Date: April 15/14

Section 8.3 – DERIVATIVES OF GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1. Find the derivative of $y = b^{f(x)}$.

$y = b^{f(x)}$

take the natural logarithm of both sides

$\ln y = \ln b^{f(x)}$

$\ln y = f(x) \ln b$ ← constant multiple

diff. w.r.t. x

$\frac{1}{y} \cdot \frac{dy}{dx} = f'(x) \cdot \ln b$

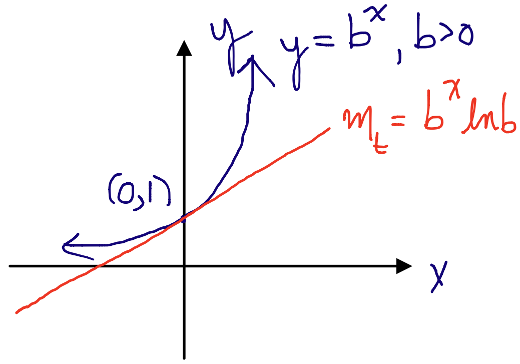
$\frac{dy}{dx} = y \cdot \ln b \cdot f'(x)$

$\because y = b^{f(x)}$

$\therefore \frac{dy}{dx} = b^{f(x)} \cdot \ln b \cdot f'(x)$

If $y = b^x$ then $\frac{dy}{dx} = b^x \cdot \ln b \cdot 1$

If $y = b^{f(x)}$ then $\frac{dy}{dx} = b^{f(x)} \cdot \ln b \cdot f'(x)$



2. Find the derivative of $y = \log_b f(x)$.

$y = \log_b f(x)$

Using the change of base identity

$y = \frac{\ln f(x)}{\ln b}$

constant multiple

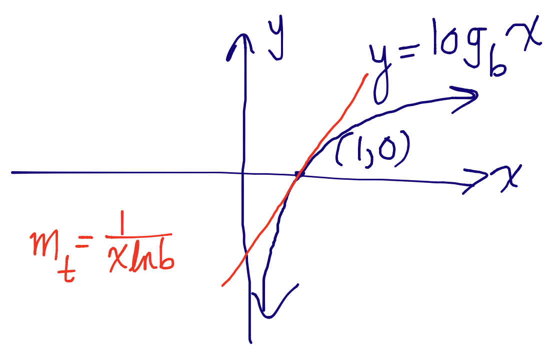
$y = \frac{1}{\ln b} \cdot \ln f(x)$

$\frac{dy}{dx} = \frac{1}{\ln b} \cdot \frac{1}{f(x)} \cdot f'(x)$

$\frac{dy}{dx} = \frac{1}{f(x) \cdot \ln b} \cdot f'(x)$

If $y = \log_b x$ then $\frac{dy}{dx} = \frac{1}{x \ln b} \cdot 1$

If $y = \log_b f(x)$ then $\frac{dy}{dx} = \frac{1}{f(x) \cdot \ln b} \cdot f'(x)$



Ex. 1. Find the derivative of each of the following.

a) $y = e^x + x^e - \ln x$

$$\frac{dy}{dx} = e^x \cdot 1 + e x^{e-1} \cdot 1 - \frac{1}{x} \cdot 1$$

$$\frac{dy}{dx} = e^x + e x^{e-1} - \frac{1}{x}$$

b) $y = 3^x + x^3 - \log_3 x$

$$\frac{dy}{dx} = 3^x \ln 3 \cdot 1 + 3x^2 \cdot 1 - \frac{1}{x \ln 3} \cdot 1$$

$$\frac{dy}{dx} = 3^x \ln 3 + 3x^2 - \frac{1}{x \ln 3}$$

c) $s = 10^{\sqrt{t^2-1}}$

$$s = 10^{(t^2-1)^{\frac{1}{2}}}$$

$$\frac{ds}{dt} = 10^{\sqrt{t^2-1}} \cdot \ln 10 \cdot \frac{1}{\cancel{2}}(t^2-1)^{-\frac{1}{2}} (\cancel{2}t)$$

$$\frac{ds}{dt} = \frac{10^{\sqrt{t^2-1}} \cdot \ln 10 \cdot t}{\sqrt{t^2-1}}$$

d) $f(x) = \log_4 \sqrt{3x^2 - 5}$ ☆

$$f(x) = \log_4 (3x^2-5)^{\frac{1}{2}}$$

$$f(x) = \frac{1}{2} \cdot \log_4 (3x^2-5)$$

$$f'(x) = \frac{1}{\cancel{2}_{1}} \cdot \frac{1}{(3x^2-5) \cdot \ln 4} \cdot (\cancel{6}^{3} x)$$

$$f'(x) = \frac{3x}{(3x^2-5) \cdot \ln 4}$$

e) $y = \log\left(\frac{1-x}{1+x}\right)$ ☆

$$y = \log_{10}(1-x) - \log_{10}(1+x)$$

$$\frac{dy}{dx} = \frac{1}{(1-x) \cdot \ln 10} \cdot (-1) - \frac{1}{(1+x) \cdot \ln 10} \cdot (1)$$

$$= \boxed{\frac{-1}{(1-x) \cdot \ln 10}} - \boxed{\frac{1}{(1+x) \cdot \ln 10}}$$

$$= \frac{-1(1+x) - 1(1-x)}{(1-x)(1+x) \cdot \ln 10}$$

$$= \frac{-1 - x - 1 + x}{(1-x)(1+x) \cdot \ln 10}$$

$$\therefore \frac{dy}{dx} = \frac{-2}{(1-x^2) \cdot \ln 10}$$

Ex. 2. Find the equation of the tangent to $g(x) = x^3 \cdot \underbrace{\log_2 x^3}$ at $x = 1$.

☆ $g(x) = x^3 \cdot 3\log_2 x$

$g(x) = 3x^3 \cdot \log_2 x$

$m_t = g'(x)$

$= 9x^2 \cdot \log_2 x + \frac{1}{x \ln 2} \cdot 3x^3$

$\therefore m_t = 9x^2 \cdot \log_2 x + \frac{3x^2}{\ln 2}$

at $x = 1$

$m_t = 9 \cdot (0) + \frac{3}{\ln 2}$; $g(1) = 1 \cdot \log_2 1 = 0$

$m_t = \frac{3}{\ln 2}$

$m_t = \frac{3}{\ln 2}$; $(1, 0)$; $b =$ __

$0 = \frac{3}{\ln 2}(1) + b$

$-\frac{3}{\ln 2} = b$

$\therefore$ the equation of the tangent is $y = \frac{3}{\ln 2} x - \frac{3}{\ln 2}$

HW: p. 316 # 1 to 4, 5a, 6, 7ab

Date: April 16/14 **Section 8.5 – LOGARITHMIC DIFFERENTIATION**

Warm-up: Find the slope of the tangent to $y = \dfrac{(x^4+1)\sqrt{x+2}}{\sqrt[3]{2x^2+2x+1}}$ at $x = -1$.

$y = (x^4+1)\cdot(x+2)^{\frac{1}{2}}\cdot(2x^2+2x+1)^{-\frac{1}{3}}$

$\frac{dy}{dx} = 4x^3\cdot(x+2)^{\frac{1}{2}}\cdot\underbrace{(2x^2+2x+1)^{-\frac{1}{3}}} + \frac{1}{2}(x+2)^{-\frac{1}{2}}(x^4+1)(2x^2+2x+1)^{-\frac{1}{3}} - \frac{1}{3}(2x^2+2x+1)^{-\frac{4}{3}}(4x+2)(x^4+1)(x+2)^{\frac{1}{2}}$

at $x = -1$

$\frac{dy}{dx} = (-4)(1)(1) + \frac{1}{2}(1)(2)(1) - \frac{1}{3}(1)(-2)(2)(1)$

$= -3 + \frac{4}{3}$

$= -\frac{5}{3}$ $\therefore m_t = -\frac{5}{3}$ at $x = -1$.

Recall:

If $y = x^n$	then $\frac{dy}{dx} = nx^{n-1}$	If $y = [f(x)]^n$	then $\frac{dy}{dx} = n[f(x)]^{n-1}\cdot f'(x)$
If $y = e^x$	then $\frac{dy}{dx} = e^x$	If $y = e^{f(x)}$	then $\frac{dy}{dx} = e^{f(x)}\cdot f'(x)$
If $y = b^x$	then $\frac{dy}{dx} = b^x \ln b$	If $y = b^{f(x)}$	then $\frac{dy}{dx} = b^{f(x)}\cdot\ln b\cdot f'(x)$
If $y = \ln x$	then $\frac{dy}{dx} = \frac{1}{x}$	If $y = \ln f(x)$	then $\frac{dy}{dx} = \frac{1}{f(x)}\cdot f'(x)$
If $y = \log_b x$	then $\frac{dy}{dx} = \frac{1}{x\ln b}$	If $y = \log_b f(x)$	then $\frac{dy}{dx} = \frac{1}{f(x)\cdot\ln b}\cdot f'(x)$

Ex. 1. Differentiate each of the following.

a) $y = \pi x + x^{\pi} - \pi^x$

$\frac{dy}{dx} = \pi + \pi x^{\pi-1} - \pi^x \ln\pi$

b) $f(x) = 2x^{2\sqrt{3}} - \log_2 x$

$f'(x) = 4\sqrt{3}\, x^{2\sqrt{3}-1} - \frac{1}{x\ln 2}$

Note: If $y = x^{f(x)}$ or if $y = [f(x)]^{g(x)}$, the derivative is found by first taking the natural logarithm of both sides and then differentiating with respect to x. This is the technique of ***logarithmic differentiation***.

Ex. 2. Use the technique of ***logarithmic differentiation*** to find the derivative for each of the following:

a) $f(x) = x^x$

$\ln f(x) = \ln x^x$

$\ln f(x) = x\cdot\ln x$

diff. w.r.t. x

$\frac{1}{f(x)}\cdot f'(x) = 1\cdot\ln x + \frac{1}{\cancel{x}}\cdot\cancel{x}$

$f'(x) = f(x)\cdot[\ln x + 1]$

$f'(x) = x^x(\ln x + 1)$

b) $y = (\cos x)^{\sin x}$

$\ln y = \ln(\cos x)^{\sin x}$

$\ln y = \sin x\cdot\ln(\cos x)$

diff. w.r.t. x

$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x\cdot\ln(\cos x) + \frac{1}{\cos x}\cdot(-\sin x)\cdot\sin x$

$\frac{dy}{dx} = y\left(\cos x\ln(\cos x) - \tan x\cdot\sin x\right)$

$\frac{dy}{dx} = (\cos x)^{\sin x}\left[\cos x\cdot\ln\cos x - \tan x\cdot\sin x\right]$

Ex. 3. If $y = (x^2 - 1)^{\sqrt{x}}$ find $\dfrac{dy}{dx}$.

$$\ln y = \ln (x^2-1)^{\sqrt{x}}$$

$$\ln y = x^{\frac{1}{2}} \cdot \ln (x^2-1)$$

diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \cdot \ln(x^2-1) + \frac{1}{x^2-1} \cdot 2x^1 \cdot x^{\frac{1}{2}}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \cdot \ln(x^2-1) + \frac{4}{2}x^{\frac{3}{2}} \cdot \frac{1}{x^2-1}$$

"y"

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\left[\ln(x^2-1) + \frac{4x^2}{x^2-1}\right] \cdot (x^2-1)^{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{(x^2-1)^{\sqrt{x}}}{2\sqrt{x}}\left[\ln(x^2-1) + \frac{4x^2}{x^2-1}\right]$$

Ex. 4. Find the equation of the normal to $y = \dfrac{(x^4+1)\sqrt{x+2}}{\sqrt[3]{2x^2+2x+1}}$ at $x = -1$, using ***logarithmic differentiation***.

$$y = \frac{(2)(1)}{(1)} = 2$$

$$\ln y = \ln\left[\frac{(x^4+1)(x+2)^{\frac{1}{2}}}{(2x^2+2x+1)^{\frac{1}{3}}}\right]$$

$$\ln y = \ln(x^4+1) + \ln(x+2)^{\frac{1}{2}} - \ln(2x^2+2x+1)^{\frac{1}{3}}$$

$$\ln y = \ln(x^4+1) + \frac{1}{2}\ln(x+2) - \frac{1}{3}\ln(2x^2+2x+1)$$

diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{4x^3}{x^4+1} + \frac{1}{2}\cdot\frac{1}{x+2} - \frac{1}{3}\cdot\frac{4x+2}{2x^2+2x+1}$$

Find $\frac{dy}{dx}$ if $x=-1$ & $y=2$

$$\frac{1}{2}\frac{dy}{dx} = -2 + \frac{1}{2} - \frac{1}{3}\cdot(-2)$$

$$\frac{dy}{dx} = 2\left[-2 + \frac{1}{2} + \frac{2}{3}\right]$$

$$\frac{dy}{dx} = -4 + 1 + \frac{4}{3}$$

$$\frac{dy}{dx} = -\frac{5}{3}$$

For normal,

$m_n = +\frac{3}{5}$; $(-1, 2)$; $b =$ ____

Find b

$$2 = \frac{3}{5}(-1) + b$$

$$\frac{10}{5} + \frac{3}{5} = b$$

$$\frac{13}{5} = b$$

$\therefore$ the equation of the normal at $x=-1$ is $y = \frac{3}{5}x + \frac{13}{5}$ or $3x - 5y + 13 = 0$.

HW: p. 326 # 1, 3b, 2, 3ac, 4, 6

Date: Apr. 17/14

Applications of Exponential and Logarithmic Functions

Ex. 1. The position of a particle that moves on a straight line is given by $s(t) = t^{\frac{1}{t}}$ for $t > 0$.

a) Find the velocity.

b) At what time, t, is the velocity zero?

a) $s(t) = t^{\frac{1}{t}}$

use logarithmic differentiation

$\ln s(t) = \ln t^{\frac{1}{t}}$

$\ln s(t) = t^{-1} \ln t$

diff. wrt t

$\frac{1}{s(t)} \cdot s'(t) = -t^{-2} \ln t + \frac{1}{t} \cdot t^{-1}$

$\frac{1}{s(t)} \cdot s'(t) = -t^{-2} \ln t + t^{-2}$

$s'(t) = t^{\frac{1}{t}} \left[-t^{-2} (\ln t - 1) \right]$

$s'(t) = -t^{-2+\frac{1}{t}} (\ln t - 1)$

$\therefore v(t) = -t^{-2+\frac{1}{t}} (\ln t - 1)$

b) Find t if $v(t) = 0$

$\because -t^{-2+\frac{1}{t}} \neq 0$

$\therefore \ln t - 1 = 0$

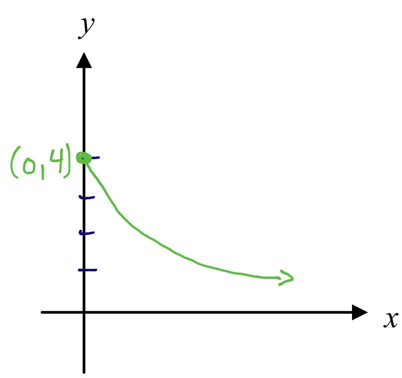
$\ln t = 1$

$t = e^1$

$\therefore t = e$

$\therefore$ the velocity is 0 at $t = e$.

Ex. 2. A playground slide has the shape of the curve $y = 4e^{-\frac{x}{2}}$. If the horizontal component of the velocity for a person on the slide is $\frac{dx}{dt} = 2\ m/s$ when $x = 1$, find the vertical component of the velocity $\frac{dy}{dt}$ at that instant.



$y = 4e^{-\frac{x}{2}}$

$\frac{dx}{dt} = 2 m/s$; Find $\frac{dy}{dt}$ when $x = 1$

$y = 4e^{-\frac{x}{2}}$

diff wrt t:

$\frac{dy}{dt} = 4e^{-\frac{x}{2}} \left(-\frac{1}{2} \frac{dx}{dt} \right)$

at $x = 1$; $\frac{dx}{dt} = 2$

$\frac{dy}{dt} = 4e^{-\frac{1}{2}} \left(-\frac{1}{2} \cdot 2 \right)$

$\frac{dy}{dt} = -\frac{4}{\sqrt{e^1}}$

$\therefore$ the vertical component of the velocity is exactly $\frac{-4}{\sqrt{e^1}}$ m/s or -2.43 m/s (approx)

Ex. 3. Find the maximum and minimum values of $f(x) = \ln(e^x + 4e^{-x})$ for $-\ln 2 \le x \le \ln 4$.

$f(x) = \ln(e^x + 4e^{-x})$

$f'(x) = \frac{1}{e^x + 4e^{-x}} \cdot [e^x + 4e^{-x}(-1)]$

$f'(x) = \frac{e^x - 4e^{-x}}{e^x + 4e^{-x}}$

for max/min, $f'(x) = 0$

$\frac{0}{1} = \frac{e^x - 4e^{-x}}{e^x + 4e^{-x}}$

$0 = e^x - 4e^{-x}$

$0 = e^{-x}(e^{2x} - 4)$

$e^{-x} \ne 0$, $e^{2x} = 4$

$2x = \ln 4$

$x = \frac{1}{2}\ln 4$

$x = \ln 4^{\frac{1}{2}}$

$x = \ln 2$

$f(-\ln 2) = \ln(e^{-\ln 2} + 4e^{\ln 2})$

$= \ln(e^{\ln 2^{-1}} + 4(2))$

$= \ln(\frac{1}{2} + 8)$

$= \ln(\frac{17}{2})$

$f(\ln 2) = \ln(e^{\ln 2} + 4e^{-\ln 2})$

$= \ln(2 + 4(\frac{1}{2}))$

$= \ln 4$

$f(\ln 4) = \ln(e^{\ln 4} + 4e^{-\ln 4})$

$= \ln(4 + 4(\frac{1}{4}))$

$= \ln 5$

x	$f(x)$
$-\ln 2$	$\ln\frac{17}{2} \doteq 2.14$
$\ln 2$	$\ln 4 \doteq 1.39$
$\ln 4$	$\ln 5 \doteq 1.61$

$\therefore$ The maximum value of $f(x)$ is $\ln(\frac{17}{2})$ and the minimum value is $\ln 4$.

Ex. 4. The effectiveness of studying for a test depends on how many hours a student studies. Some experiments showed that if the effectiveness, E, is put on a scale of 0 to 10, then

$E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$, where t is the number of hours spent studying for an examination.

If a student has up to 30 h that he can spend studying, how many hours should he study for maximum effectiveness? $0 \le t \le 30$

$E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$

$E'(t) = 0.5\left[e^{-\frac{t}{20}} + e^{-\frac{t}{20}} \cdot (\frac{-1}{20})t\right]$

For max, $E'(t) = 0$

$0 = e^{-\frac{t}{20}} - \frac{1}{20}te^{-\frac{t}{20}}$

$\times 20)$ $0 = 20e^{-\frac{t}{20}} - te^{-\frac{t}{20}}$

$0 = e^{-\frac{t}{20}}(20 - t)$

$e^{-\frac{t}{20}} \ne 0$, $20 - t = 0$

$\therefore t = 20$

t	$E(t)$
0	5
20	8.68
30	8.35

$\therefore$ The student should study for 20 hours to maximize his effectiveness.

HW: p. 303 #13, 14; p. 310 #11 to 13; p. 316 #8; p. 320 #2a., 4, 7
Review Exercise p. 330 #1 to 13