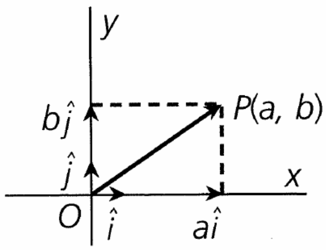
Date:_____________

UNIT 8 – ALGEBRAIC VECTORS AND APPLICATIONS

Section 5.1 – Coordinate Systems and Algebraic Vectors

A. Two-Dimensional Vectors $\Re^2$



Any vector in the plane can be translated so that its initial point lies at the ***origin***.

If the coordinates of P are (a , b), then $\overrightarrow{OP} = (\ a,\ b\ )$ is called the ***position vector*** and a and b are the ***components*** of the vector.

$\therefore (\ a, b\ )$ means a ***point*** or a ***vector***.

Let $\hat{i}$ and $\hat{j}$ represent ***unit vectors***.

$\hat{i} = (\quad , \quad)$ and $\hat{j} = (\quad , \quad)$ with $\vec{0} = (\quad , \quad)$.

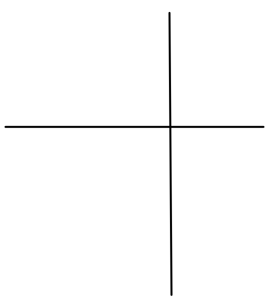
$\hat{i}$ and $\hat{j}$ are the ***standard basis vectors*** in $\Re^2$.

We can represent $\overrightarrow{OP}$ in terms of the ***standard basis vectors***, where $\overrightarrow{OP} =$ __________ .

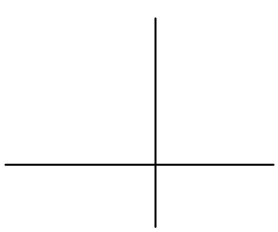
Every vector in $\Re^2$ can be represented ***algebraically*** or ***geometrically***.

Algebraic Form	Geometric Form
Ordered pair notation: $\overrightarrow{OP} = (\ a\ ,\ b\ )$	***Magnitude:*** $\lvert\overrightarrow{OP}\rvert$, where $\lvert\overrightarrow{OP}\rvert = \sqrt{a^2 + b^2}$
or	*and*
Unit vector notation: $\overrightarrow{OP} = a\hat{i} + b\hat{j}$	***Direction:*** θ , where θ is measured counter-clockwise
Note: $(\ a\ , b\ ) = a\hat{i} + b\hat{j}$	from the positive x-axis to the line of the vector.

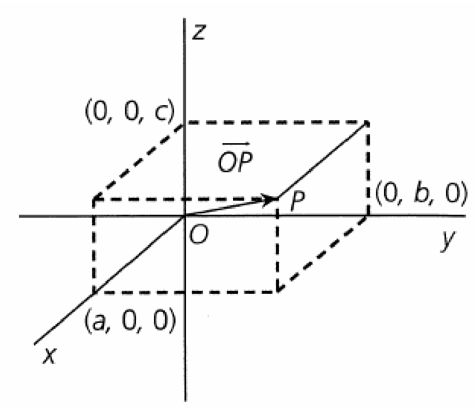
Ex. 1. Given $\lvert\vec{u}\rvert = 8$ and $\theta = 210^\circ$, express $\vec{u}$ as an ***algebraic vector*** in the form: **i)** $(\ a\ , b\ )$ **ii)** $a\hat{i} + b\hat{j}$



Ex. 2. Given $\overrightarrow{OP} = (-3,\ 4)$, express $\overrightarrow{OP}$ as a ***geometric vector*** by stating its *magnitude* and *direction*.



B. Three-Dimensional Vectors $\Re^3$



Any vector in the plane can be translated so that its initial point lies at the ***origin***.

If the coordinates of P are (a, b, c), then $\overrightarrow{OP} = (a, b, c)$ is called the ***position vector***, and a, b, and c are the ***components*** of the vector.

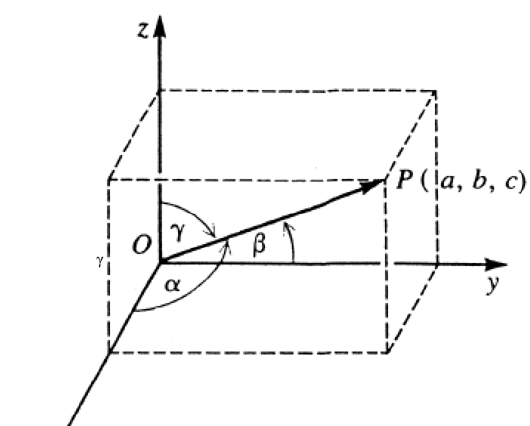
$\therefore$ (a, b, c) means a ***point*** (a, b, c) or a ***vector*** (a, b, c).

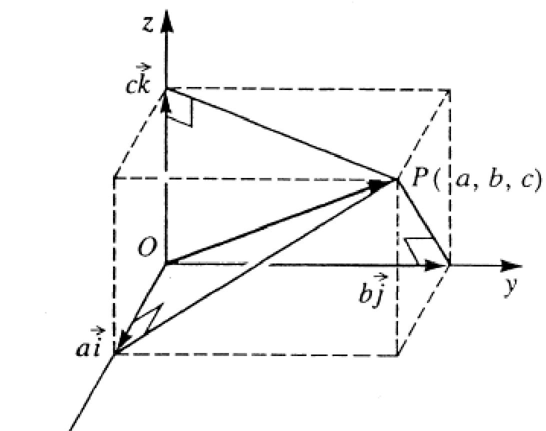
Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ represent ***unit vectors*** in the positive x, y and z directions.

$\hat{i} = (\quad,\quad,\quad)$, $\hat{j} = (\quad,\quad,\quad)$ and $\hat{k} = (\quad,\quad,\quad)$ with $\vec{0} = (\quad,\quad,\quad)$.

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are the ***standard basis vectors*** in $\Re^3$. We can represent $\overrightarrow{OP}$ in terms of the ***standard basis vectors***, where $\overrightarrow{OP}$ = ______________________.

Every vector in $\Re^3$ can be represented ***algebraically*** or ***geometrically***.





The ***direction angles*** of a vector $\overrightarrow{OP} = (a, b, c)$ the angles ___, ___ and ____ that $\overrightarrow{OP}$ makes with the positive x, y and z - axes respectively.

a, b and c are called the ***direction numbers***.

The ***direction cosines*** of a vector $\overrightarrow{OP} = (a, b, c)$ are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

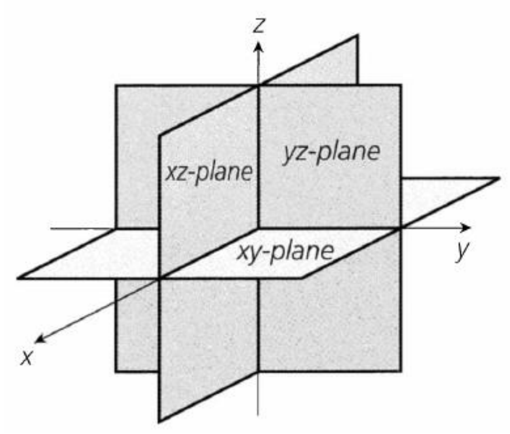
$\cos\alpha = \text{———}$ $\quad\cos\beta = \text{———}$ $\quad\cos\gamma = \text{———}$

Every vector in $\Re^2$ can be represented ***algebraically*** or ***geometrically***.

Algebraic Form	**Geometric Form**
Ordered triple notation: $\overrightarrow{OP} = (a, b, c)$	***Magnitude:*** $\left\|\overrightarrow{OP}\right\|$, where $\left\|\overrightarrow{OP}\right\| = \sqrt{a^2 + b^2 + c^2}$
or	*and*
Unit vector notation: $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$	***Direction angles:*** α , β and γ that $\overrightarrow{OP}$ makes with the positive x, y and z - axes respectively.
Note: $(a\ , b, c) = a\hat{i} + b\hat{j} + c\hat{k}$	

Note: A ***unit vector*** in the direction of $\overrightarrow{OP}$ is:

$\because$ a unit vector has a length of 1,



A ***plane*** in space that contains ***two*** of the coordinate axes is known as a ***coordinate plane***. The plane containing the *x*- and *y*- axes, for instance, is called the *xy*-plane. The other two coordinate planes are named similarly. A point such as $(-4,0,1)$, which has a *y*-coordinate of 0, lies in the *xz*-plane.

To plot a point $P\ (a,b,c)$ in space, move *a* units from the origin in the *x* direction, *b* units in the *y* direction, and then *c* units in the *z* direction. Be sure each move is made along a line parallel to the corresponding axis. Drawing a rectangular box will help you to see the three-dimensional aspect of such diagrams.

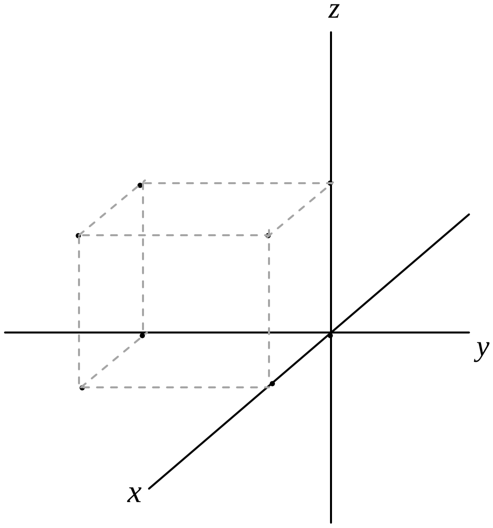
Ex. 3. Given vector $\vec{v}=(2,-5,4)$,

a) graph $\vec{v}$.
b) find the magnitude of $\vec{v}$.
c) find the direction cosines.
d) find the direction angles.
e) find $\hat{v}$

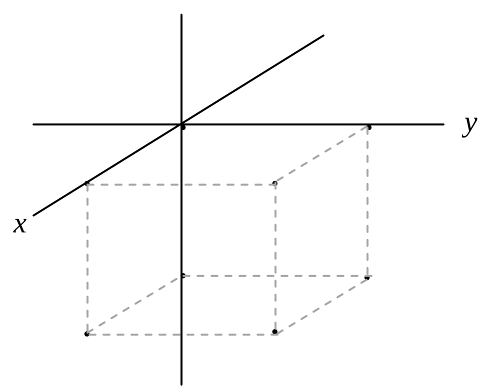
Ex. 4. Given vector $\overrightarrow{OP}=3\hat{i}+5\hat{j}-4\hat{k}$,

a) graph $\overrightarrow{OP}$.
b) find the magnitude of $\overrightarrow{OP}$.
c) find the direction cosines.
d) find the direction angles.
e) find a unit vector in the direction opposite to $\overrightarrow{OP}$.

a)



a)



HW: p. 166 #1 to 5, 6ab, 7bc, 8 to 12, 13acef, 14, 16 to 20, 22, 23 & Worksheet #1 to 4

Date:______________ **WORKSHEET on Direction Cosines and Angles**

1. Find the angle that $(2, 3, -2)$ makes with each of the coordinate axes.

2. Find the direction cosines of $\vec{u} = (a, b, c)$ where $\vec{u}$ is a unit vector.

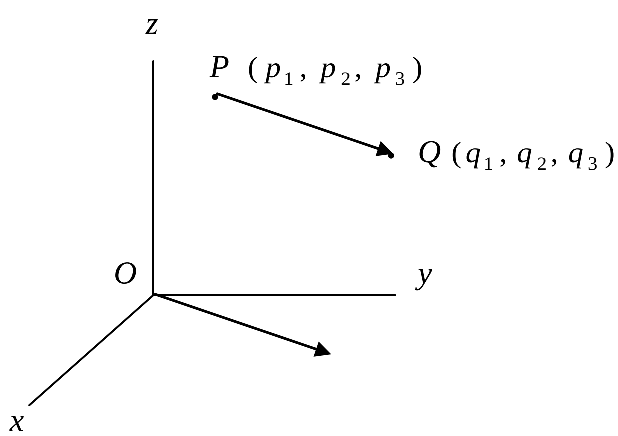
3. The vector $\vec{v} = (1, \sqrt{2}, c)$ makes an angle of 60° with the positive z-axis. Determine the angles that $\vec{v}$ makes with the positive x-axis and the positive y-axis. Explain how many answers there are.

4. Determine the angle that $\vec{j} = (0, 1, 0)$ makes with each of the coordinate axes.

Answers: **1.** 61°, 43°, 119° **2.** a, b, c **3.** 60°, 45°; one answer **4.** 90°, 0°, 90°

Date:______________ **Section 5.2 – Operations With Algebraic Vectors**

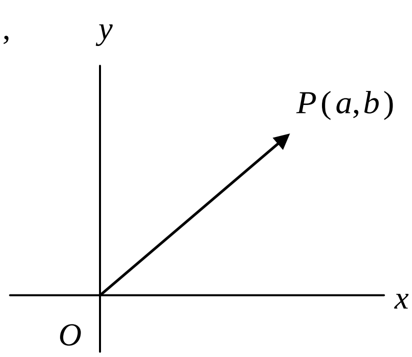
I The Vector Joining Two Points



$\overrightarrow{PQ} =$

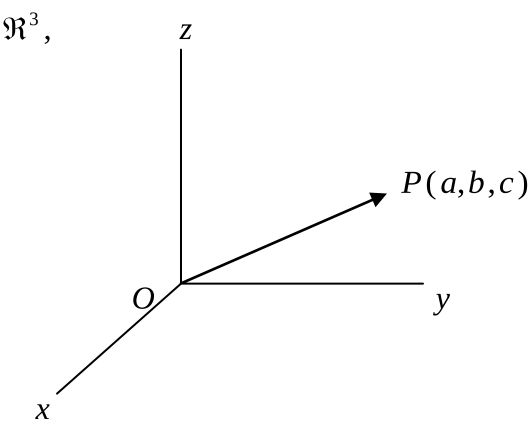
II The Magnitude of a Vector

In $\Re^2$,



$\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2}$

In $\Re^3$,



$\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2 + c^2}$

Ex. 1. Given the points $A(1,1,2)$, $B(2,-1,3)$ and $C(4,1,5)$, find:

a) $\overrightarrow{OA} + \overrightarrow{OB}$

b) $2\overrightarrow{OB} - 3\overrightarrow{OC}$

c) $\overrightarrow{BC}$

d) $\overrightarrow{CA}$

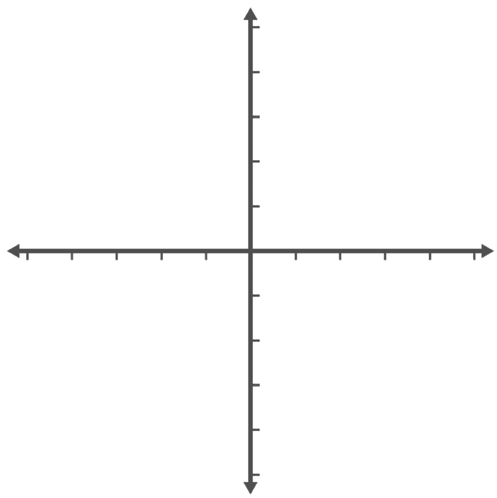
Ex. 2. If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - \hat{k}$, find $\left|\vec{a} - \vec{b}\right|$.

Ex. 3. Given the points $P(4,3,5)$ and $Q(1,-2,5)$, find $\left|\overrightarrow{PQ}\right|$.

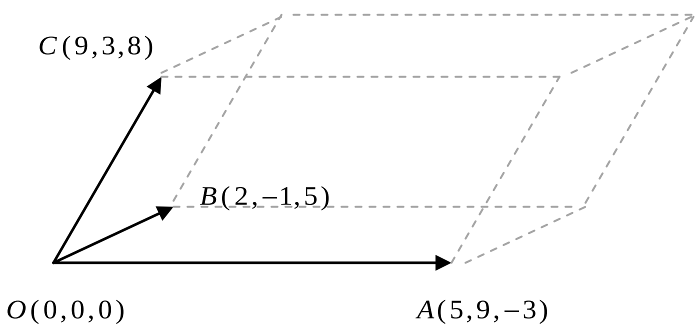
Ex. 4. Given the points $A(5,-1)$, $B(-3,4)$ and $C(13,-6)$, show that A, B and C are ***collinear using vectors***.

Note: A, B and C are ***collinear*** if ______ is a ***scalar multiple*** of _______.

Ex. 5. If quadrilateral *ABCD* is a parallelogram with vertices $A(-5,3)$, $B(5,2)$ and $C(7,-8)$, find the coordinates of *D*, ***using vectors***.



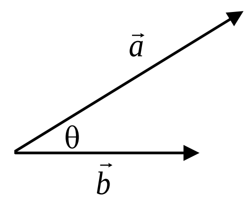
Ex. 6. If $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are three edges of a ***parallelepiped*** where O is $(0,0,0)$, A is $(5,9,-3)$, B is $(2,-1,5)$ and C is $(9,3,8)$. Find the coordinates of the other four vertices, *D*, *E*, *F* and *G*.



HW: p. 172 #2agij, 3ab, 4f, 5d, 6c, 8, 9c, 12, 13ad, 14ab, 15, 16

Date:_____________ **Section 5.3 – The Dot Product of Two Vectors**

A. The Dot Product in Vector Form



The ***dot product*** of any two vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

where θ is the angle between the vectors.

Note: The ***dot product*** of two vectors is a ***scalar***.

Ex. 1. Complete the following.

a) If the angle, θ, between the vectors is ***acute*** then ___ $< \theta <$ ___ and $\vec{a} \cdot \vec{b}$ _______.

b) If the angle, θ, between the vectors is ***obtuse*** then ___ $< \theta <$ _______ and $\vec{a} \cdot \vec{b}$ _______.

c) If the angle, θ, between the vectors is ***right*** then $\theta =$ ____ and $\vec{a} \cdot \vec{b}$ _______.

Ex. 2. If $|\vec{a}| = 5$, $|\vec{b}| = 6$ and $\theta = 60^\circ$, then find $\vec{a} \cdot \vec{b}$.

Ex. 3. If $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, determine each of the following:

a) $\hat{i} \cdot \hat{i}$

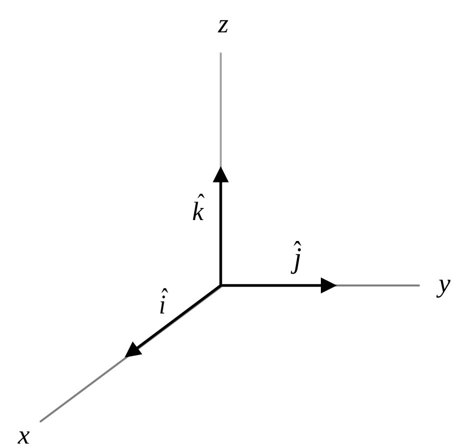
d) $\hat{i} \cdot \hat{j}$

b) $\hat{j} \cdot \hat{j}$

e) $\hat{i} \cdot \hat{k}$

c) $\hat{k} \cdot \hat{k}$

f) $\hat{j} \cdot \hat{k}$



B. The Dot Product in Component Form

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$\vec{a} \cdot \vec{b}$

$= (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$

$= (\qquad\qquad\qquad) \cdot (\qquad\qquad\qquad)$

$= a_1b_1(\hat{i} \cdot \hat{i}) + a_1b_2(\hat{i} \cdot \hat{j}) + a_1b_3(\hat{i} \cdot \hat{k}) + a_2b_1(\hat{j} \cdot \hat{i}) + a_2b_2(\hat{j} \cdot \hat{j}) + a_2b_3(\hat{j} \cdot \hat{k}) + a_3b_1(\hat{k} \cdot \hat{i}) + a_3b_2(\hat{k} \cdot \hat{j}) + a_3b_3(\hat{k} \cdot \hat{k})$

$= a_1b_1(\quad) + a_1b_2(\quad) + a_1b_3(\quad) + a_2b_1(\quad) + a_2b_2(\quad) + a_2b_3(\quad) + a_3b_1(\quad) + a_3b_2(\quad) + a_3b_3(\quad)$

$=$

$\therefore$ the ***dot product*** of any two vectors $\vec{a}$ and $\vec{b}$ in component form is

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Ex. 4. Find $\vec{a} \cdot \vec{b}$ if $\vec{a} = (1, 2, -3)$ and $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$.

Ex. 5. Determine whether or not $\vec{u} = (1, 2, 3)$ and $\vec{v} = (3, -4, -2)$ are ***perpendicular***.

Ex. 6. Find the angle θ between the vectors $\vec{a} = (2, -1, 4)$ and $\vec{b} = (-3, 1, 2)$.

Ex. 7. For what values of p will the vectors of $\vec{a} = (1, p, 2)$ and $\vec{b} = (3, -9, 6)$ be

i) ***collinear***?

ii) ***perpendicular***?

Ex. 8. Find a vector ***perpendicular*** to

i) $(5, -2)$

ii) $(4, -1, 2)$

C. Properties of the Dot Product

1. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ ***Commutative Law*** ***2.** $\vec{a}\cdot\vec{a}=|\vec{a}|^2$

3.** $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ ***Distributive Law **4.** $(k\vec{a})\cdot\vec{b}=\vec{a}\cdot(k\vec{b})=k(\vec{a}\cdot\vec{b})$ ***Associative Law***

***** We will prove these properties of the dot product by example.

Ex. 9. If $\vec{a}=(-2,3,1)$, $\vec{b}=(5,6,-7)$ and $\vec{c}=(3,-2,4)$, verify that

a) $\vec{a}\cdot\vec{a}=|\vec{a}|^2$

b) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

Ex. 10. If $\vec{a}$ and $\vec{b}$ are distinct ***unit vectors*** and the angle between them is 120°, calculate $(2\vec{a}+3\vec{b})\cdot(4\vec{a}-5\vec{b})$.

HW: p. 178 #1, 2a, 4a, 5, 6, 7b, 8d, 11, 12c, 13, 14ac, 15-18, 20
REVIEW: p. 194 #1, 2, 3, 5 to 11

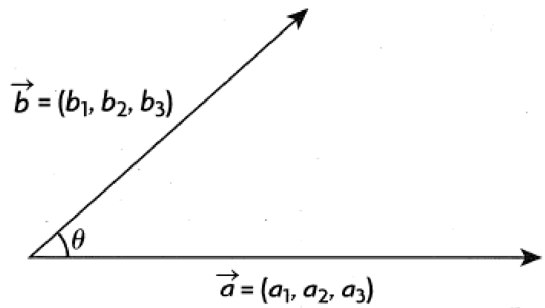
Date:______________

Section 5.4 – The Cross Product of Two Vectors

Recall the Dot Product:

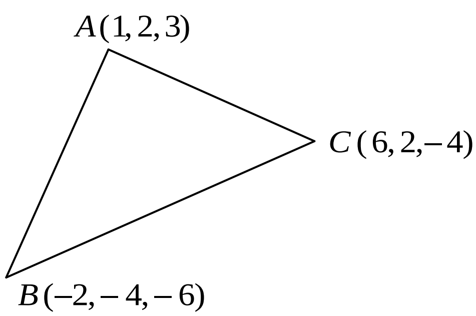
When two vectors are placed tail to tail, as shown,

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$
- $\cos \theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

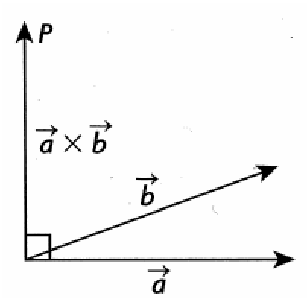


Warm-up

A triangle has vertices $A(1,2,3)$, $B(-2,-4,-6)$ and $C(6,2,-4)$.
Find the measure of $< A$ using vectors.



I. Cross Product



The **cross product**, $\vec{a} \times \vec{b}$, of two vectors $\vec{a}$ and $\vec{b}$ in $\Re^3$ is a ***vector*** that is ***perpendicular*** to both $\vec{a}$ and $\vec{b}$.

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{v} = (x, y, z)$ be any vector perpendicular to **both** $\vec{a}$ and $\vec{b}$.

So, $\vec{a} \cdot \vec{v} = $ ____ and $\vec{b} \cdot \vec{v} = $ ____

$(a_1, a_2, a_3) \cdot (x, y, z) = 0$ ① $(b_1, b_2, b_3) \cdot (x, y, z) = 0$ ②

Solve for *x*, *y* and *z*.

$$a_1 x + a_2 y + a_3 z = 0 \quad ①$$
$$b_1 x + b_2 y + b_3 z = 0 \quad ②$$

Eliminate *z* Eliminate *y*

①$\times b_3$ $a_1 b_3 x + a_2 b_3 y + a_3 b_3 z = 0$

②$\times a_3$ $a_3 b_1 x + a_3 b_2 y + a_3 b_3 z = 0$

Subtract $(a_1 b_3 - a_3 b_1)x + (a_2 b_3 - a_3 b_2)y = 0$

$$(a_2 b_3 - a_3 b_2)y = (a_3 b_1 - a_1 b_3)x$$

$$\frac{y}{(a_3 b_1 - a_1 b_3)} = \frac{x}{(a_2 b_3 - a_3 b_2)}$$

Ex. 1. Find a vector perpendicular to both $\vec{a} = (-3, 5, 1)$ and $\vec{b} = (2, -1, 7)$.

Ex. 2. Find a ***unit vector*** perpendicular to both $\vec{a} = (3, 4, -1)$ and $\vec{b} = (2, -1, 3)$.

II. Prove: $\left|\vec{a}\times\vec{b}\right| = |\vec{a}||\vec{b}|\sin\theta$

Proof:

Let $\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

The formula for ***cross product*** is

$\vec{a}\times\vec{b} =$

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}$$

$$\left|\vec{a}\times\vec{b}\right|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

The right-hand side is expanded and then factored to give

$$\left|\vec{a}\times\vec{b}\right|^2 = (a_1{}^2 + a_2{}^2 + a_3{}^2)(b_1{}^2 + b_2{}^2 + b_3{}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$\because |\vec{a}|^2 =$, $|\vec{b}|^2 =$ and $\vec{a}\cdot\vec{b} =$

$\therefore \left|\vec{a}\times\vec{b}\right|^2 =$

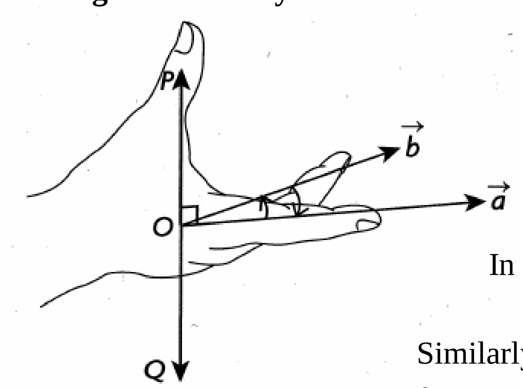
III. Properties of Cross Product

1. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ ***Anti-commutative Law***
2. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ ***Distributive Law***
3. $k(\vec{a}\times\vec{b}) = (k\vec{a})\times\vec{b} = \vec{a}\times(k\vec{b}),\ k\in\Re$ ***Associative Law***

SUMMARY:

- The ***cross product*** $\vec{a} \times \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ in $\Re^3$ is the vector that is perpendicular to both $\vec{a}$ and $\vec{b}$.
- $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$
- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$
- Vectors $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ form a ***right-handed system*** where $\vec{a} \times \vec{b}$ points in the opposite direction of $\vec{b} \times \vec{a}$.

IV. Right-handed System



$\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are two vectors perpendicular to $\vec{a}$ and $\vec{b}$.

The direction of $\vec{a} \times \vec{b}$ can be found by placing the extended fingers of your **right hand** on $\vec{a}$ and curling them towards $\vec{b}$ through an angle less than 180°. Your thumb points in the direction of $\vec{a} \times \vec{b}$.

In this case $\vec{a} \times \vec{b} =$ _____ and is directed _____________ the page.

Similarly, the direction of $\vec{b} \times \vec{a}$ can be found by placing the extended fingers of your **right hand** on $\vec{b}$ and curling them towards $\vec{a}$ through an angle less than 180°. Your thumb points in the direction of $\vec{b} \times \vec{a}$.

In this case $\vec{b} \times \vec{a} =$ _____ and is directed _____________ the page.

Ex. 3. If $|\vec{a}| = 4$, $|\vec{b}| = 10$ and the angle between $\vec{a}$ and $\vec{b}$ is 60°, find the ***exact*** value of $|\vec{a} \times \vec{b}|$.

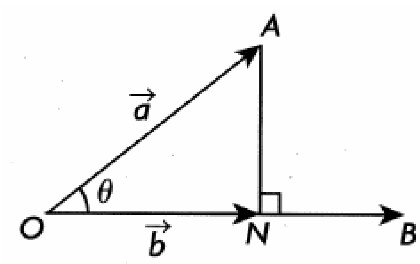
Ex. 4. If $|\vec{u}| = 3$, $|\vec{v}| = 6$ and the angle between $\vec{u}$ and $\vec{v}$ is 330°, find the ***exact*** value of $|\vec{u} \times \vec{v}|$.

HW: p. 185 #2 to 7, 15b

Date:______________ **Section 5.5 – Applications of Dot and Cross Products**

I. Projections: A *projection* is formed by dropping a perpendicular from an object onto a line or plane. The shadow of an object is a physical example of a projection.

The projection of one vector onto another can be pictured below.



The **Vector Projection** of $\vec{a}$ onto $\vec{b}$ is the vector $\overrightarrow{ON}$

and

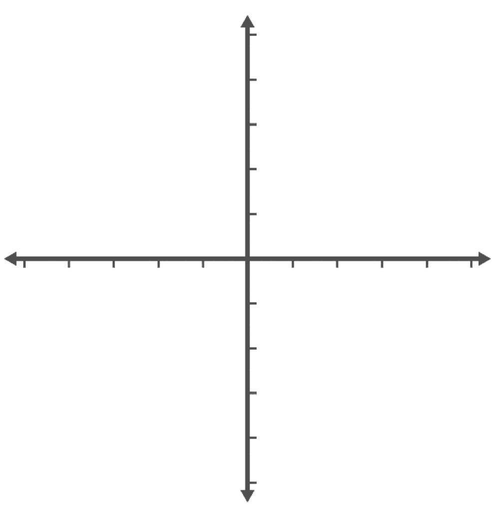
the **Scalar Projection** of $\vec{a}$ onto $\vec{b}$ is the *signed magnitude* of the vector projection $\overrightarrow{ON}$.

We will develop formulas for each type of projection.

i) Scalar Projection of $\vec{a}$ onto $\vec{b}$

ii) Vector Projection of $\vec{a}$ onto $\vec{b}$

Ex. 1. Find the scalar and vector projections of $\vec{a} = (4,3)$ onto $\vec{b} = (-4,1)$.



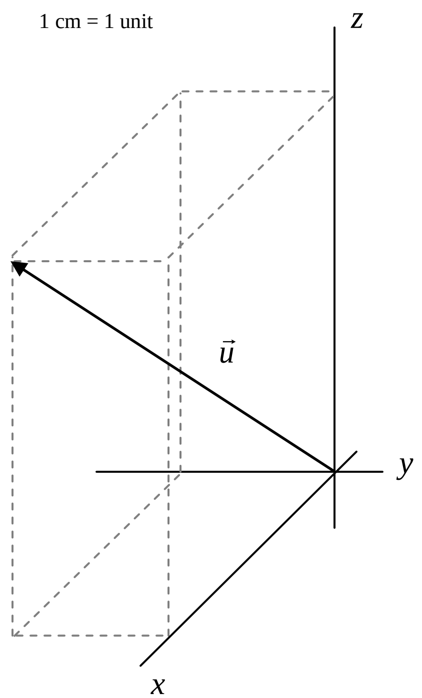
SUMMARY OF PROJECTIONS:

Scalar Projections	**Vector Projections**	**Magnitudes**
$SP_{\vec{a}\,on\,\vec{b}} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{b}\rvert}$	$VP_{\vec{a}\,on\,\vec{b}} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{b}\rvert}\hat{b}$ or $VP_{\vec{a}\,on\,\vec{b}} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{b}\rvert^2}\vec{b}$	$\left\lvert SP_{\vec{a}\,on\,\vec{b}}\right\rvert = \dfrac{\lvert\vec{a}\cdot\vec{b}\rvert}{\lvert\vec{b}\rvert}$
$SP_{\vec{b}\,on\,\vec{a}} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert}$	$VP_{\vec{b}\,on\,\vec{a}} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert}\hat{a}$ or $VP_{\vec{b}\,on\,\vec{a}} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert^2}\vec{a}$	$\left\lvert SP_{\vec{b}\,on\,\vec{a}}\right\rvert = \dfrac{\lvert\vec{a}\cdot\vec{b}\rvert}{\lvert\vec{a}\rvert}$

Ex. 2. Given $\vec{a} = (1,6,3)$ and $\vec{b} = (1,4,5)$, find the vector projection of $\vec{b}$ onto $\vec{a}$, and its magnitude.

Ex. 3. Graph $\vec{u} = (3,-2,5)$ and find the vector projections of $\vec{u}$ onto each of the coordinate axes and coordinate planes.

1 cm = 1 unit



i) $VP_{\vec{u}\text{ on }x-axis} =$

ii) $VP_{\vec{u}\text{ on }y-axis} =$

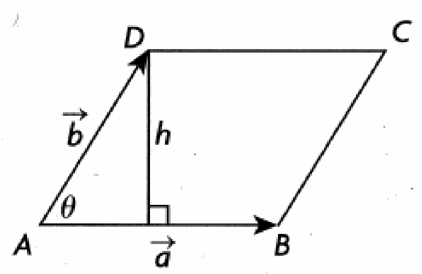
iii) $VP_{\vec{u}\text{ on }z-axis} =$

iv) $VP_{\vec{u}\text{ on }xy-plane} =$

v) $VP_{\vec{u}\text{ on }xz-plane} =$

vi) $VP_{\vec{u}\text{ on }yz-plane} =$

II. Area of a Parallelogram



$Area = base \times height$

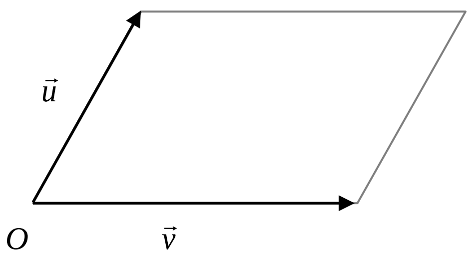
$= |\vec{a}|h$

Find *h*.

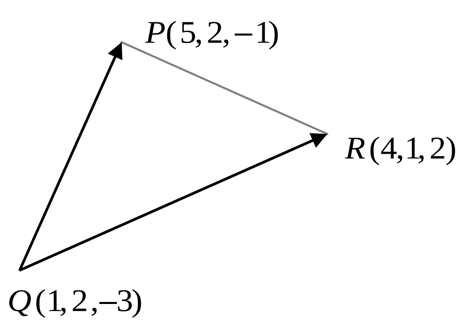
$\frac{h}{|\vec{b}|} = \sin\theta$, so $h = |\vec{b}|\sin\theta$

$\therefore A = |\vec{a}||\vec{b}|\sin\theta$ or $A = |\vec{a} \times \vec{b}|$

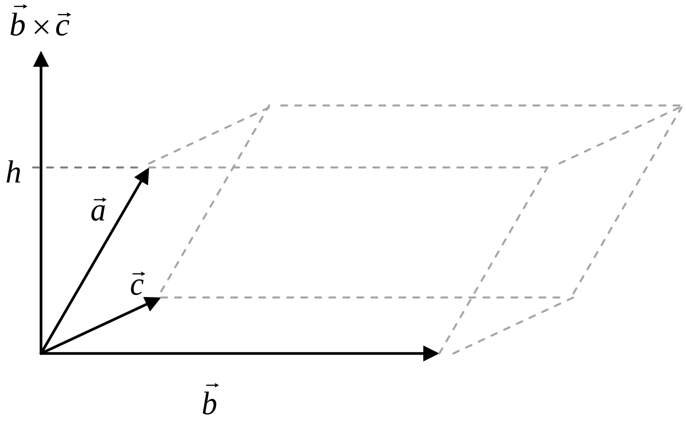
Ex. 4. Calculate the *exact* area of the parallelogram with sides $\vec{u} = (-6, 4, 5)$ and $\vec{v} = (9, -6, 2)$.



Ex. 5. Find the *exact* area of the triangle with vertices $P(5, 2, -1)$, $Q(1, 2, -3)$ and $R(4, 1, 2)$,



III. Volume of a Parallelepiped



$$Volume = Area_{base} \times height$$
$$= A_{parallelogram} \times h$$
$$= \left|\vec{b}\times\vec{c}\right| h$$

Find h, where h is the ***magnitude*** of the ***vector projection*** of $\vec{a}$ onto $\vec{b}\times\vec{c}$.

So, $h = \dfrac{\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|}{\left|\vec{b}\times\vec{c}\right|}$

$$\therefore V = \left|\vec{b}\times\vec{c}\right| \frac{\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|}{\left|\vec{b}\times\vec{c}\right|} \quad or$$

$$V = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$$

SUMMARY OF GEOMETRIC APPLICATIONS:

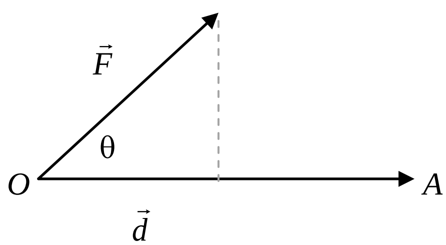
Parallelogram	**Triangle**	**Parallelepiped**
$A = \left\|\vec{a}\times\vec{b}\right\|$	$A = \frac{1}{2}\left\|\vec{a}\times\vec{b}\right\|$	$V = \left\|\vec{a}\cdot(\vec{b}\times\vec{c})\right\|$

Note: ***Vectors*** $\vec{a}$**,** $\vec{b}$ ***and*** $\vec{c}$ ***must be drawn tail to tail.***

HW: p. 192 #1-8; p. 194 #4, 12-14

Date:______________ **Section 5.5 – More Applications of Dot and Cross Products**

IV. Work: In everyday life, the word *work* is applied to any form of activity that requires physical exertion or mental effort.
In physics, **work** is done whenever a force acting on an object causes a displacement of the object from one position to another.



Suppose a force $\vec{F}$ moves an object from *O* to *A*.

$\vec{F}$ is the force acting on an object measured in newtons (N)
$\vec{d}$ is the displacement caused by the force, measured in metres (m)
θ is the angle between the force and the displacement
W is the work done, measured in newton-metres, or joules (J)

Work is defined as the product of the distance an object has been displaced and the component of the force along the line of displacement. ***Work** is a **scalar quantity***.

$$W = |\vec{d}| \, SP_{\vec{F} \, on \, \vec{d}}$$

$$W = |\vec{d}| \frac{\vec{F} \cdot \vec{d}}{|\vec{d}|}$$

$$\therefore W = \vec{F} \cdot \vec{d} \quad or \quad W = |\vec{F}||\vec{d}|\cos\theta$$

SUMMARY OF WORK: The **work** done by a force is defined as the dot product.

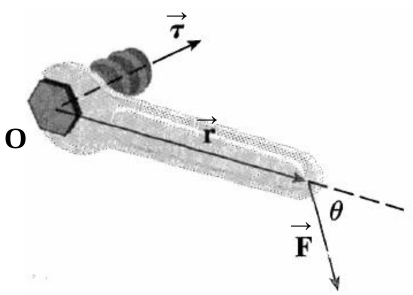
algebraic form	*or*	**geometric form**
$W = \vec{F} \cdot \vec{d}$		$W = |\vec{F}||\vec{d}|\cos\theta$

Ex. 1. A crate on a ramp is hauled 8 m up the ramp under a constant force of 15 N, applied at an angle of 30° to the ramp. Find the *exact* work done.

Ex. 2. Find the *exact* work done by a 7-N force in moving an object from *A* (3, 2) to *B* (7, 5) when the force acts at an angle of 30° to $\overrightarrow{AB}$. The distance is in metres.

Ex. 3. Find the work done by a 24-N force in the direction of $\vec{v} = (1, 2, 2)$ when it moves an object from $A(2,-4,1)$ to $B(10,3,-1)$. The distance is in metres.

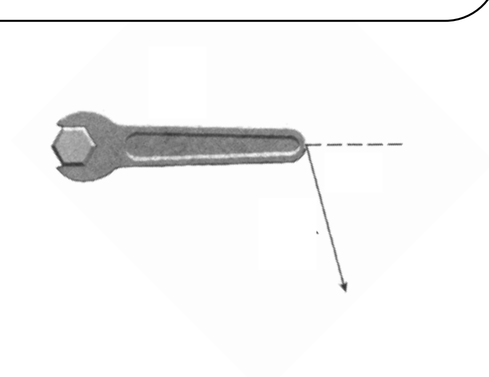
V. Torque: Sometimes instead of a force causing a change in position, a force causes an object to turn about a point or an axis. Examples are tightening a bolt using a wrench or applying a force to a bicycle pedal to make the crank arm rotate.
This turning effect of a force is called ***torque***. ***Torque*** is a ***vector*** quantity.



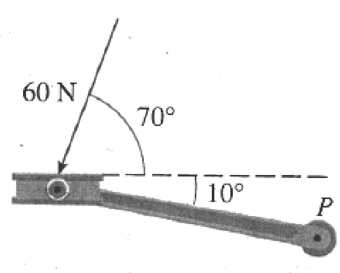
The ***torque*** caused by a force is defined as the cross product $\vec{T} = \vec{r} \times \vec{F}$ and its magnitude is $|\vec{T}| = |\vec{r} \times \vec{F}|$ or $|\vec{T}| = |\vec{r}||\vec{F}|\sin\theta$.

$\vec{F}$ is the applied force, $\vec{r}$ is the vector determined by the lever arm acting from the axis of rotation and θ is the angle between the force and lever arm.
Note: The magnitude of torque is measured in N-m or J.

Ex. 4. A 40-N force is applied to the end of a 25 cm wrench with which it makes an angle of 105°. Calculate the magnitude of the torque about the centre of the bolt.



Ex. 5. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of torque about *P*.



HW: p. 193 #9bc, 10 to 18 ; REVIEW for TEST: p. 197 #1 to 6;