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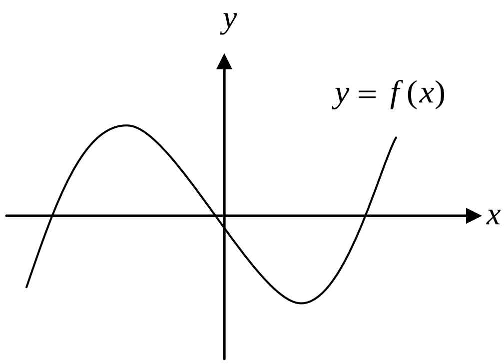
UNIT 6: CURVE SKETCHING

Section 9.1 – Increasing and Decreasing Functions
Section 9.2 – Critical Points, Relative Maxima and Minima

CRITICAL POINTS: ***are points where the derivative is 0 or undefined.***

Note: *All relative maxima and minima (local extrema) are critical points, but not all critical points are maxima or minima.*



For local max/min $f'(x) = 0$ ***or*** $y' = 0$.

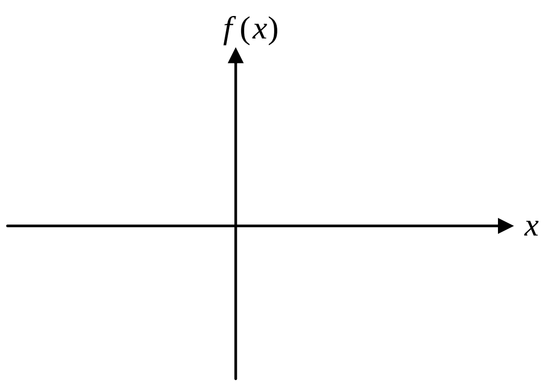
$f(x)$ *is said to be* ***increasing*** *if as x increases,* $f(x)$ *increases.*

For an increasing function, $f'(x) > 0$ ***or*** $y' > 0$.

$f(x)$ *is said to be* ***decreasing*** *if as x increases,* $f(x)$ *decreases.*

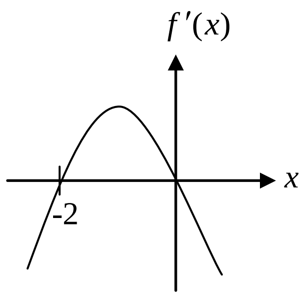
For a decreasing function, $f'(x) < 0$ ***or*** $y' < 0$.

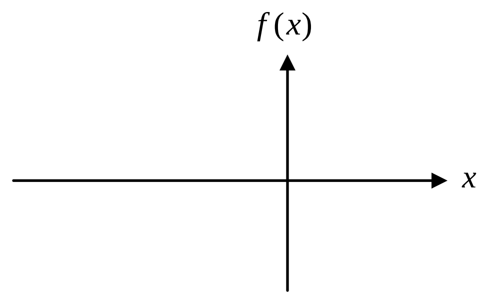
Ex. 1. Sketch the graph of the function $f(x)$ that satisfies $f'(x) > 0$ for $x < -2$ and $x > 3$ and $f'(x) < 0$ for $-2 < x < 3$.



Interval	$x < -2$	$-2 < x < 3$	$x > 3$
$f'(x)$			
$f(x)$			

Ex. 2. Sketch the graph of $f(x)$, assuming $f(0) = 1$ and given the graph of the derivative $f'(x)$.



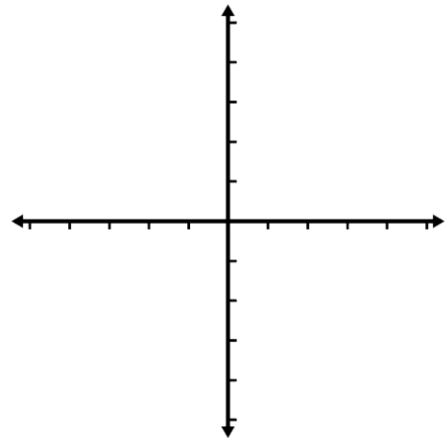


Interval			
$f'(x)$			
$f(x)$			

Ex. 3. Find all ***critical points*** of the following functions. Identify the ***critical points*** as ***local maxima***, ***local minima*** *or* ***neither*** and sketch the graph.

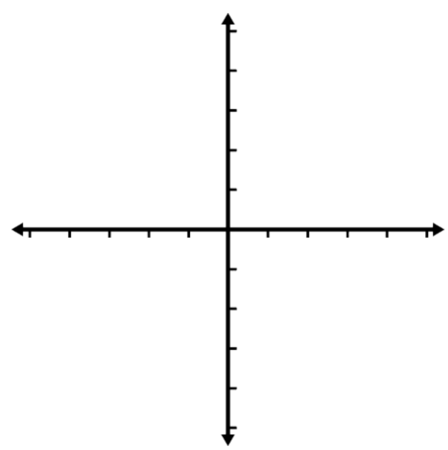
a) $f(x) = x^3 - 3x + 2$

x	$f(x)$	$f'(x)$



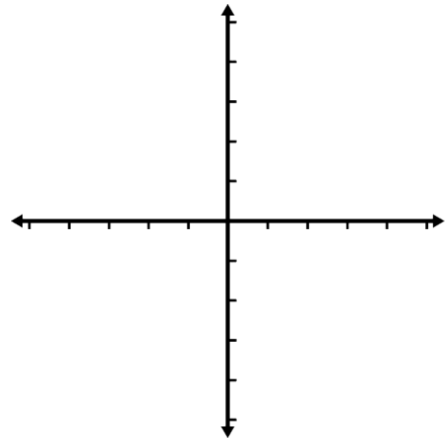
b) $f(x) = (1 - x^2)^2 - 2$

x	$f(x)$	$f'(x)$



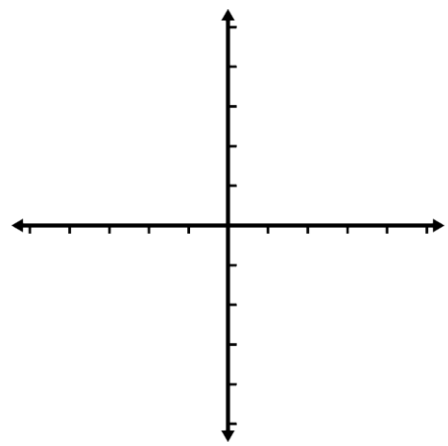
c) $f(x) = |x|$

x	$f(x)$	$f'(x)$



d) $f(x) = x^3$

x	$f(x)$	$f'(x)$



Ex. 4. The function $f(x) = x^3 + ax + b$ has a turning point at $(-2, 6)$. Find the constants a and b and the function $f(x)$.

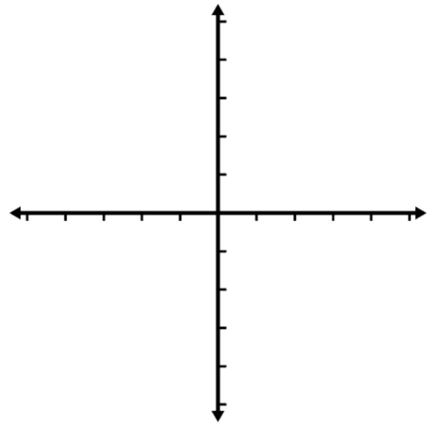
HW: p. 342 # 4, 8, 9, 11ab; p. 350 #2, 7bde, 9, 10

Date:________________

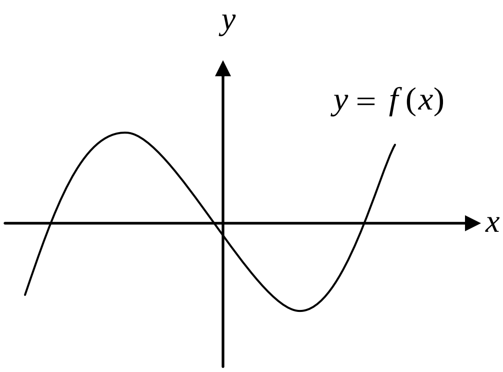
Section 9.4 – Concavity and Points of Inflection

Ex. 1. Find and classify all ***local extrema*** *(local maxima and minima)* of the function $f(x) = x^3 - 3x^2$**.** Illustrate your results graphically.

x	$f(x)$	$f'(x)$



Applications of the Second Derivative for Curve Sketching: $f''(x)$ ***or*** y''



Note: **1.** $f(x)$ is ***concave up*** if $f''(x) > 0$ ***or*** $y'' > 0$.

2. $f(x)$ is ***concave down*** if $f''(x) < 0$ ***or*** $y'' < 0$.

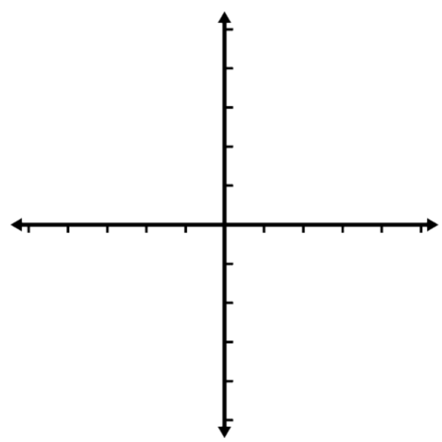
3. For ***inflection points*** $f''(x) = 0$ ***or*** $y'' = 0$

Second Derivative Test for Local Extrema

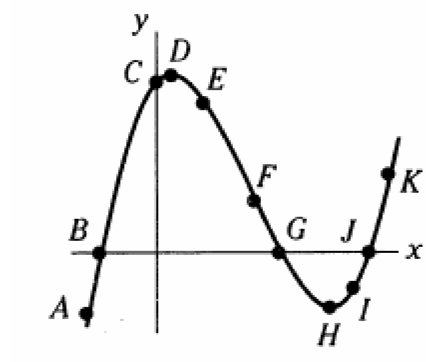
For a ***local maximum***, $f'(x) = 0$ and $f''(x) < 0$**.**
For a ***local minimum***, $f'(x) = 0$ and $f''(x) > 0$**.**

Ex. 2. Find and classify all ***local extrema*** of the function $f(x) = 3x^5 - 25x^3 + 60x$ using the ***second derivative test***. Illustrate your results graphically.

x	$f(x)$	$f'(x)$	$f''(x)$



Ex. 3. Identify the point on the graph that satisfies the following conditions:



a) $y > 0,\ y' > 0,\ y'' < 0$ ________

b) $y > 0,\ y' < 0,\ y'' < 0$ ________

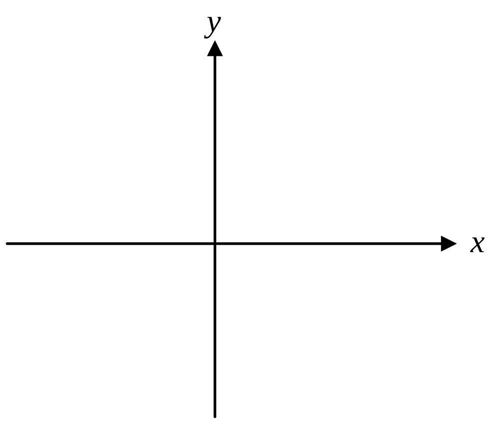
c) $y < 0,\ y' > 0,\ y'' > 0$ ________

d) $y < 0,\ y' = 0,\ y'' < 0$ ________

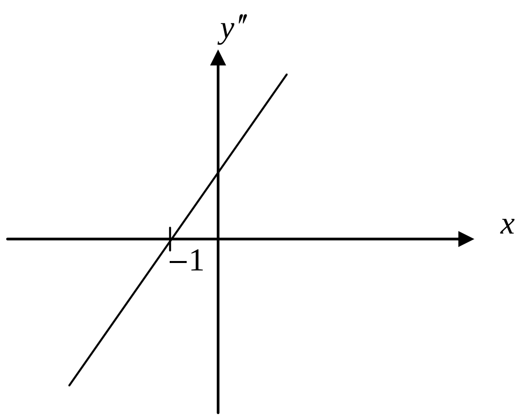
e) $y = 0,\ y' > 0,\ y'' < 0$ ________

f) $y > 0,\ y' < 0,\ y'' = 0$ ________

Ex. 4. Sketch the graph of a function that satisfies the following conditions:
$f''(x) > 0$ on the interval $x < 2$; $f''(x) < 0$ on the interval $x > 2$; $f(2) = 0$; $f'(2) > 0$



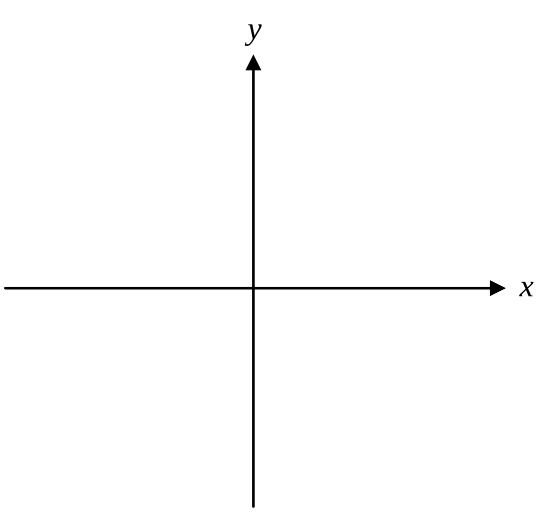
Ex. 5.



a) On what intervals is the graph of $f(x)$
i) concave up? **ii)** concave down?

b) List the x – coordinates of all points of inflection.

c) Make a rough sketch of a possible graph of $f(x)$, assuming that $f(0) = 1$.



Ex. 6. Determine the constants *a*, *b*, *c* and *d* so that the curve $y = ax^3 + bx^2 + cx + d$ has a ***point of inflection*** at the *origin* and a ***local maximum*** at the point $(2, 4)$.

Ex. 7. Evaluate $f''(x)$ at the indicated value, and state whether the graph lies above or below the tangent for $f(x) = \dfrac{1}{2x^2 + 1}$, $x = 1$. Sketch the graph near the point where $x = 1$.

HW: p. 369 #1, 2acd, 3acd, 4abc & (#9 Unit 4 Day 2 Worksheet, 4b Unit 4 Review Days 1-3); p. 369 #5ab, 8i), 9, 10 do not sketch, 11

Date:_______________

Graphing Polynomial Functions

Warm-up:

Given $f(x)=\dfrac{1}{2x^2+1}$, $f'(x)=\dfrac{-4x}{(2x^2+1)^2}$ and $f''(x)=\dfrac{4(6x^2-1)}{(2x^2+1)^3}$ determine the intervals where the function is **i)** *increasing, decreasing* **ii)** *concave up, concave down*.

i)

Interval		
$f'(x)$		
$f(x)$		

ii)

Interval			
$f''(x)$			
$f(x)$			

Sketching the Graph of a Polynomial or Rational Function

1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

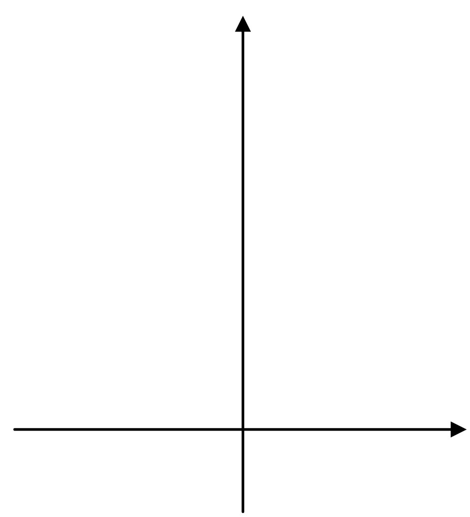
The second derivative can also be used to identify local maxima and minima.

4. Calculate the values of *y* that correspond to critical points and points of inflection. Use the information above to sketch the graph.

Remember that you will not use all the steps in every situation! Use only the steps that are necessary to give you a good idea of what the graph will look like.

Ex. Graph $f(x) = 3x^4 - 4x^3$ by finding and labeling all *intercepts*, *local max/min* and *inflection* points. Identify intervals for which the function is *increasing*, *decreasing*, *concave up* and *concave down*.

x	$f(x)$	$f'(x)$	$f''(x)$



HW: Worksheet on Graphing Polynomial Functions

Date:____________

WORKSHEET on Sketching Polynomial Functions

Determine the following information for each given polynomial function:

- the y-intercept
- the x-intercept(s)
- the local maximum and minimum point(s)
- the interval(s) where the function is increasing
- the interval(s) where the function is decreasing
- the point(s) of inflection
- the interval(s) where the function is concave up
- the interval(s) where the function is concave down
- provide a clearly labeled sketch

State exact values for increasing/decreasing intervals and concave up/down intervals. State the exact value for an x-value of an ordered pair (local max/min and POIs). You may state the y-value of an ordered pair (local max/min and POIs) as a decimal number accurate to two decimal places.

1. $f(x) = x^3 - 2x^2 - 2x - 3$

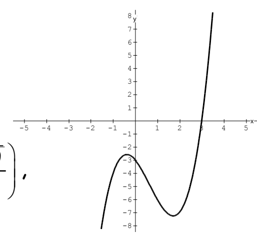
2. $f(x) = x^3 - 3x^2 + 3x - 1$

3. $f(x) = x^4 + x^3 + x + 1$

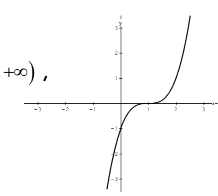
4. $f(x) = -x^4 - x^3 + 3x^2 + x - 2$

Answers:

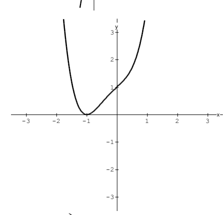
1. y-int -3, x-int 3, local max $\left(\frac{2-\sqrt{10}}{3}, -2.58\right)$, local min $\left(\frac{2+\sqrt{10}}{3}, -7.27\right)$,
increasing $x \in \left(-\infty, \frac{2-\sqrt{10}}{3}\right) \cup \left(\frac{2+\sqrt{10}}{3}, +\infty\right)$, decreasing $x \in \left(\frac{2-\sqrt{10}}{3}, \frac{2+\sqrt{10}}{3}\right)$,
POI $\left(\frac{2}{3}, -4.93\right)$, concave up $x \in \left(\frac{2}{3}, +\infty\right)$, concave down $x \in \left(-\infty, \frac{2}{3}\right)$



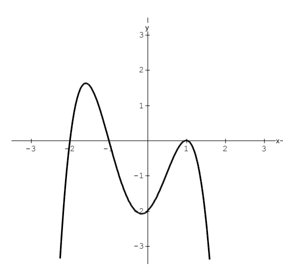
2. y-int -1, x-int 1, no local max, no local min, increasing $x \in (-\infty, 1) \cup (1, +\infty)$,
decreasing interval – none, POI(1,0), concave up $x \in (1, +\infty)$,
concave down $x \in (-\infty, 1)$



3. y-int 1, x-int -1, no local max, local min (-1,0), increasing $x \in (-1, +\infty)$,
decreasing $x \in (-\infty, -1)$, POIs (0,1) & $\left(\frac{-1}{2}, 0.44\right)$,
concave up $x \in \left(-\infty, \frac{-1}{2}\right) \cup (0, +\infty)$, concave down $x \in \left(\frac{-1}{2}, 0\right)$



4. y-int -2, x-int -2,-1,1, local max (1,0) & $\left(\frac{-7-\sqrt{33}}{8}, 1.62\right)$, local min $\left(\frac{-7+\sqrt{33}}{8}, -2.08\right)$,
increasing $x \in \left(-\infty, \frac{-7-\sqrt{33}}{8}\right) \cup \left(\frac{-7+\sqrt{33}}{8}, 1\right)$, decreasing $x \in \left(\frac{-7-\sqrt{33}}{8}, \frac{-7+\sqrt{33}}{8}\right) \cup (1, +\infty)$,
POI (-1,0) & $\left(\frac{1}{2}, -0.9.375\right)$, concave up $x \in \left(-1, \frac{1}{2}\right)$, concave down $x \in (-\infty, -1) \cup \left(\frac{1}{2}, +\infty\right)$



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Graphing Rational Functions

Vertical Asymptotes and Infinite Limits

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a ***vertical asymptote*** at $x=c$ if

- $q(c)=0$ and $p(c)\neq 0$

  Note: If $q(c)=0$ and $p(c)=0$ then a ***hole*** is created at $x=c$

- $\lim\limits_{x\to c^-} f(x)=+\infty$, $\lim\limits_{x\to c^-} f(x)=-\infty$, $\lim\limits_{x\to c^+} f(x)=+\infty$ or $\lim\limits_{x\to c^+} f(x)=-\infty$

Horizontal Asymptotes and Limits at Infinity

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a ***horizontal asymptote***, if the *degree of the numerator is less than or equal to the degree of the denominator*.

It is the horizontal line that is approached as *x* tends towards $+\infty$ and or $-\infty$.
If $\lim\limits_{x\to+\infty} f(x)=L$ or $\lim\limits_{x\to-\infty} f(x)=L$, we say that the line $y=L$ is the ***horizontal asymptote***.

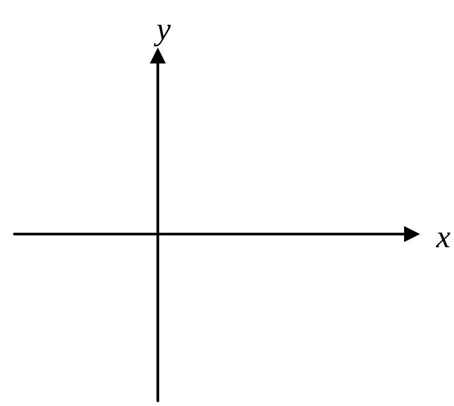
Oblique Asymptotes and Limits at Infinity

A rational function of the form $f(x)=\dfrac{p(x)}{q(x)}$ has a ***linear oblique asymptote***, if the *degree of the numerator is exactly one more than the degree of the denominator*.

It is the non-horizontal line of the form $y=mx+b$ that is approached as *x* tends towards $+\infty$ and or $-\infty$.

Ex. 1. Given the following function, check for discontinuities and state the equation of any asymptotes. Conduct limit tests to determine the behaviour of the curve near the asymptotes and then sketch the curve.

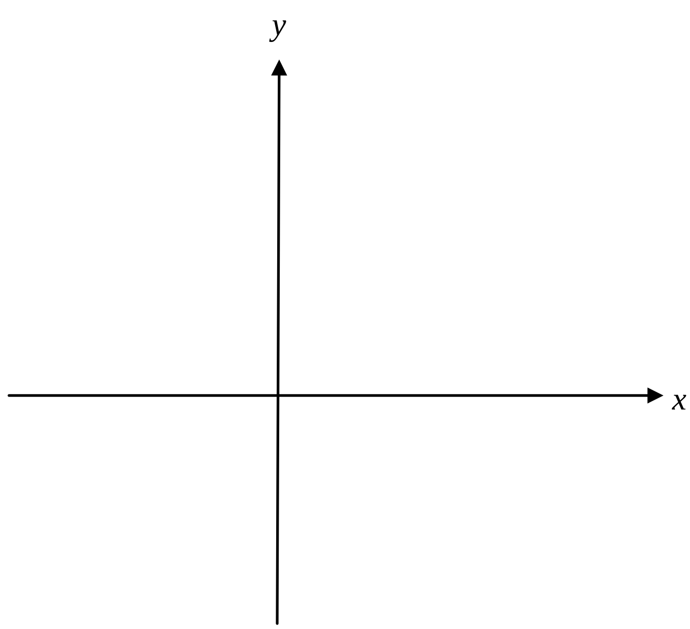
$$f(x)=\frac{2-x}{x^2-5x+6}$$



Ex. 2. Graph the following by finding any asymptotes, holes, intercepts, max/min and inflection points. Identify intervals for which the function is increasing, decreasing, concave up and concave down.

a) $f(x) = \dfrac{x+2}{x-5}$

x	$f(x)$	$f'(x)$	$f''(x)$



Summary: $f(x)$ is:

increasing *for*

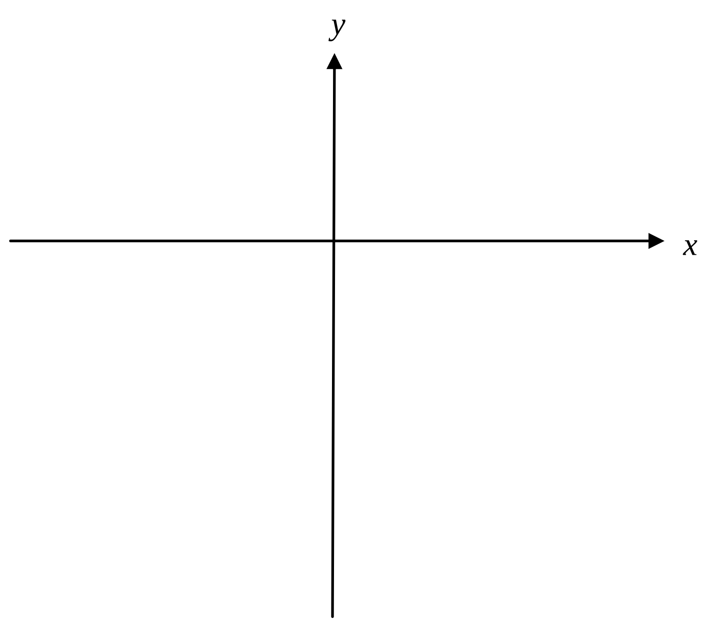
decreasing *for*

concave up *for*

concave down *for*

b) $g(x) = \dfrac{x^2}{x+1}$

x	$g(x)$	$g'(x)$	$g''(x)$



Summary: $g(x)$ is:
increasing *for*

decreasing *for*

concave up *for*

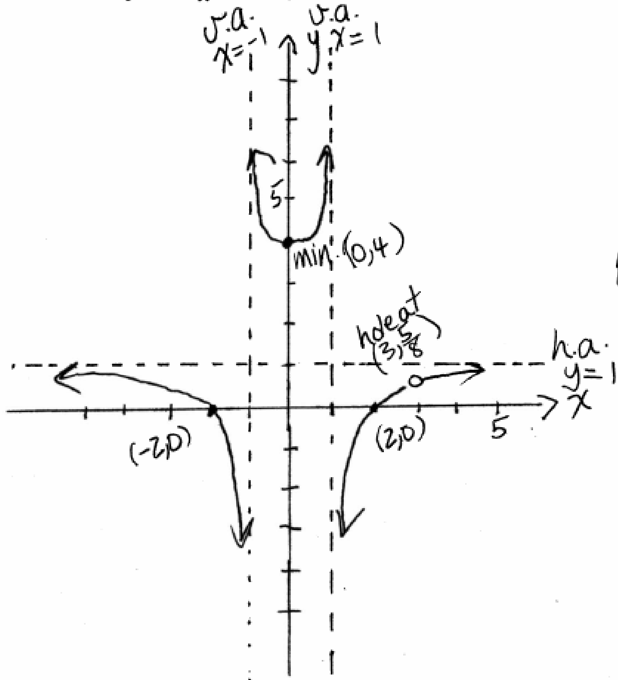
concave down *for*

HW: Graph the following by finding any asymptotes, holes, intercepts, max/min and inflection points. Identify intervals for which the function is increasing, decreasing, concave up and concave down.

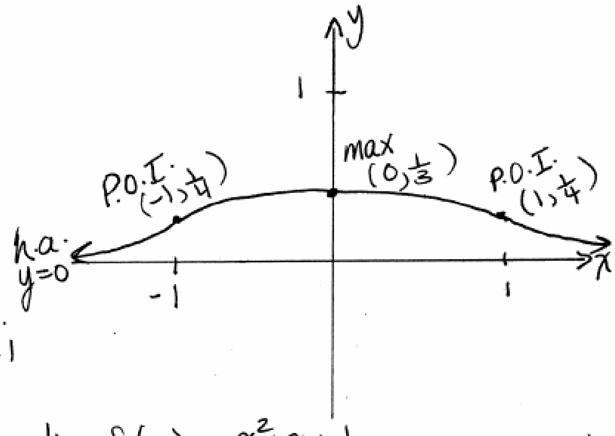
a) $y = \dfrac{x^3 - 3x^2 - 4x + 12}{x^3 - 3x^2 - x + 3}$ **b)** $y = \dfrac{1}{x^2 + 3}$ **c)** $f(x) = \dfrac{x^2}{1 - x^2}$ **d)** $f(x) = \dfrac{x^2 + x + 1}{x}$ **e)** $y = \dfrac{1 + x - x^2}{x - 1}$

Unit 6: Day 4   H.W. Graphing Rational Functions

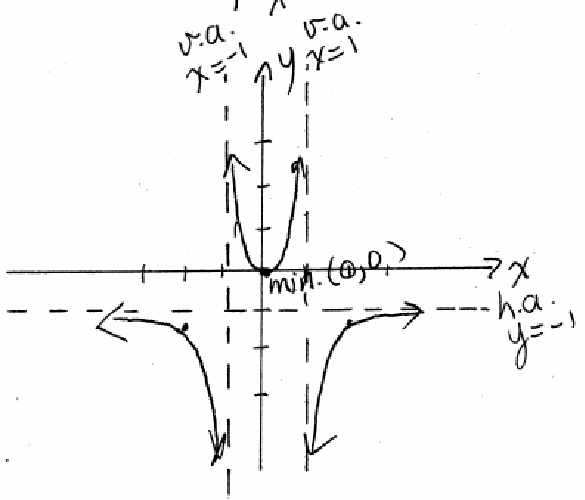
a) $y = \dfrac{x^3 - 3x^2 - 4x + 12}{x^3 - 3x^2 - x + 3}$



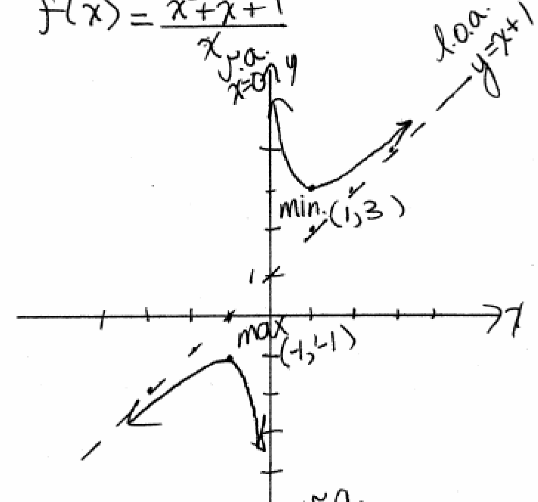
b) $y = \dfrac{1}{x^2+3}$



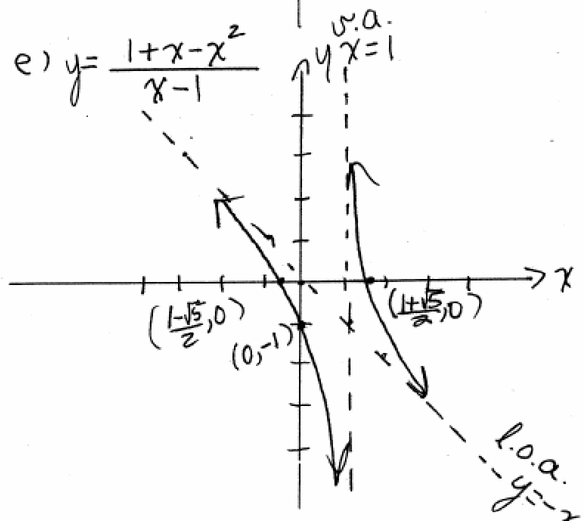
c) $f(x) = \dfrac{x^2}{1 - x^2}$



d) $f(x) = \dfrac{x^2 + x + 1}{x}$



e) $y = \dfrac{1 + x - x^2}{x - 1}$



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Additional Curve Sketching Techniques

Ex. 1. For each of the following, sketch the graph of a function $y = f(x)$, that satisfies the given conditions.

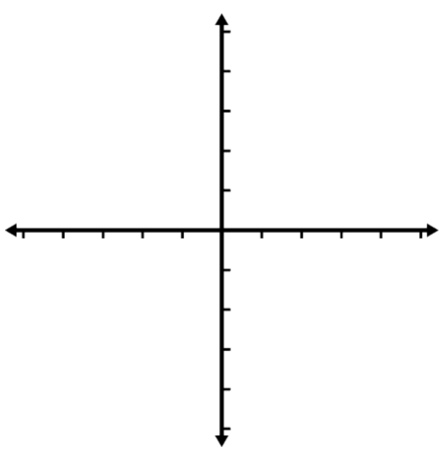
a) $f(-2) = 4$; $f(-1) = 2$; $f(0) = 1$; $f'(-2) = 0$; $f'(-1) = -3$; $f'(0) = 0$; $f''(-1) = 0$; $f''(x) < 0$ for all $x < -1$; $f''(x) > 0$ for all $x > -1$.

b) $\lim\limits_{x \to +\infty} f(x) = 0$; $\lim\limits_{x \to -\infty} f(x) = 0$; there is a vertical asymptote at $x = -1$ and $x = 3$; $f(1) = 1$; $f'(x) > 0$ for $x < -1$ or $x > 3$; $f'(x) < 0$ for $-1 < x < 3$; $f''(x) > 0$ for $x < -1$ or $-1 < x < 1$; $f''(x) < 0$ for $1 < x < 3$ or $x > 3$.

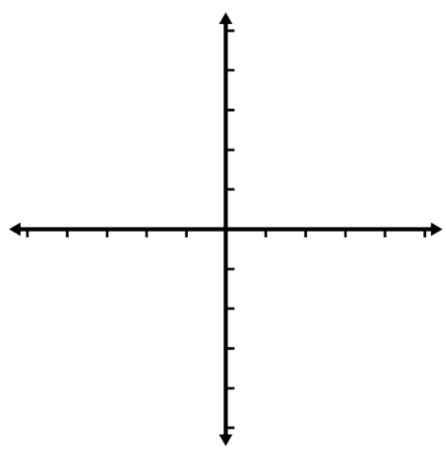
c) $f(0) = 0$; $f(1) = 1$; $f(-1) = 1$; $f'(1) = \frac{2}{3}$; $f'(-1) = -\frac{2}{3}$; $f(x)$ is decreasing for all $x < 0$; $f(x)$ is concave down for all $x \neq 0$; there is a cusp at $x = 0$.

d) $f(-1) = -2$; $f(1) = 2$; $\lim\limits_{x \to 0^-} f(x) = -\infty$; $\lim\limits_{x \to 0^+} f(x) = \infty$; there is a maximum point at $x = -1$; there is a minimum point at $x = 1$; $f(x)$ is concave down for $x < 0$; $f(x)$ is concave up for $x > 0$; $y = x$ is an oblique asymptote on both sides of the y-axis.

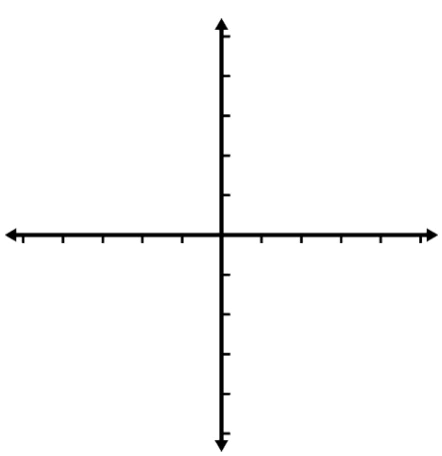
a)



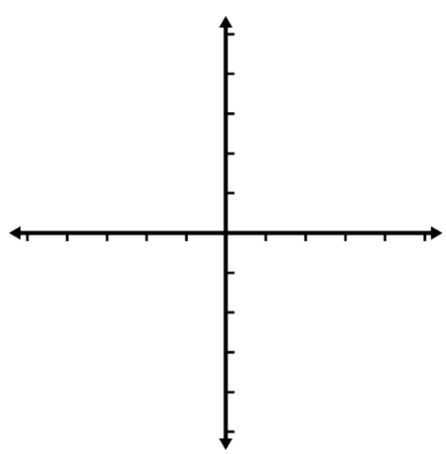
b)



c)



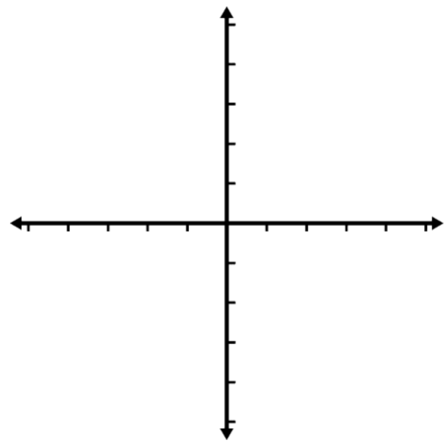
d)



Ex. 2. Use the algorithm for curve sketching to graph each of the following.

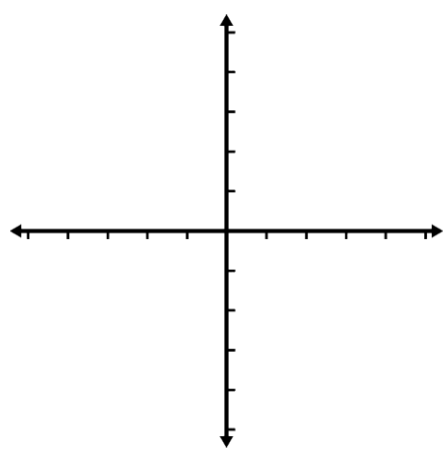
a) $y = \sqrt[3]{x-1}$

x	y	y'	y''



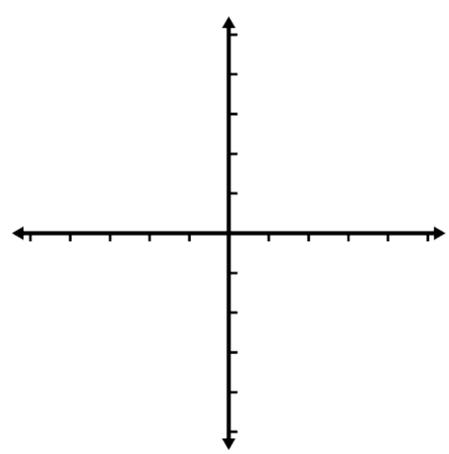
b) $f(x) = (3x-6)^{\frac{2}{3}}$

x	$f(x)$	$f'(x)$	$f''(x)$



c) $f(x) = e^{-x^2}$

x	$f(x)$	$f'(x)$	$f''(x)$



HW: Following Ex. 2 from this note, graph each function below:

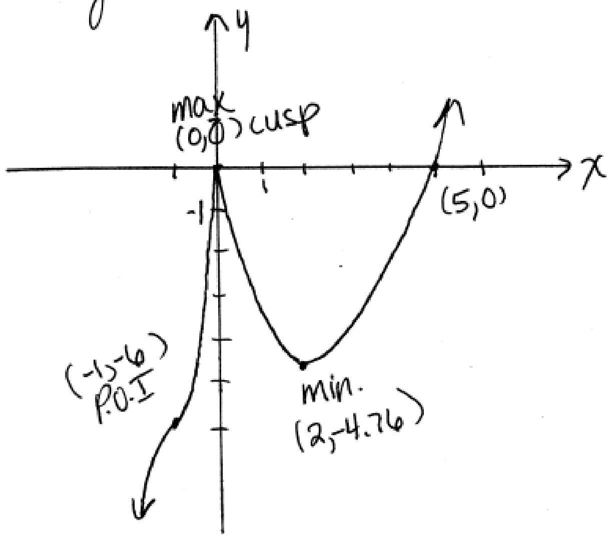
1. $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$ **2.** $f(x) = x \cdot e^{-x}$ **3.** $y = (\ln x)^2$ **4.** $y = 2x - \tan x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

5. $f(x) = 2\cos x - \sin 2x, \ -\pi \le x \le \pi$

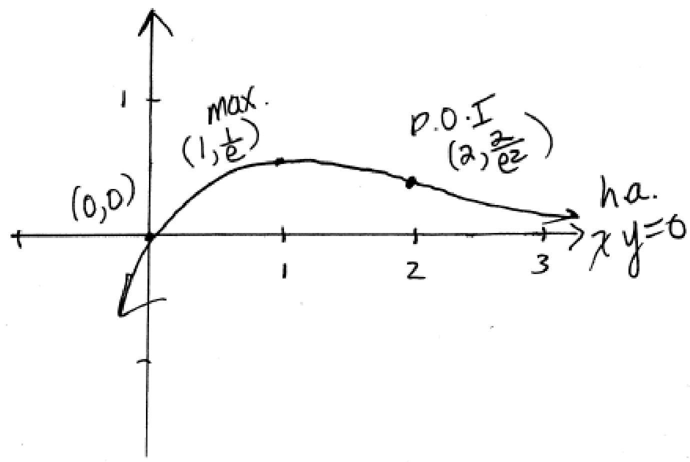
REVIEW for TEST: p. 378 #2, 4 to 8, 9i, 10, 11, 12abcdef; p. 382 #8, 9

Unit 6: Day 5 H.W. Graphs

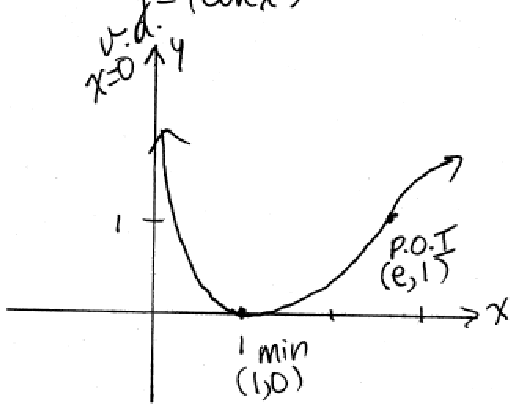
1. $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$



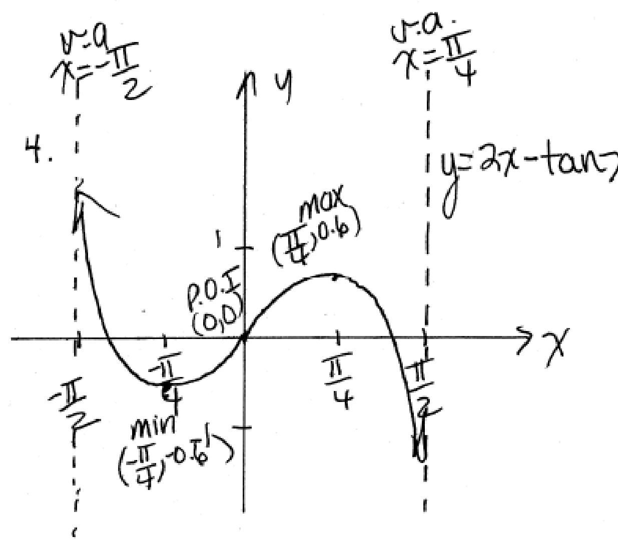
2. $f(x) = x \cdot e^{-x}$



3. $y = (\ln x)^2$



4. $y = 2x - \tan x$



5. $f(x) = 2\cos x - \sin 2x$, $-\pi \le x \le \pi$