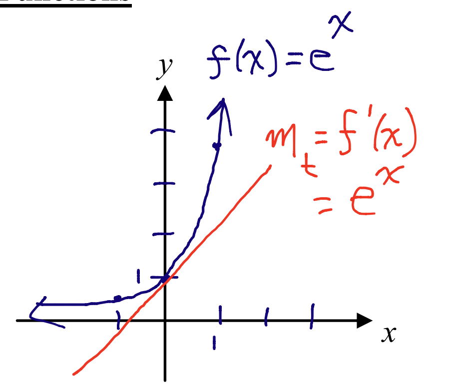
Date: April 11/14 **UNIT 5: DERIVATIVES OF EXPONENTIAL & LOGARITHMIC FUNCTIONS**

Section 8.1 – Derivatives of Exponential Functions

Problem: Find the derivative of the exponential function $f(x) = e^x$ from *first principles*.



$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$

$$= \lim_{h\to 0} \left[e^x \cdot \frac{e^h - 1}{h}\right]$$

$$= \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1$$

$$= e^x$$

h	$\frac{e^h - 1}{h}$
0.1	$\doteq 1.05$
0.01	$\doteq 1.005$
0.001	$\doteq 1.0005$
0.000 1	$\doteq 1.0005$

Summary:

If $y = e^x$ then $\frac{dy}{dx} = e^x$

If $y = e^{f(x)}$ then $\frac{dy}{dx} = e^{f(x)} \cdot f'(x)$

Ex. 1. Find the derivatives of the following.

a) $y = e^{x^2 - x}$

$$\frac{dy}{dx} = e^{x^2 - x}(2x - 1)$$

b) $y = \sqrt{e^{\frac{4}{x}}} \rightarrow y = \left(e^{\frac{4}{x}}\right)^{\frac{1}{2}} \rightarrow y = e^{\frac{2}{x}}$

$$y = e^{2x^{-1}}$$

$$\frac{dy}{dx} = e^{\frac{2}{x}} \cdot (-2x^{-2})$$

$$\frac{dy}{dx} = \frac{-2e^{\frac{2}{x}}}{x^2}$$

c) $f(x) = \tan 2x \cdot e^{2x}$

$$f'(x) = \sec^2 2x \cdot 2 \cdot e^{2x} + e^{2x} \cdot 2 \cdot \tan 2x$$

$$f'(x) = 2e^{2x}\sec^2 2x + 2e^{2x}\tan 2x$$

or

$$f'(x) = 2e^{2x}(\sec^2 2x + \tan 2x)$$

d) $s(t) = \frac{e^{3t}}{1 - e^{3t}}$

$$s'(t) = \frac{e^{3t} \cdot 3(1 - e^{3t}) - (-e^{3t} \cdot 3) \cdot e^{3t}}{(1 - e^{3t})^2}$$

$$= \frac{3e^{3t} - \cancel{3e^{6t}} + \cancel{3e^{6t}}}{(1 - e^{3t})^2}$$

$$\therefore s'(t) = \frac{3e^{3t}}{(1 - e^{3t})^2}$$

Ex. 2. Given $f(x)=e^{\sqrt{1-x}}$, determine $f'(-3)$.

$f(x)=e^{(1-x)^{\frac{1}{2}}}$

$f'(x)=e^{\sqrt{1-x}}\cdot\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$f'(x)=\frac{-e^{\sqrt{1-x}}}{2\sqrt{1-x}}$

$f'(-3)=\frac{-e^{2}}{4}$

Ex. 3. Find the slope of the normal to the curve $y-e^{xy}=0$ at the point $(0,1)$.

$y-e^{x\cdot y}=0$

diff. w.r.t. x

$\frac{dy}{dx}-e^{x\cdot y}\left(1\cdot y+\frac{dy}{dx}\cdot x\right)=0$

at $x=0$ & $y=1$

$\frac{dy}{dx}-e^{0}(1+0)=0$

$\frac{dy}{dx}-1=0$

$\frac{dy}{dx}=1$

at $(0,1)$,

$m_t=1$

$\therefore m_n=-1$

Ex. 4. Determine the equations of the tangent and normal to $\boxed{y=\frac{e^x}{x^2}}$, $x\neq 0$, at $x=2$.

$y=\frac{e^{x}}{x^{2}}$

$\frac{dy}{dx}=\frac{e^{x}\cdot 1\cdot x^{2}-2x\cdot e^{x}}{(x^{2})^{2}}$

$\frac{dy}{dx}=\frac{x^{2}e^{x}-2xe^{x}}{x^{4}}$

at $x=2$

$\frac{dy}{dx}=\frac{4e^{2}-4e^{2}}{16}$ & $y=\frac{e^{2}}{4}$

$\frac{dy}{dx}=0$

For tangent

$m_t=0$; $\left(2,\frac{e^{2}}{4}\right)$

$\therefore$ the horizontal tangent is $y=\frac{e^{2}}{4}$

& the vertical normal passing through $\left(2,\frac{e^{2}}{4}\right)$ is

$x=2$.

HW: pg. 302 # 1 to 5, 6a, 7 to 10, 12ab, 15, 16

Date: April 14/14

Section 8.2 – The Derivative of the Natural Logarithmic Function

Find the derivative of $y = \ln x$**.**

$y = \ln x$

$y = \log_e x$

$x = e^y$

diff. w.r.t. x

$1 = e^y \cdot \frac{dy}{dx}$

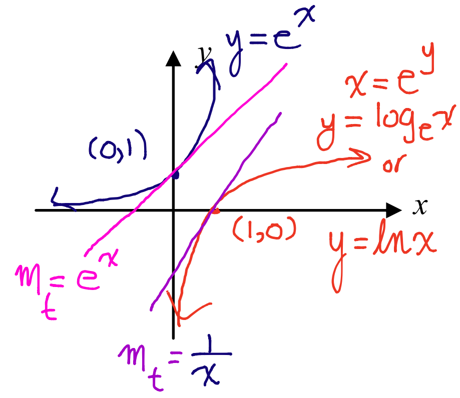
$\frac{dy}{dx} = \frac{1}{e^y}$

$\therefore y = \ln x$

$\frac{dy}{dx} = \frac{1}{e^{\ln x}}$

$\frac{dy}{dx} = \frac{1}{e^{\log_e x}}$

$\frac{dy}{dx} = \frac{1}{x}$



Summary:

If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

$e^{\ln x} =$

If $y = \ln u$ then $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$

If $y = \ln f(x)$ then $\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

Properties of Natural Logarithms:

i) $\ln 1 = 0$ **ii)** $\ln e = 1$ **iii)** $\ln e^x = x$ **iv)** $e^{\ln x} = x$

Natural Logarithm Laws: $x, y > 0$

1. $\ln(x \cdot y) = \ln x + \ln y$ ***multiplication***

2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ ***division***

3. $\ln x^n = n \cdot \ln x$ ***power***

Ex. 1. Differentiate each of the following. (In some cases making use of the logarithm laws and exponent
laws will make the process of differentiation easier.)

a) $f(x) = \ln(\cos x)$

$f'(x) = \frac{1}{\cos x} \cdot (-\sin x)$

$f'(x) = -\frac{\sin x}{\cos x}$

$f'(x) = -\tan x$

b) $y = e^{2\ln x}$ ☆

$y = e^{\ln x^2}$

$y = e^{\log_e x^2}$

$y = x^2$

$\frac{dy}{dx} = 2x$

c) $y = \ln x^5$ ☆

$\frac{dy}{dx} = \frac{1}{x^5} \cdot 5x^4$ or $y = 5\ln x$

$\frac{dy}{dx} = \frac{5}{x}$

$\frac{dy}{dx} = 5 \cdot \frac{1}{x}$

$\frac{dy}{dx} = \frac{5}{x}$

d) $y = e^{\sqrt{x}} \cdot \ln\sqrt{x}$ ☆

$y = e^{x^{\frac{1}{2}}} \cdot \underbrace{\ln x^{\frac{1}{2}}}$

$y = e^{x^{\frac{1}{2}}} \cdot \frac{1}{2}\ln x$

$\frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}} \cdot \frac{1}{2}\ln x + \frac{1}{2} \cdot \frac{1}{x} \cdot e^{\sqrt{x}}$

$= \frac{1}{4}x^{-\frac{1}{2}}e^{\sqrt{x}}\ln x + \frac{2}{4}x^{-1}e^{\sqrt{x}}$

$= \frac{1}{4}x^{-1}e^{\sqrt{x}}\left(x^{\frac{1}{2}}\ln x + 2\right)$

$\therefore \frac{dy}{dx} = \frac{e^{\sqrt{x}}(\sqrt{x} \cdot \ln x + 2)}{4x}$

e) $h(x)=\dfrac{\ln x}{x^3}$

$$h'(x)=\frac{\frac{1}{x}\cdot x^3-3x^2\cdot\ln x}{(x^3)^2}$$

$$=\frac{x^2-3x^2\ln x}{x^6}\quad \frac{\div x^2}{\div x^2}$$

$$\therefore h'(x)=\frac{1-3\ln x}{x^4}$$

f) $s(t)=\ln\sqrt{\dfrac{1-t}{1+t}}$ $\rightarrow s(t)=\ln\dfrac{(1-t)^{\frac{1}{2}}}{(1+t)^{\frac{1}{2}}}$

$$s(t)=\ln(1-t)^{\frac{1}{2}}-\ln(1+t)^{\frac{1}{2}}$$

$$s(t)=\frac{1}{2}\ln(1-t)-\frac{1}{2}\ln(1+t)$$

$$s'(t)=\frac{1}{2}\cdot\frac{1}{1-t}\cdot(-1)-\frac{1}{2}\cdot\frac{1}{1+t}\cdot 1$$

$$s'(t)=\boxed{\frac{-1}{2(1-t)}}-\boxed{\frac{1}{2(1+t)}}$$

$$=\frac{-1(1+t)-1(1-t)}{2(1-t)(1+t)}$$

$$=\frac{-1-\cancel{t}-1+\cancel{t}}{2(1-t)(1+t)}$$

$$=\frac{-2}{2(1-t)(1+t)}$$

$$\therefore s'(t)=\frac{-1}{1-t^2}$$

Ex. 2. Find an equation for the normal line to the curve $y\cdot\ln x=e^x-y$ at the point $(1,e)$

$$y\cdot\ln x=e^x-y$$

diff w.r.t. x

$$\frac{dy}{dx}\cdot\ln x+\frac{1}{x}\cdot 1\cdot y=e^x\cdot 1-\frac{dy}{dx}$$

Find $\frac{dy}{dx}$ at $x=1$, $y=e$

$$\frac{dy}{dx}\cdot\underbrace{\ln 1}+e=e-\frac{dy}{dx}$$

$$\frac{dy}{dx}(0)+e=e-\frac{dy}{dx}$$

$$\frac{dy}{dx}=e-e$$

$$\frac{dy}{dx}=0$$

If $m_t=0$ then the normal is vertical and has the equation $x=1$ if it passes through $(1,e)$.

Ex. 3. If $f(x) = x(\ln x)^2$, determine: $(\ln x)^2 \neq \underbrace{\ln x^2}$

a) $f'(x)$

b) the equation of the tangent at $x = \frac{1}{e}$

c) all points at which the graph of $f(x)$ has a horizontal tangent line.

a) $f(x) = x \cdot (\ln x)^2$

$f'(x) = 1 \cdot (\ln x)^2 + 2(\ln x)^1 \cdot \frac{1}{x} \cdot 1 \cdot x$

$f'(x) = (\ln x)^2 + 2(\ln x)$

or $f'(x) = (\ln x)(\ln x + 2)$

b) $m_t = f'(\frac{1}{e})$ & $f(\frac{1}{e}) = \frac{1}{e}(\underbrace{\ln e^{-1}})^2$

$= (\underbrace{\ln e^{-1}})^2 + 2(\underbrace{\ln e^{-1}})$ $= \frac{1}{e}(-1)^2$

$= (-1)^2 + 2(-1)$ $= \frac{1}{e}$

$= -1$

For the tangent,

$m_t = -1$; $(\frac{1}{e}, \frac{1}{e})$; $b =$ ____

Find b: $\frac{1}{e} = -1(\frac{1}{e}) + b$

$\frac{1}{e} = -\frac{1}{e} + b$

$\frac{2}{e} = b$

$\therefore$ the equation of the tangent at $x = \frac{1}{e}$ is

$y = -x + \frac{2}{e}$.

c) $m_t = 0$

$f'(x) = 0$

$\ln x(\ln x + 2) = 0$

$\therefore \ln x = 0$ or $\ln x + 2 = 0$

$\log_e x = 0$ $\ln x = -2$

$x = e^0$ $x = e^{-2}$

$x = 1$ $f(e^{-2}) = e^{-2}(\ln e^{-2})^2$

$f(1) = 1(\ln 1)^2$ $= e^{-2}(-2)^2$

$= 0$ $= \frac{4}{e^2}$

HW: p. 309 # 1, 3, 4, 5ab, 6, 7a, 8, 9a, 10, 14, 15

$\therefore$ the tangent is horizontal at the points $(1, 0)$ & $(\frac{1}{e^2}, \frac{4}{e^2})$