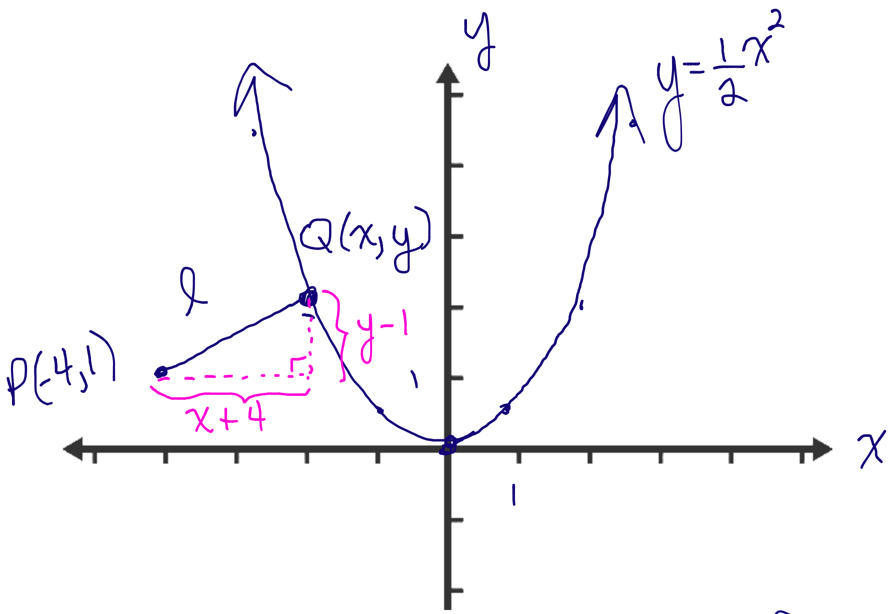
Date: Mar. 26/14

Section 5.5 – OPTIMIZATION PROBLEMS continued

Ex. 1. Find the point on the parabola $2y = x^2$ that is closest to the point $(-4, 1)$.

$2y = x^2$

$y = \frac{1}{2}x^2$



minimize the length, l, of PQ

$l^2 = (x+4)^2 + (y-1)^2$

$\because y = \frac{1}{2}x^2$

$l^2 = (x+4)^2 + \left(\frac{1}{2}x^2 - 1\right)^2$

diff. w.r.t. x

$\cancel{2}l\frac{dl}{dx} = \cancel{2}(x+4) + \cancel{2}\left(\frac{1}{2}x^2 - 1\right)(x)$

For max/min, $\frac{dl}{dx} = 0$

$0 = \cancel{x} + 4 + \frac{1}{2}x^3 - \cancel{x}$

$-4 = \frac{1}{2}x^3$

$-8 = x^3$

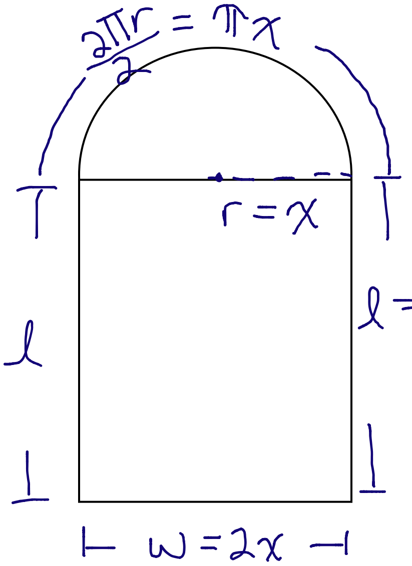
$x = -2$

if $x = -2$, $y = \frac{1}{2}(-2)^2$

$y = 2$

$\therefore (-2, 2)$ is the point on the parabola that is closest to $(-4, 1)$

Ex. 2. A Norman window is in the form of a rectangle surmounted by a semi-circle with a diameter equal to the width of the rectangle. If the perimeter of the window is 14 m, what is the maximum area of the window to the nearest tenth of a m^2.



$l = \frac{14 - 2x - \pi x}{2}$

Find l in terms of x

$P = 14$ m

$2l + 2x + \pi x = 14$

$2l = 14 - 2x - \pi x$

$l = \frac{14 - 2x - \pi x}{2}$

maximize the area, A, in m^2

$A = lw + \frac{1}{2}\pi r^2$

$A = \cancel{2}x\left(\frac{14 - 2x - \pi x}{\cancel{2}}\right) + \frac{1}{2}\pi(x)^2$

$A = 14x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$

$A = 14x - 2x^2 - \frac{\pi}{2}x^2$

$\frac{dA}{dx} = 14 - 4x - \pi x$

For max/min $\frac{dA}{dx} = 0$

$0 = 14 - 4x - \pi x$

$4x + \pi x = 14$

$x(4 + \pi) = 14$

$x = \frac{14}{4 + \pi}$

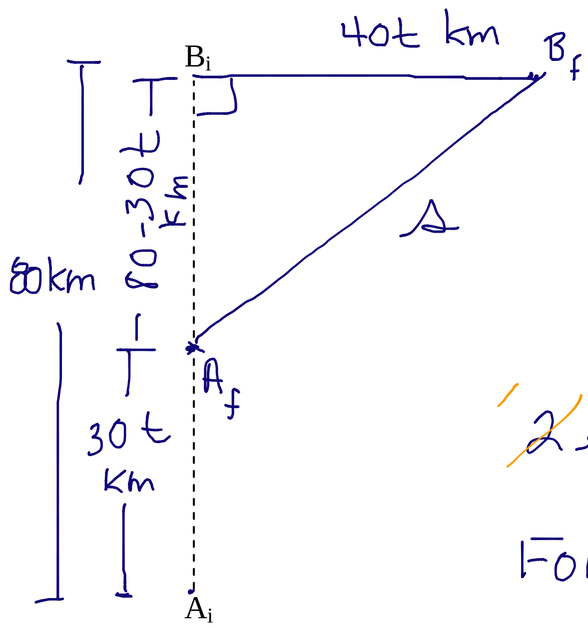
$x \doteq 1.96$

If $x \doteq 1.96$

$A \doteq 13.7$

$\therefore$ the maximum area of the window is about 13.7 m^2.

Ex. 3. At 1:00 p.m., ship A was 80 km south of ship B. Ship A is sailing north at 30 km/h and ship B is sailing east at 40 km/h. Find when the distance between the ships is at a minimum.



Minimize the distance, s, between the ships in km, t hours after 1:00 p.m.

$$s^2 = (80-30t)^2 + (40t)^2$$

diff. w.r.t t

$$\cancel{2} s \frac{ds}{dt} = \cancel{2}(80-30t)(-30) + \cancel{2}(40t)(40)$$

For max/min

$$\frac{ds}{dt} = 0$$

$$0 = -2400 + 900t + 1600t$$

$$2400 = 2500t$$

$$\frac{24}{25} = t$$

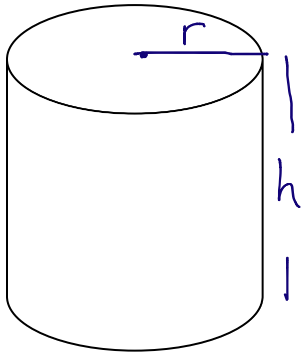
$$t = 0.96$$

$\therefore$ the distance between the ships is at a minimum at 0.96 hours after 1:00 p.m. or at 1:58 p.m. approximately.

Ex. 4. A manufacturer wishes to produce cylindrical fruit juice cans with a capacity of 250 ml.

a) What dimensions will minimize the amount of material required for a can? (1 ml = 1 cm^3) Answer to the nearest tenth of a cm.

b) What exact ratio of height to diameter will minimize the amount of material required.



$h = \frac{250}{\pi r^2}$

Find h in terms of r.

$V = 250\ cm^3$

$\pi r^2 h = 250$

$h = \frac{250}{\pi r^2}$

a) minimize the surface area, S.A. in cm^2

$S.A. = A_{top} + A_{bottom} + A_{sides}$

$S.A. = 2\pi r^2 + 2\pi r h$

$S.A. = 2\pi r^2 + 2\cancel{\pi} r \left(\frac{250}{\cancel{\pi} r^2}\right)$

$S.A. = 2\pi r^2 + \frac{500}{r}$

$S.A = 2\pi r^2 + 500r^{-1}$

$\frac{dS.A.}{dr} = 4\pi r - 500r^{-2}$

For max/min,

$\frac{dS.A.}{dr} = 0$

$4\pi r - \frac{500}{r^2} = 0$

$4\pi r^3 - 500 = 0$

$r^3 = \frac{125}{\pi}$

$r = \sqrt[3]{\frac{125}{\pi}}$

$r \doteq 3.41$

if $r \doteq 3.41$,

$h \doteq 6.84$

$\therefore$ the amount of material required is minimized if the radius and height are approximately 3.4 cm and 6.8 cm respectively.

b) Find $\frac{h}{d}$

$\frac{h}{d}$

$= \frac{\frac{250}{\pi r^2}}{2r}$

$= \frac{250}{\pi r^2} \cdot \frac{1}{2r}$

$= \frac{125}{\pi r^3}$

if $r^3 = \frac{125}{\pi}$

$\frac{h}{d} = \frac{125}{\cancel{\pi}\left(\frac{125}{\cancel{\pi}}\right)}$

$= \frac{1}{1}$

$\therefore$ the $h:d = 1:1$ for a minimum surface area.

HW: p. 207 #10 to 18

Date: Mar. 28/14

Section 5.6 – Optimizing in Economics and Science

Ex. 1. A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is $7.00 per person. Market research shows that 40 fewer people would ride the train for each $0.10 increase in the fare. If the capacity of the train is 2600 passengers, and contracts with the rail employees require that at least 1600 passengers be carried, what fare should the railway charge to get the largest possible revenue?

Maximize the revenue, R, in $.

Let x represent the number of fare increases.

$R = \text{no. of passengers} \times \text{fare}$

$R = (2000 - 40x)(7 + 0.1x)$

$R = 14000 - 80x - 4x^2$

$\frac{dR}{dx} = -80 - 8x$

For max/min $\frac{dR}{dx} = 0$

$-80 - 8x = 0$

$-8x = 80$

$x = -10$

Restricted domain

$1600 \le 2000 - 40x \le 2600$

$-400 \le -40x \le 600$

$10 \ge x \ge -15$

or

$-15 \le x \le 10$

x	R
-15	14300
-10	14400 ← max
10	12800

For maximum revenue.

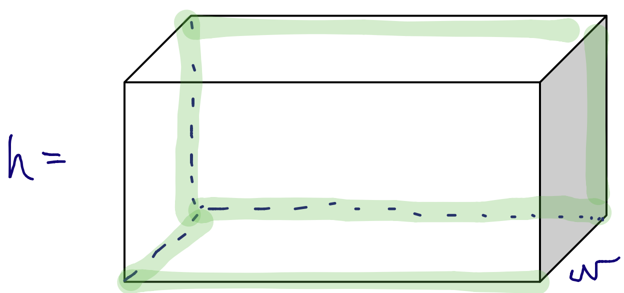
$x = -10$

$\text{fare} = 7 + 0.1(-10)$

$= 6$

$\therefore$ the fare should be $6 to get the largest possible revenue.

Ex. 2. The base of a chest is a rectangle which is twice as long as it is wide. The top, front, and sides are made of oak, and the back and base are made of pine. The chest has a volume of 12.25 m^3 and oak costs three times as much as pine. Find the dimensions of the box for which the lumber has the lowest cost. (Neglect any effects due to the thickness of the sides, top, or base.)



$l = 2w$

Minimize the cost, C, in \$

Let pine cost \$ k/m^2

& oak cost \$ 3k/m^2

$C = k \cdot S.A._{pine} + 3k \cdot S.A._{oak}$

$C = k(2w^2 + 2wh) + 3k(2w^2 + 2wh + 2wh)$

$C = 2kw^2 + 2kwh + 6kw^2 + 6kwh + 6kwh$

$C = 8kw^2 + 14kwh$

if $h = \frac{6.125}{w^2}$

$C = 8kw^2 + 14kw\left(\frac{6.125}{w^2}\right)$

$C = 8kw^2 + 85.75k\,w^{-1}$

$\frac{dC}{dw} = 16kw - 85.75kw^{-2}$

For max/min, $\frac{dC}{dw} = 0$

$16kw - \frac{85.75k}{w^2} = 0$

$\div$ by k

$16w^3 - 85.75 = 0$

$w^3 = \frac{85.75}{16}$

$w = 1.75$

Find h in terms of w

$V = 12.25 \text{ m}^3$

$lwh = 12.25$

$(2w) \cdot w \cdot h = 12.25$

$2w^2h = 12.25$

$h = \frac{12.25}{2w^2}$

$h = \frac{6.125}{w^2}$

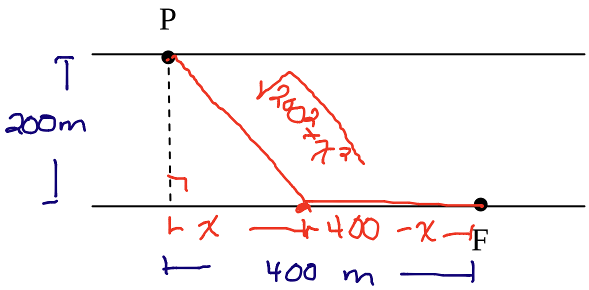
if $w = 1.75$

$l = 3.5$

$h = 2$

$\therefore$ the lumber has the lowest cost if the chest measures 3.5 m $\times$ 1.75 m $\times$ 2 m.

Ex. 3. A powerhouse, P, is on one bank of a straight river 200 m wide, and a factory, F, is on the opposite bank, 400 m downstream from P. The cable must be run from the powerhouse to the factory at a cost of \$6/m under water and \$4/m on land. What path should be chosen so that the cost is minimized?



Minimize the cost, C, in \$.

$C = 6 \times d_{water} + 4 \times d_{land}$

$C = 6\sqrt{40000 + x^2} + 4(400 - x)$

$C = 6(40000 + x^2)^{\frac{1}{2}} + 1600 - 4x$

$\frac{dC}{dx} = 3(40000 + x^2)^{-\frac{1}{2}}(2x) - 4$

$\frac{dC}{dx} = \underbrace{6x(40000 + x^2)^{-\frac{1}{2}}} - 4$

Domain

$0 \le x \le 400$

For max/min, $\frac{dC}{dx} = 0$

$\frac{6x}{\sqrt{x^2 + 40000}} - 4 = 0$

x	C
0	2800
$\sqrt{32000}$	2494.4 min
400	2683.3

$\div$ by 2

$\frac{3x}{\sqrt{x^2 + 40000}} = \frac{2}{1}$

$3x = 2\sqrt{x^2 + 40000}$

$(3x)^2 = \left(2\sqrt{x^2 + 40000}\right)^2$

$9x^2 = 4(x^2 + 40000)$

$9x^2 = 4x^2 + 160000$

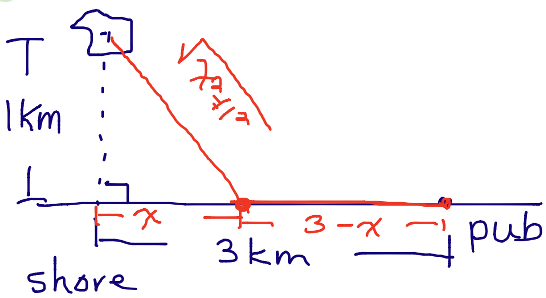
$5x^2 = 160\,000$

$x^2 = 32\,000$

$x \doteq 178.9$

$\therefore$ the cost is minimized if the cable comes out of the water on the opposite bank approximate 178.9 m downstream from P.

Ex. 4. A man lives on an island 1 km from the mainland. His favourite pub is 3 km along the shore from the point on the shore closest to the island. The man can paddle his canoe at 3 km/h and can jog at 5 km/h. Determine where he should land so as to reach the pub in the shortest possible time.



x	t (hrs)	
0	$\frac{14}{15}$	(56 min.)
$\frac{3}{4}$	$\frac{13}{15}$	minimum (52 min)
3	$\frac{\sqrt{10}}{3}$	(1 hr. 3 min)

$d = s \times t$

$\frac{d}{s} = t$

minimize time, t, in h

$$t = t_{\text{paddling}} + t_{\text{jogging}}$$

$$= \frac{d_{\text{paddled}}}{s_{\text{paddled}}} + \frac{d_j}{s_j}$$

$$= \frac{\sqrt{x^2+1}}{3} + \frac{3-x}{5}$$

$$t = \frac{1}{3}(x^2+1)^{\frac{1}{2}} + \frac{3}{5} - \frac{1}{5}x$$

$$\frac{dt}{dx} = \frac{1}{\cancel{6}\,3}(x^2+1)^{-\frac{1}{2}}(\cancel{2}x) - \frac{1}{5}$$

$$\frac{dt}{dx} = \frac{x}{3\sqrt{x^2+1}} - \frac{1}{5}$$

For max/min, $\frac{dt}{dx} = 0$

$$\frac{x}{3\sqrt{x^2+1}} - \frac{1}{5} = 0$$

$$\frac{x}{3\sqrt{x^2+1}} = \frac{1}{5}$$

$$5x = 3\sqrt{x^2+1}$$

$$(5x)^2 = (3\sqrt{x^2+1})^2$$

$$25x^2 = 9(x^2+1)$$

$$25x^2 = 9x^2 + 9$$

$$16x^2 = 9$$

$$x^2 = \frac{9}{16}$$

$$x = \frac{3}{4}$$

$\therefore$ to reach the pub in the shortest possible time, he should start jogging $2\frac{1}{4}$ km from the pub.

HW: p. 214 #2 to 9, 12a, 14;

Review for Test: p. 219 #10, 14, 23 to 31; p. 224 #8, 10, 11