Date: April 29/14

Graphing Rational Functions

Vertical Asymptotes and Infinite Limits

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a ***vertical asymptote*** at $x=c$ if

- $q(c)=0$ and $p(c)\neq 0$

 Note: If $q(c)=0$ and $p(c)=0$ then a ***hole*** is created at $x=c$

- $\lim\limits_{x\to c^-} f(x)=+\infty$, $\lim\limits_{x\to c^-} f(x)=-\infty$, $\lim\limits_{x\to c^+} f(x)=+\infty$ or $\lim\limits_{x\to c^+} f(x)=-\infty$

Horizontal Asymptotes and Limits at Infinity

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a ***horizontal asymptote***, if the *degree of the numerator is less than or equal to the degree of the denominator*.

It is the horizontal line that is approached as x tends towards $+\infty$ and or $-\infty$.
If $\lim\limits_{x\to+\infty} f(x)=L$ or $\lim\limits_{x\to-\infty} f(x)=L$, we say that the line $y=L$ is the ***horizontal asymptote***.

Oblique Asymptotes and Limits at Infinity

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a ***linear oblique asymptote***, if the *degree of the numerator is exactly one more than the degree of the denominator*.

It is the non-horizontal line of the form $y=mx+b$ that is approached as x tends towards $+\infty$ and or $-\infty$.

Ex. 1. Given the following function, check for discontinuities and state the equation of any asymptotes. Conduct limit tests to determine the behaviour of the curve near the asymptotes and then sketch the curve.

$f(x)=\dfrac{2-x}{x^2-5x+6}$

$f(x)=\dfrac{-\cancel{(x-2)}^{1}}{(x-3)\cancel{(x-2)}_{1}}, x\neq 3 \text{ \& } 2$

$f(x)=\dfrac{-1}{x-3}, x\neq 3, 2$

hole @ $(2,1)$

V.A. $x=3$

$\lim\limits_{x\to 3^-}\dfrac{-1}{x-3} = \boxed{+}\infty$ or $\lim\limits_{x\to 3^+}\dfrac{-1}{x-3} = \boxed{-}\infty$

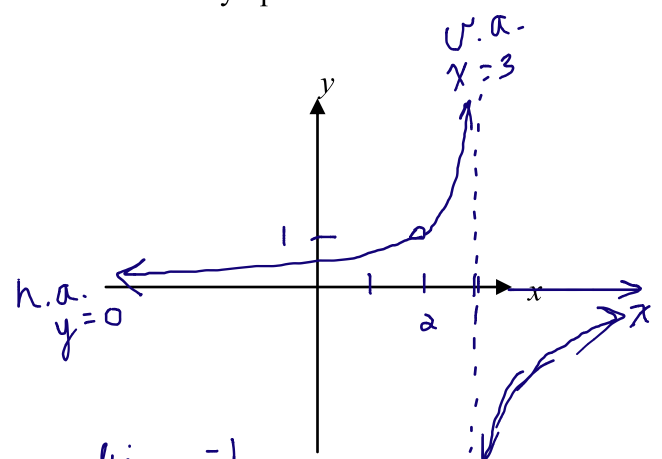
H.A. $y=0$

$\lim\limits_{x\to -\infty}\dfrac{-1}{x-3}$ $\lim\limits_{x\to +\infty}\dfrac{-1}{x-3}$

$=\lim\limits_{x\to -\infty}\dfrac{-\frac{1}{x}}{1-\frac{3}{x}} = 0$

$=\dfrac{0}{1-0}$

$=0$



Ex. 2. Graph the following by finding any asymptotes, holes, intercepts, max/min and inflection points. Identify intervals for which the function is increasing, decreasing, concave up and concave down.

a) $f(x) = \dfrac{x+2}{x-5}$; $f'(x) = \dfrac{-7}{(x-5)^2}$; $f''(x) = \dfrac{14}{(x-5)^3}$

$f'(x) = \dfrac{1(x-5) - 1(x+2)}{(x-5)^2}$

$f'(x) = \dfrac{-7}{(x-5)^2}$

$f'(x) = -7(x-5)^{-2}$

$f''(x) = 14(x-5)^{-3}(1)$

Intercepts

$\boxed{\begin{array}{l} x\text{-int is } -2 \\ y\text{-int is } -\frac{2}{5} \end{array}}$

Asymptotes

$\boxed{\text{V.A. is } x = 5}$

For h.a.,

$\lim_{x \to \pm\infty} f(x)$

$= \lim_{x \to \pm\infty} \dfrac{x+2}{x-5}$

$= \lim_{x \to \pm\infty} \dfrac{1 + \frac{2}{x}}{1 - \frac{5}{x}}$

$= 1$

$\boxed{\therefore \text{HA is } y = 1}$

For Critical pts.

$f'(x) = 0$

$\dfrac{-7}{(x-5)^2} = 0$

no solution

$\therefore$ no max/min

For P.O.I

$f''(x) = 0$

no solution

Does f(x) cross H.A.? NO!

$f(x) = \text{H.A.}$

$\dfrac{x+2}{x-5} = \dfrac{1}{1}$

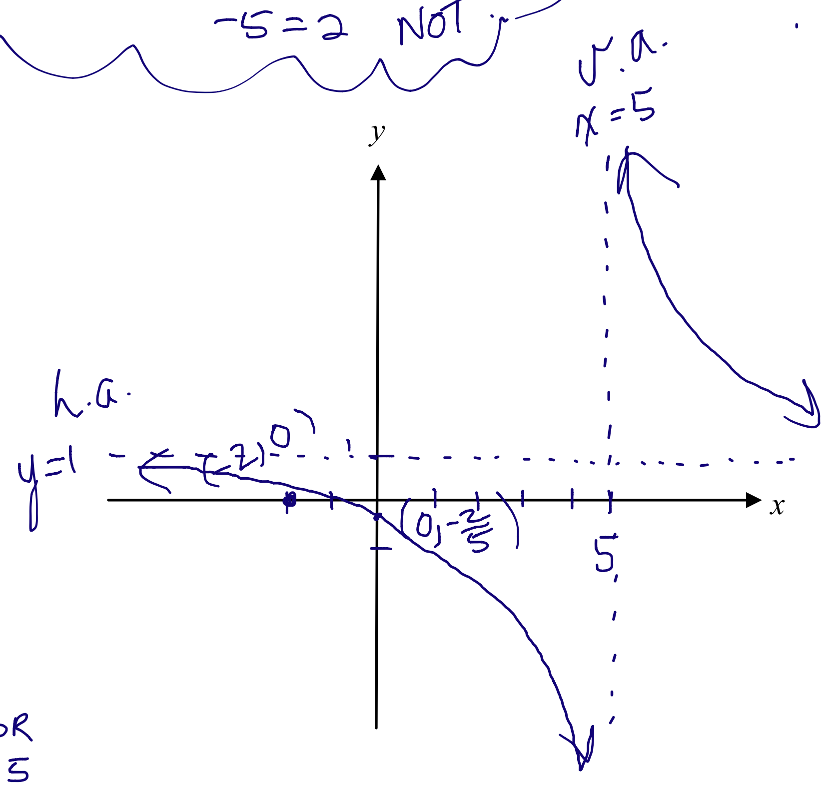
$x - 5 = x + 2$

$-5 = 2$ NOT!

$f'(x) = \dfrac{-7}{(x-5)^2}$

$f''(x) = \dfrac{14}{(x-5)^3}$

x	$f(x)$	$f'(x)$	$f''(x)$	
$x<5$		$-$	$-$	
5	$\times$	$\times$	$\times$	
$x>5$		$-$	$+$	



Summary: $f(x)$ is:

increasing *for* no x-values

decreasing *for* $x \in \mathbb{R}, x \neq 5$ OR $x < 5$ or $x > 5$

concave up *for* $x > 5$

concave down *for* $x < 5$

b) $g(x) = \dfrac{x^2}{x+1}$

$$\begin{array}{r} x - 1 \\ x+1 \overline{\smash{)}\, x^2 + 0x + 0} \\ \underline{x^2 + x} \downarrow \\ -x + 0 \\ \underline{-x - 1} \\ 1 \end{array}$$

Rewrite as a mixed rational

$\boxed{g(x) = (x-1) + \dfrac{1}{x+1}}$

$g(x) = x - 1 + (x+1)^{-1}$

$g'(x) = 1 - (x+1)^{-2}$

$g''(x) = 2(x+1)^{-3}$

$g''(x) = \dfrac{2}{(x+1)^3}$

$\boxed{\begin{array}{l} x\text{-int is } 0 \\ y\text{-int is } 0 \end{array}}$

$\boxed{\begin{array}{l} \text{v.a is } x = -1 \\ \text{l.o.a. is} \\ y = x - 1 \end{array}}$

For critical pts.

$g'(x) = 0$

$\boxed{1 - \dfrac{1}{(x+1)^2}} = 0$

$(x+1)^2 - 1 = 0$

$(x+1)^2 = 1$

$x + 1 = \pm 1$

$x = -1 - 1$ or $x = -1 + 1$

$x = -2 \qquad x = 0$

$\boxed{\begin{array}{l} \therefore (-2, -4) \text{ is a local maximum} \\ \& \ (0,0) \text{ is a local minimum} \end{array}}$

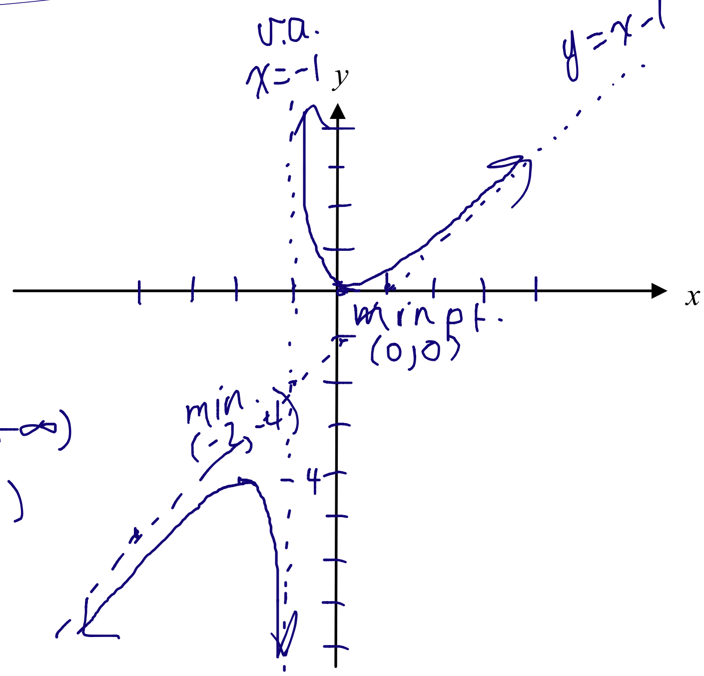
For P.O.I.

$g''(x) = 0$

no solution

$\therefore$ no inflection pts.

x	$g(x)$	$g'(x)$	$g''(x)$	
$x < -2$		$+$	$-$	
-2	-4	0	$-$	max.
-1	$\times$	$\times$	$\times$	
0	0	0	$+$	min.
$x > 0$		$+$	$+$	



Summary: $g(x)$ is:

***increasing** for* $x \in (-\infty, -2) \cup (0, +\infty)$

***decreasing** for* $x \in (-2, -1) \cup (-1, 0)$

***concave up** for* $x \in (-1, +\infty)$

***concave down** for* $x \in (-\infty, -1)$

H.W. Graph the following by finding any asymptotes, holes, intercepts, max/min and inflection points. Identify intervals for which the function is increasing, decreasing, concave up and concave down.

a) $y = \dfrac{x^3 - 3x^2 - 4x + 12}{x^3 - 3x^2 - x + 3}$ **b)** $y = \dfrac{1}{x^2+3}$ **c)** $f(x) = \dfrac{x^2}{1-x^2}$ **d)** $f(x) = \dfrac{x^2+x+1}{x}$ **e)** $y = \dfrac{1+x-x^2}{x-1}$

holes !!

$f(x) = (x+1) + \dfrac{1}{x}$

Date:______________ **ADDITIONAL CURVE SKETCHING TECHNIQUES**

Ex. 1. For each of the following, sketch the graph of a function $y = f(x)$, that satisfies the given conditions.

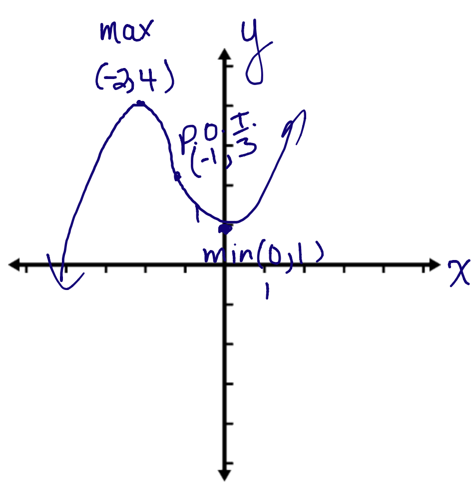
a) $f(-2) = 4$; $f(-1) = 2$; $f(0) = 1$; $f'(-2) = 0$; $f'(-1) = -3$; $f'(0) = 0$; $f''(-1) = 0$; $f''(x) < 0$ for all $x < -1$; $f''(x) > 0$ for all $x > -1$.

b) $\lim_{x \to +\infty} f(x) = 0$; $\lim_{x \to -\infty} f(x) = 0$; there is a vertical asymptote at $x = -1$ and $x = 3$; $f(1) = 1$; $f'(x) > 0$ for $x < -1$ or $x > 3$; $f'(x) < 0$ for $-1 < x < 3$; $f''(x) > 0$ for $x < -1$ or $-1 < x < 1$; $f''(x) < 0$ for $1 < x < 3$ or $x > 3$.

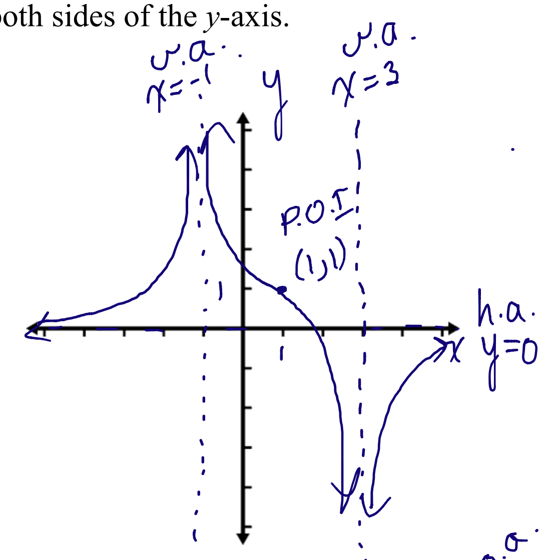
c) $f(0) = 0$; $f(1) = 1$; $f(-1) = 1$; $f'(1) = \frac{2}{3}$; $f'(-1) = -\frac{2}{3}$; $f(x)$ is decreasing for all $x < 0$; $f(x)$ is concave down for all $x \neq 0$; there is a cusp at $x = 0$.

d) $f(-1) = -2$; $f(1) = 2$; $\lim_{x \to 0^-} f(x) = -\infty$; $\lim_{x \to 0^+} f(x) = \infty$; there is a maximum point at $x = -1$; there is a minimum point at $x = 1$; $f(x)$ is concave down for $x < 0$; $f(x)$ is concave up for $x > 0$; $y = x$ is an oblique asymptote on both sides of the y-axis.

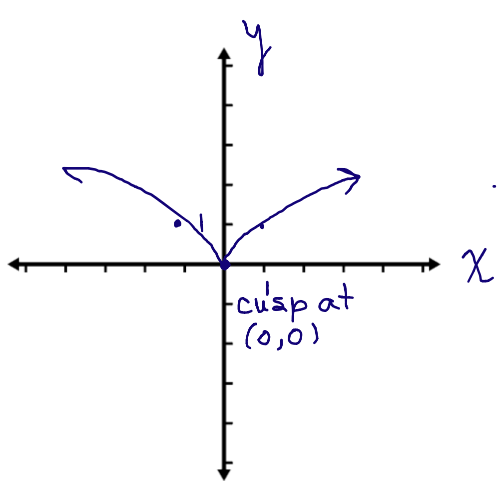
a)



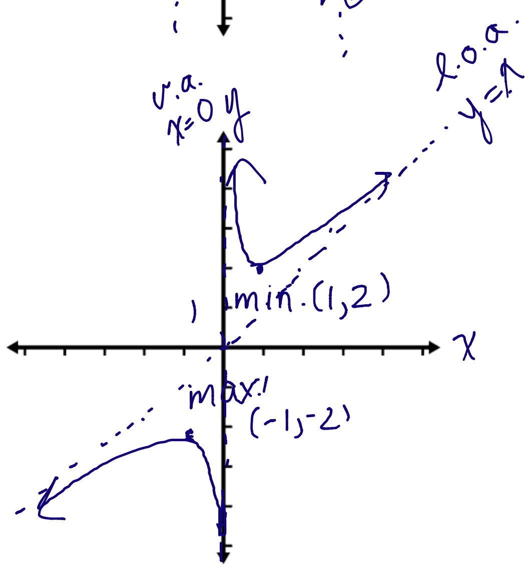
b)



c)



d)



Ex. 2. Use the algorithm for curve sketching to graph each of the following.

a) $y=\sqrt[3]{x-1}$

$y=(x-1)^{\frac{1}{3}}$

$y'=\frac{1}{3}(x-1)^{-\frac{2}{3}}$

$y''=-\frac{2}{9}(x-1)^{-\frac{5}{3}}$

$y'=\frac{1}{3\sqrt[3]{(x-1)^2}}$

$y''=\frac{-2}{9\sqrt[3]{(x-1)^5}}$

$D=\{x\in\mathbb{R}\}$

For x-int, $y=0$

$\sqrt[3]{x-1}=0$

$x-1=0^3$

$x=1$

$\therefore$ x-int. is 1

For y-int., $x=0$

$y=\sqrt[3]{-1}$

$y=-1$

$\therefore$ y-int is -1

For critical pts.,

$y'=0$ or y' is undefined

no solution $\quad x=1$

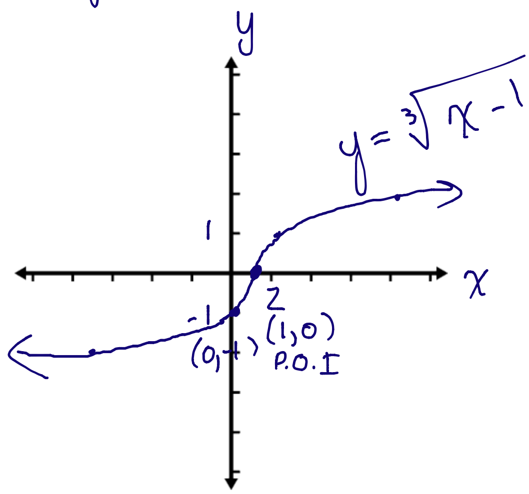
$\therefore$ the critical pt. is $(1,0)$ a P.O.I.

For P.O.I,

$y''=0$

no solution

x	y	y'	y''	
$x<1$		+	+	inc. con. up
1	0	X	X	P.O.I
$x>1$		+	−	increasing concave down



b) $f(x)=(3x-6)^{\frac{2}{3}}$ or $f(x)=\sqrt[3]{(3x-6)^2}$; $D=\{x\in\mathbb{R}\}$

$f'(x)=\frac{2}{3}(3x-6)^{-\frac{1}{3}}\cdot(3)$

$f'(x)=2(3x-6)^{-\frac{1}{3}}$

$f''(x)=\frac{-2}{3}(3x-6)^{-\frac{4}{3}}(3)$

$f''(x)=-2(3x-6)^{-\frac{4}{3}}$

$f'(x)=\frac{2}{\sqrt[3]{3x-6}}$

$f''(x)=\frac{-2}{\sqrt[3]{(3x-6)^4}}$

For x-int, $f(x)=0$

$(3x-6)^{\frac{2}{3}}=0 \quad ^{\frac{3}{2}}$

$3x-6=0$

$3x-6=0$

$x=2$

$\therefore$ x-int is 2

For y-int, $x=0$

$f(0)=\sqrt[3]{(-6)^2}$

$=\sqrt[3]{36}$

$\doteq 3.3$

$\therefore$ y-int is $\sqrt[3]{36}$

For critical pts,

$f'(x)=0$ or $f'(x)$ is undefined

no solution $\quad x=2$

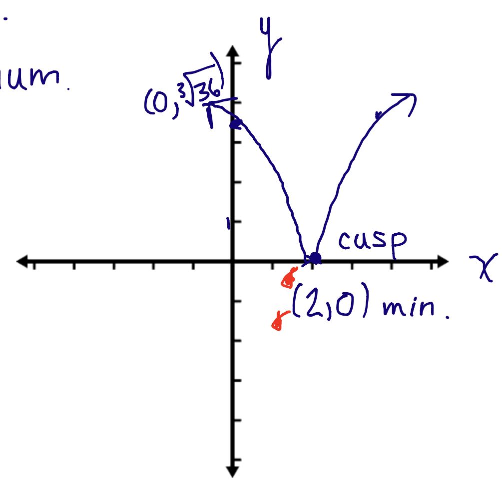
$\therefore$ the critical pt. is $(2,0)$ a local minimum.

For P.O.I.,

$f''(x)=0$

no solution

x	$f(x)$	$f'(x)$	$f''(x)$	
$x<2$		−	−	decreasing concave down
min. 2	0	X	X	
$x>2$		+	−	increasing concave down



c) $f(x)=e^{-x^2}$; $f(x)=\dfrac{1}{e^{x^2}}$

$f'(x)=e^{-x^2}(-2x)$

$f'(x)=\underbrace{-2x}\cdot\underbrace{e^{-x^2}}$

$f''(x)=-2\cdot e^{-x^2}+e^{-x^2}(-2x)\cdot(-2x)$

$=-2e^{-x^2}+4x^2e^{-x^2}$

$=4x^2e^{-x^2}-2e^{-x^2}$

$\therefore f''(x)=2e^{-x^2}(2x^2-1)$

$f'(x)=\dfrac{-2x}{e^{x^2}}$

$f''(x)=\dfrac{2(2x^2-1)}{e^{x^2}}$

$f\left(-\frac{1}{\sqrt{2}}\right)=\dfrac{1}{e^{\frac{1}{2}}}$

$=\dfrac{1}{\sqrt{e}}$

$f\left(+\frac{1}{\sqrt{2}}\right)=\dfrac{1}{\sqrt{e}}$

$D=\{x\in\mathbb{R}\}$

For x-int,
$f(x)=0$
$\dfrac{1}{e^{x^2}}=0$
no sol$^{\underline{n}}$
$\therefore$ no x-int.

For y-int, $x=0$
$f(0)=1$
$\therefore$ y-int is 1

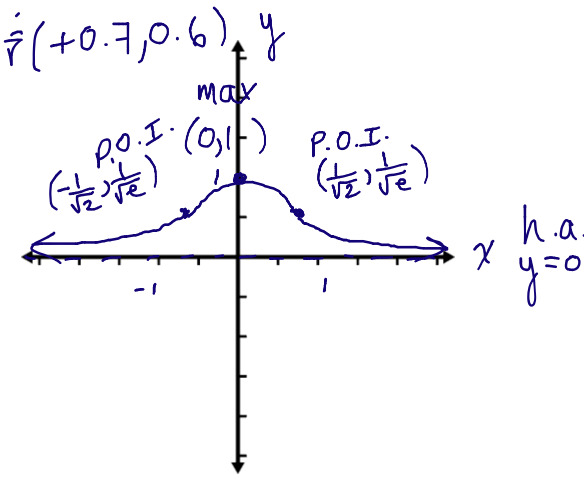
For inflection pts,
$f''(x)=0$
$2x^2-1=0$
$x^2=\frac{1}{2}$
$x=\pm\frac{1}{\sqrt{2}}$
$\therefore$ the P.O.I are
$\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{e}}\right)$ & $\doteq(-0.7,0.6)$
$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{e}}\right)$ or $\doteq(+0.7,0.6)$

For h.a,
$\lim\limits_{x\to\pm\infty} f(x)$
$=\lim\limits_{x\to\pm\infty}\dfrac{1}{e^{x^2}}$
$=0$

$\therefore$ h.a is $y=0$.

For critical pts,
$f'(x)=0$
$-2x=0$
$x=0$
$f(0)=1$
$\therefore$ the critical pt. $(0,1)$ is a local maximum.

x	$f(x)$	$f'(x)$	$f''(x)$	
$x<-\frac{1}{\sqrt{2}}$		+	+	
$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{e}}$	+	0	P.O.I
☆ 0	1	0	−	max.
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{e}}$	−	0	P.O.I
$x>\frac{1}{\sqrt{2}}$		−	+	



H.W. Following Ex. 2 from this note, graph each function below:

1. $y=x^{\frac{5}{3}}-5x^{\frac{2}{3}}$ **2.** $f(x)=x\cdot e^{-x}$ **3.** $y=(\ln x)^2$ **4.** $y=2x-\tan x,\ -\frac{\pi}{2}<x<\frac{\pi}{2}$

5. $f(x)=2\cos x-\sin 2x,\ -\pi\le x\le\pi$

Review: pg. 378 #2, 4 to 8, 9i, 10, 11, 12abcdef; pg. 382 #8, 9