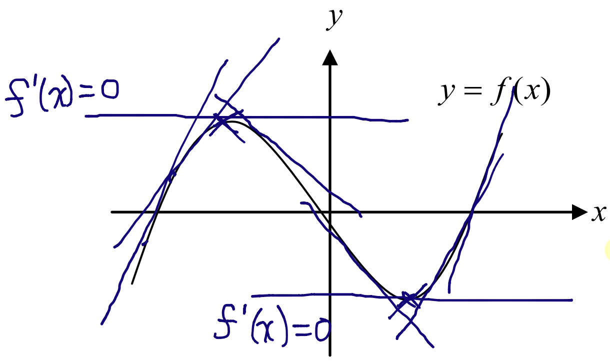
Date: April 24/14

UNIT 6: CURVE SKETCHING

Section 9.1 – Increasing and Decreasing Functions
Section 9.2 – Critical Points, Relative Maxima and Minima

CRITICAL POINTS: ***are points where the derivative is 0 or undefined.***

Note: *All relative maxima and minima (local extrema) are critical points, but not all critical points are maxima or minima.*



For local max/min $f'(x)=0$ ***or*** $y'=0$.

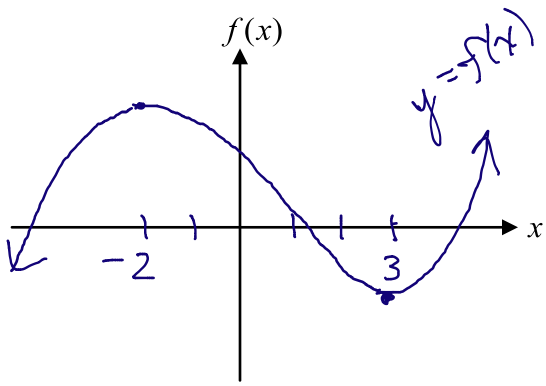
$f(x)$ *is said to be* ***increasing*** *if as x increases,* $f(x)$ *increases.*

For an increasing function, $f'(x)>0$ ***or*** $y'>0$.

$f(x)$ *is said to be* ***decreasing*** *if as x increases,* $f(x)$ *decreases.*

For a decreasing function, $f'(x)<0$ ***or*** $y'<0$.

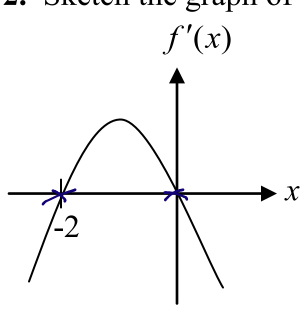
Ex. 1. Sketch the graph of the function $f(x)$ that satisfies $f'(x)>0$ for $x<-2$ and $x>3$ and $f'(x)<0$ for $-2<x<3$.

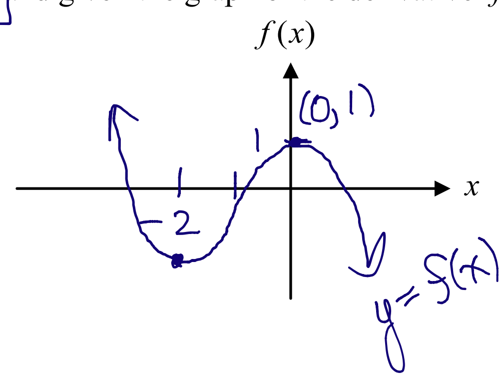


Interval	$x<-2$	$-2<x<3$	$x>3$
$f'(x)$	+	−	+
$f(x)$	increasing	decreasing	increasing

at $x=-2$ local max.   at $x=3$ local min.

Ex. 2. Sketch the graph of $f(x)$, assuming $\boxed{f(0)=1}$ and given the graph of the derivative $f'(x)$.





Interval	$x<-2$	$-2<x<0$	$x>0$
$f'(x)$	−	+	−
$f(x)$	dec.	inc.	dec.

at $x=-2$ local min.   at $x=0$ local max (0,1)

Ex. 3. Find all ***critical points*** of the following functions. Identify the ***critical points*** as ***local maxima***, ***local minima** or **neither*** and sketch the graph.

a) $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

For max/min.

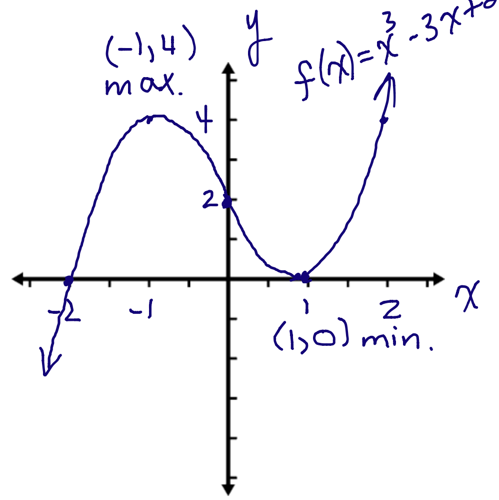
$f'(x) = 0$

$3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

x	$f(x)$	$f'(x)$
-2	0	$+$
-1	4	0 max
0	2	$-$
1	0	0 min.
2	4	$+$



$\therefore$ the critical points are
$(-1,4)$ a local maximum and
$(1,0)$ a local minimum.

b) $f(x) = (1 - x^2)^2 - 2$

$f'(x) = 2(1-x^2)(-2x)$

$f'(x) = -4x(1-x^2)$

For critical pts.

$f'(x) = 0$

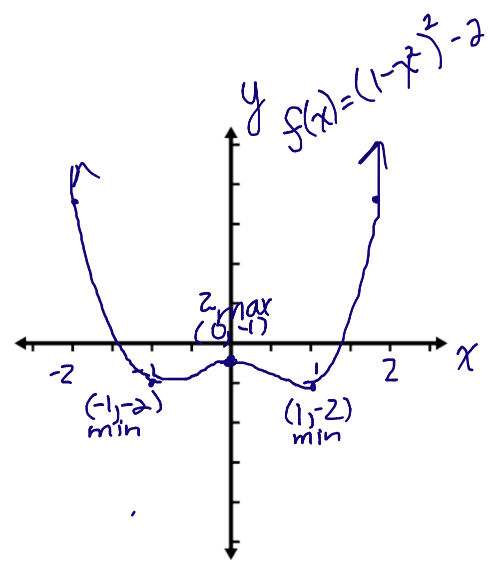
$-4x(1-x^2) = 0$

$\therefore x = 0$ or $1 - x^2 = 0$

$x^2 = 1$

$x = \pm 1$

x	$f(x)$	$f'(x)$
-2	7	$-$
-1	-2	0 min.
$-\frac{1}{2}$		$+$
0	-1	0 max
$\frac{1}{2}$		$-$
1	-2	0 min.
2	7	$+$



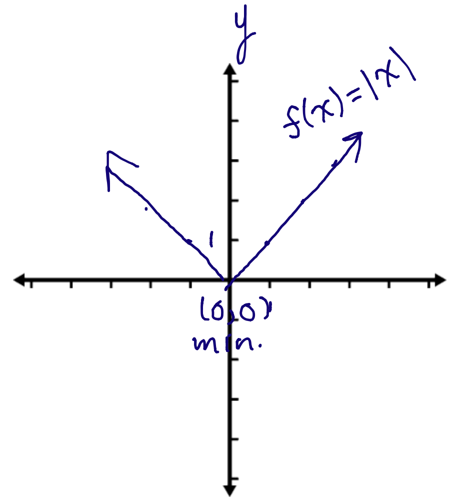
$\therefore$ the critical points are
$(-1,-2)$ & $(1,-2)$ both local minima
and $(0,-1)$ a local maximum.

c) $f(x)=|x|$

$$f(x)=\begin{cases}-x & \text{if } x<0\\ x & \text{if } x\geq 0\end{cases}$$

$$f'(x)=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x>0\end{cases}$$

x	$f(x)$	$f'(x)$
$x<0$		$-$
0	0	$\times$ min.
$x>0$		$+$



$\because f'(0)$ d.n.e.

$\therefore$ $(0,0)$ is a critical pt., a local minimum.

d) $f(x)=x^3$

$f'(x)=3x^2$

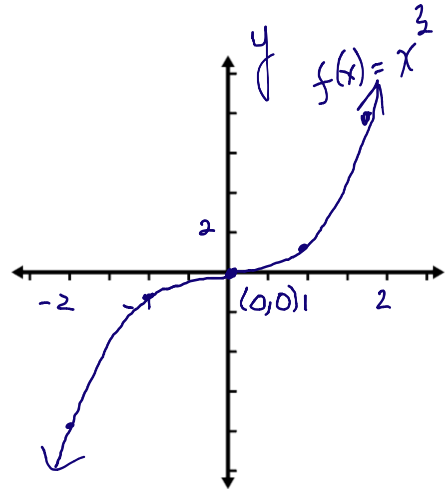
For critical pts.

$f'(x)=0$

$3x^2=0$

$x=0$

x	$f(x)$	$f'(x)$
$x<0$		$+$
0	0	
$x>0$		$+$



$\therefore$ $(0,0)$ is a critical pt, neither a maximum or minimum.

Ex. 4. The function $f(x)=x^3+ax+b$ has a turning point at $(-2,6)$. Find the constants a and b and the function $f(x)$.

$f(x)=x^3+ax+b$

$f'(x)=3x^2+a$

pt. $(-2,6)$ ①

$f(-2)=6$

$(-2)^3+a(-2)+b=6$

$-8-2a+b=6$

$-2a+b=14$ ①

turning pt. at $x=-2$ ②

$f'(-2)=0$

$3(-2)^2+a=0$

$12+a=0$

$a=-12$ ②

sub $a=-12$ in ①

$-2(-12)+b=14$

$24+b=14$

$b=-10$

$\therefore a=-12,\ b=-10$

& $f(x)=x^3-12x-10$.

H.W. pg. 342 # 4, 8, 9, 11ab
pg. 350 #2, 7bde, 9, 10

Date: April 25/14 **Section 9.4 – CONCAVITY & POINTS OF INFLECTION**

Ex. 1. Find and classify all ***local extrema*** *(local maxima and minima)* of the function $f(x)=x^3-3x^2$. Illustrate your results graphically.

$f(x)=x^3-3x^2$

$f'(x)=3x^2-6x$

For max/min.

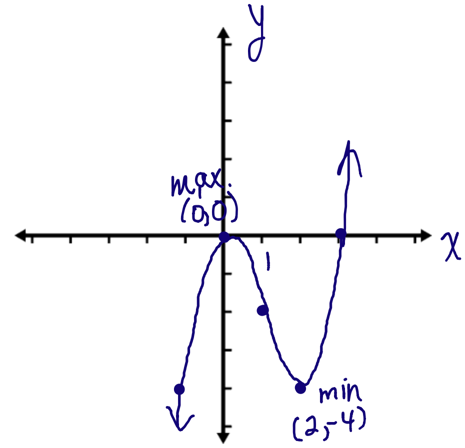
$f'(x)=0$

$3x^2-6x=0$

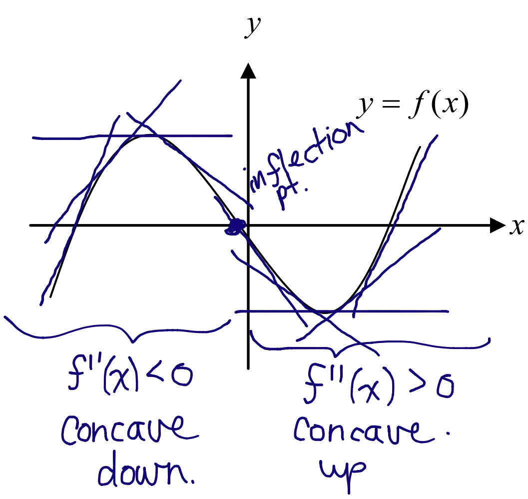
$3x(x-2)=0$

$x=0$ or $x=2$

	x	$f(x)$	$f'(x)$	
	-1	-4	+	
☆	0	0	0	max
	1	-2	–	
☆	2	-4	0	min
	3	0	+	



Applications of the Second Derivative for Curve Sketching: $f''(x)$ ***or*** y''



Note: 1. $f(x)$ is ***concave up*** if $f''(x)>0$ ***or*** $y''>0$.

2. $f(x)$ is ***concave down*** if $f''(x)<0$ ***or*** $y''<0$.

3. For ***inflection points*** $f''(x)=0$ ***or*** $y''=0$

Second Derivative Test for Local Extrema

For a ***local maximum***, $f'(x)=0$ and $f''(x)<0$.

For a ***local ~~maximum~~*** minimum, $f'(x)=0$ and $f''(x)>0$.

Ex. 2. Find and classify all ***local extrema*** of the function $f(x)=3x^5-25x^3+60x$ using the ***second derivative test***. Illustrate your results graphically.

$f(x)=3x^5-25x^3+60x$

$f'(x)=15x^4-75x^2+60$

$f''(x)=60x^3-150x$

For max/min, $f'(x)=0$

$15x^4-75x^2+60=0$

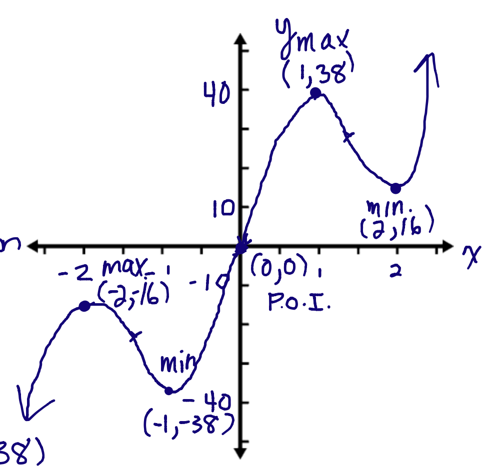
÷ by 15 $x^4-5x^2+4=0$

$(x^2-1)(x^2-4)=0$

$(x-1)(x+1)(x-2)(x+2)=0$

$\therefore x=\pm 2$ or $x=\pm 1$

	x	$f(x)$	$f'(x)$	$f''(x)$
	-2	-16	0	– ∩ max.
	-1	-38	0	+ ∪ min.
☆	0	0	+	0 inflection pt.
	1	38	0	– ∩ max
	2	16	0	+ ∪ min



$\therefore$ (-2,-16) & (1,38) are local maxima and (-1,-38) & (2,16) are local minima.

Ex. 3. Identify the point on the graph that satisfies the following conditions:

a) $y > 0,\ y' > 0,\ y'' < 0$ __C__ inc, concave down

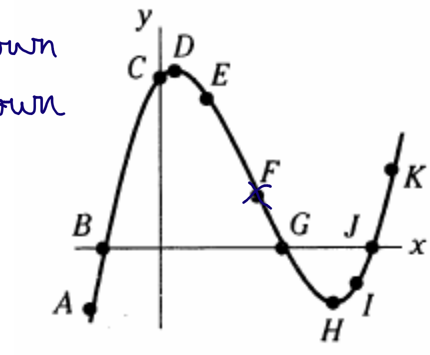
b) $y > 0,\ y' < 0,\ y'' < 0$ __E__ dec, concave down

c) $y < 0,\ y' > 0,\ y'' > 0$ __I__

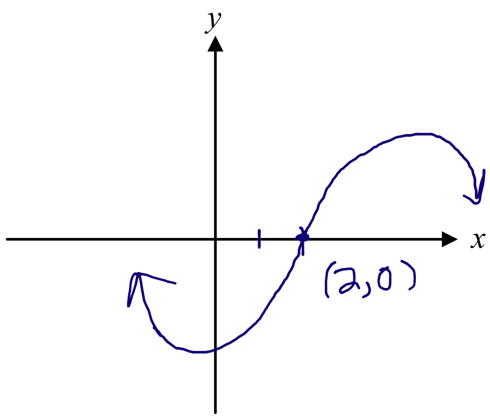
d) $y < 0,\ y' = 0,\ y'' < 0$ __none__

e) $y = 0,\ y' > 0,\ y'' < 0$ __B__

f) $y > 0,\ y' < 0,\ y'' = 0$ __F__ inflection pt.



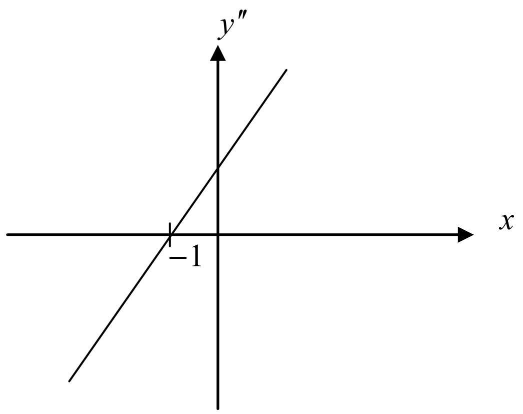
Ex. 4. Sketch the graph of a function that satisfies the following conditions:
$f''(x) > 0$ on the interval $x < 2$; $f''(x) < 0$ on the interval $x > 2$; $f(2) = 0$; $f'(2) > 0$



Interval	$x<2$	$x=2$	$x>2$
$f''(x)$	+	0	−
$f'(x)$		+	
$f(x)$	concave up	0	concave down

P.O.I at (2,0)

Ex. 5.



a) On what intervals is the graph of $f(x)$

i) concave up? $y'' > 0$ **ii)** concave down? $y'' < 0$

$x > -1$ $x < -1$

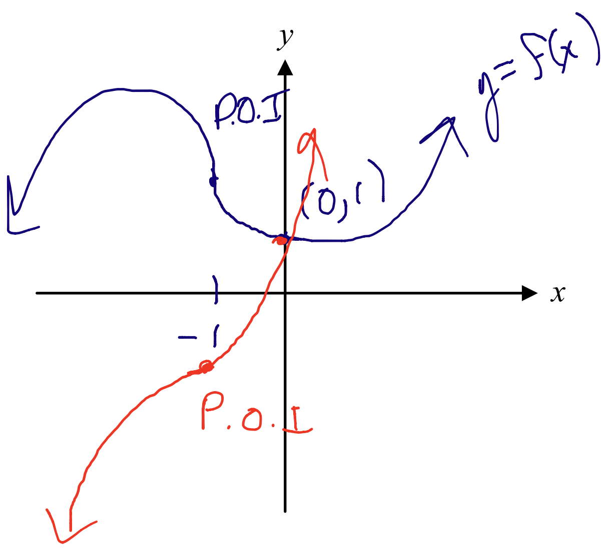
b) List the x – coordinates of all points of inflection.

$y'' = 0$ at $x = -1$

c) Make a rough sketch of a possible graph of $f(x)$, assuming that $f(0) = 1$.

if $x < -1$ concave down

if $x > -1$ Concave up



Ex. 6. Determine the constants a, b, c and d so that the curve $y = ax^3 + bx^2 + cx + d$ has a ***point of inflection*** at the *origin* and a ***local maximum*** at the point $(2,4)$.

$y = ax^3 + bx^2 + cx + d$

$y' = 3ax^2 + 2bx + c$

$y'' = 6ax + 2b$

inflection point $(0,0)$

① if $x=0, y=0$

$\boxed{d=0}$

② if $x=0$, $y''=0$

$6a(0)+2b=0$

$\boxed{b=0}$

local max at $(2,4)$

③ if $x=2$, $y=4$

$8a+4b+2c+d=4$

$\because b=0$ & $d=0$

$8a+2c=4$

$\div$ by 2 $\boxed{4a+c=2}$

④ if $x=2$, $y'=0$

$12a+4b+c=0$

$\because b=0$

$\therefore \boxed{12a+c=0}$

Solve. $\left.\begin{array}{l} 4a+c=2 \text{ ③} \\ 12a+c=0 \text{ ④} \end{array}\right\}$

③ − ④ $\frac{-8a}{-8}=\frac{2}{-8}$

$\boxed{a=-\frac{1}{4}}$

sub $a=-\frac{1}{4}$ in ③

$4\left(-\frac{1}{4}\right)+c=2$

$-1+c=2$

$\boxed{c=3}$

$\therefore a=-\frac{1}{4}$, $b=0$, $c=3$

$d=0$ and

$y=-\frac{1}{4}x^3+3x$

Ex. 7. Evaluate $f''(x)$ at the indicated value, and state whether the graph lies above or below the tangent for $f(x)=\dfrac{1}{2x^2+1}$, $x=1$. Sketch the graph near the point where $x=1$.

$f(x)=(2x^2+1)^{-1}$

$f'(x)=-(2x^2+1)^{-2}(4x)$

$f'(x)=-4x(2x^2+1)^{-2}$

$f''(x)=-4(2x^2+1)^{-2}-2(2x^2+1)^{-3}(4x)\cdot(-4x)$

$=-4(2x^2+1)^{-2}+32x^2(2x^2+1)^{-3}$

$=-4(2x^2+1)^{-3}\left[2x^2+1-8x^2\right]$

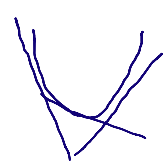
$=-4(2x^2+1)^{-3}(1-6x^2)$

$\therefore f''(x)=\dfrac{4(6x^2-1)}{(2x^2+1)^3}$

$f''(1)=\dfrac{4(5)}{27}$

$=\dfrac{20}{27}$

concave up

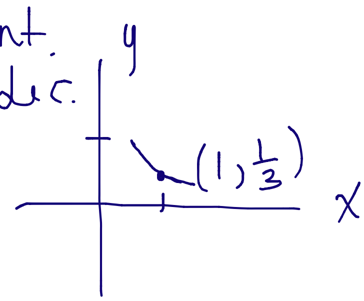


$\because f''(1)>0$

$\therefore$ the graph lies above the tangent.

$f'(1)=-\frac{4}{9}$ dec.

$f(1)=\frac{1}{3}$



H.W. pg. 369 #1, 2acd, 3acd, 4abc & (#9 Unit 4 Day 2 Worksheet, 4b Unit 4 Review Days 1-3); pg. 369 #5ab, 8i), 9, 10 do not sketch, 11

Date: April 28/14

Graphing Polynomial Functions

Warm-up:

Given $f(x)=\dfrac{1}{2x^2+1}$, $f'(x)=\dfrac{-4x}{(2x^2+1)^2}$ and $f''(x)=\dfrac{4(6x^2-1)}{(2x^2+1)^3}$ determine the intervals where the function is **i)** *increasing, decreasing* **ii)** *concave up, concave down.* For max/min.

i)

Interval	$x<0$	$x>0$
$f'(x)$	+	−
$f(x)$	increasing	decreasing

i) $f'(x)=0$

$-4x=0$

$x=0$

$\therefore$ f(x) is increasing for $x<0$ or $x\in(-\infty,0)$

& decreasing for $x>0$ or $x\in(0,+\infty)$

ii)

Interval	$x<-\frac{1}{\sqrt6}$	$-\frac{1}{\sqrt6}<x<\frac{1}{\sqrt6}$	$x>\frac{1}{\sqrt6}$
$f''(x)$	+	−	+
$f(x)$	concave up	concave down	concave up

ii) For pts. of inflection

$f''(x)=0$

$6x^2-1=0$

$x^2=\frac{1}{6}$

$x=\pm\frac{1}{\sqrt6}$

$\therefore$ f(x) is concave up for $x<-\frac{1}{\sqrt6}$ or $x>\frac{1}{\sqrt6}$

& concave down for $-\frac{1}{\sqrt6}<x<\frac{1}{\sqrt6}$.

Sketching the Graph of a Polynomial or Rational Function

1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

The second derivative can also be used to identify local maxima and minima.

4. Calculate the values of *y* that correspond to critical points and points of inflection. Use the information above to sketch the graph.

Remember that you will not use all the steps in every situation! Use only the steps that are necessary to give you a good idea of what the graph will look like.

Ex. Graph $f(x)=3x^4-4x^3$ by finding and labeling all *intercepts*, *local max/min* and *inflection* points. Identify intervals for which the function is *increasing*, *decreasing*, *concave up* and *concave down*.

$D=\{x\in\mathbb{R}\}$

$f(x)=3x^4-4x^3$

$f'(x)=12x^3-12x^2$

$f''(x)=36x^2-24x$

Intercepts

For x-int(s),

$f(x)=0$

$3x^4-4x^3=0$

$x^3(3x-4)=0$

$\boxed{\therefore x\text{-ints are } 0 \text{ \& } \tfrac{4}{3}}$

For y-int., $x=0$

$f(0)=0$

$\boxed{\therefore y\text{-int is } 0}$

For critical pts.

$f'(x)=0$

$12x^3-12x^2=0$

$12x^2(x-1)=0$

$\therefore x=0$ or $x=1$

$\boxed{\therefore (1,-1) \text{ is a local minimum}}$

For inflection pts.

$f''(x)=0$

$36x^2-24x=0$

$12x(3x-2)=0$

$x=0$ or $x=\frac{2}{3}$

$\boxed{\therefore \text{the inflection pts. are } (0,0) \text{ \& } \left(\tfrac{2}{3},-\tfrac{16}{27}\right)}$

Summary:

$f(x)$ is:

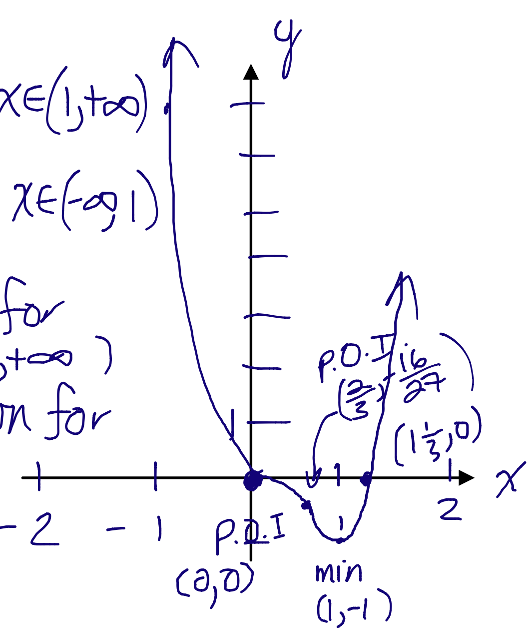
increasing for $x\in(1,+\infty)$

decreasing for $x\in(-\infty,1)$

concave up for $x\in(-\infty,0)\cup\left(\frac{2}{3},+\infty\right)$

concave down for $x\in\left(0,\frac{2}{3}\right)$

	x	$f(x)$	$f'(x)$	$f''(x)$	
	-1	7	$-$	$+$	
	0	0	0	0	P.O.I
	$0<x<\frac{2}{3}$		$-$	$-$	
	$\frac{2}{3}$	$-\frac{16}{27}$		0	P.O.I
☆	1	-1	0	$+$	min
	2	16	$+$	$+$	



H.W. Worksheet on Graphing Polynomial Functions