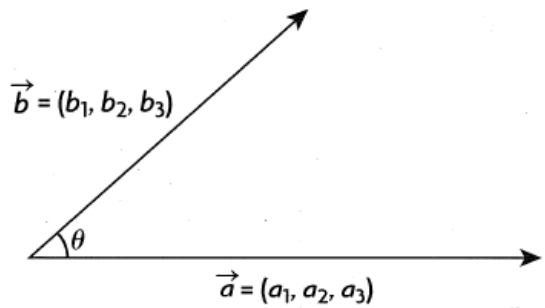
**Date:** May 20/14 **Section 5.4 – The Cross Product of Two Vectors**

***Recall the Dot Product:***

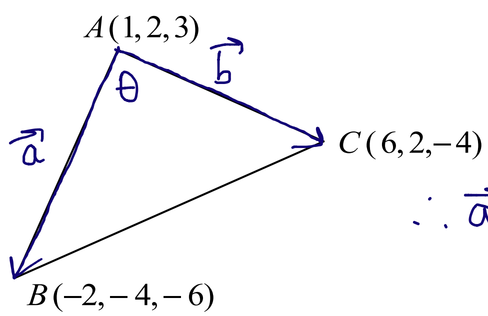
When two vectors are placed tail to tail, as shown,

- $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$




**Warm-up**

A triangle has vertices $A(1,2,3)$, $B(-2,-4,-6)$ and $C(6,2,-4)$.
Find the measure of $< A$ using vectors.




Let $\vec{a} = \vec{AB}$
$= \vec{OB} - \vec{OA}$
$= (-2,-4,-6) - (1,2,3)$
$\therefore \vec{a} = (-3,-6,-9)$

Let $\vec{b} = \vec{AC}$
$= \vec{OC} - \vec{OA}$
$= (6,2,-4) - (1,2,3)$
$\therefore \vec{b} = (5,0,-7)$

Let $\theta = < A$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{(-3,-6,-9)\cdot(5,0,-7)}{(\sqrt{9+36+81})(\sqrt{25+0+49})}$$

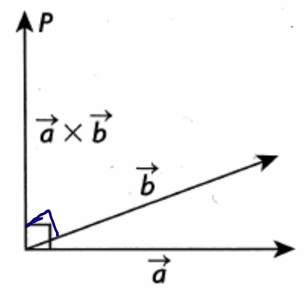
$$\cos\theta = \frac{-15+0+63}{(\sqrt{126})(\sqrt{74})}$$

$$\cos\theta = \frac{48}{(\sqrt{126})(\sqrt{74})}$$

$$\theta \doteq 60^\circ$$

$$\therefore < A \doteq 60^\circ$$

## I. Cross Product




The **cross product**, $\vec{a}\times\vec{b}$, of two vectors $\vec{a}$ and $\vec{b}$ in $\Re^3$ is a ***vector*** that is ***perpendicular*** to both $\vec{a}$ and $\vec{b}$.

Let $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$ and $\vec{v}=(x,y,z)$ be any vector perpendicular to **both** $\vec{a}$ and $\vec{b}$.

So, $\vec{a}\cdot\vec{v}=$ <u>0</u> and $\vec{b}\cdot\vec{v}=$ <u>0</u>

$(a_1,a_2,a_3)\cdot(x,y,z)=0$ ① $\qquad (b_1,b_2,b_3)\cdot(x,y,z)=0$ ②

Solve for *x*, *y* and *z*.

$a_1x+a_2y+a_3z=0$ ①

$b_1x+b_2y+b_3z=0$ ②

Eliminate *z*

① $\times b_3$ $\quad a_1b_3x+a_2b_3y+a_3b_3z=0$

② $\times a_3$ $\quad a_3b_1x+a_3b_2y+a_3b_3z=0$

Subtract $\quad (a_1b_3-a_3b_1)x+(a_2b_3-a_3b_2)y=0$

$(a_2b_3-a_3b_2)y=(a_3b_1-a_1b_3)x$

$$\frac{y}{(a_3b_1-a_1b_3)}=\frac{x}{(a_2b_3-a_3b_2)}$$

Eliminate *y*

① $\times b_2$ $\quad a_1b_2x+a_2b_2y+a_3b_2z=0$

② $\times a_2$ $\quad a_2b_1x+a_2b_2y+a_2b_3z=0$

Subtract $\quad (a_1b_2-a_2b_1)x+(a_3b_2-a_2b_3)z=0$

$(a_1b_2-a_2b_1)x=(a_2b_3-a_3b_2)z$

$$\frac{x}{(a_2b_3-a_3b_2)}=\frac{z}{(a_1b_2-a_2b_1)}$$

Let $\dfrac{x}{(a_2b_3-a_3b_2)}=\dfrac{y}{(a_3b_1-a_1b_3)}=\dfrac{z}{(a_1b_2-a_2b_1)}=k$

$\therefore x=k(a_2b_3-a_3b_2);\ y=k(a_3b_1-a_1b_3);\ z=k(a_1b_2-a_2b_1)$

*If $k=1$

$\vec{v}=\vec{a}\times\vec{b}$

$(x,y,z)=(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)$

$$\begin{matrix} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{matrix}$$

**Ex. 1.** Find a vector perpendicular to both $\vec{a}=(-3,5,1)$ and $\vec{b}=(2,-1,7)$.

$\vec{a}\times\vec{b}$

$=((5)(7)-(1)(-1),\ (1)(2)-(-3)(7),\ (-3)(-1)-5(2))$

$=(35+1,\ 2+21,\ 3-10)$

$=(36,23,-7)$

$$\begin{matrix} a_2 & a_3 & a_1 & a_2 \\ 5 & 1 & -3 & 5 \\ -1 & 7 & 2 & -1 \\ b_2 & b_3 & b_1 & b_2 \end{matrix}$$

**Ex. 2.** Find a ***unit vector*** perpendicular to both $\vec{a}=(3,4,-1)$ and $\vec{b}=(2,-1,3)$.

$$\begin{matrix} 4 & -1 & 3 & 4 \\ -1 & 3 & 2 & -1 \end{matrix}$$

$\vec{a}\times\vec{b}$

$=(12-1,\ -2-9,\ -3-8)$

$=(11,-11,-11)$

Let $\vec{v}=(1,-1,-1)$

$\hat{v}=\dfrac{1}{|\vec{v}|}\vec{v}$

$\hat{v}=\dfrac{1}{\sqrt{3}}(1,-1,-1)$

$\therefore \hat{v}=\left(\dfrac{1}{\sqrt{3}},\dfrac{-1}{\sqrt{3}},\dfrac{-1}{\sqrt{3}}\right)$

**II. Prove:** $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

**Proof:**

Let $\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$$\begin{array}{cccc} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{array}$$

The formula for ***cross product*** is

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

$$|\vec{a}\times\vec{b}| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}$$

$$|\vec{a}\times\vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

The right-hand side is expanded and then factored to give

$$|\vec{a}\times\vec{b}|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - \underbrace{(a_1b_1 + a_2b_2 + a_3b_3)^2}$$

$\because |\vec{a}|^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = a_1^2 + a_2^2 + a_3^2$, $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$ and $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\therefore |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2$$

$$= \underbrace{|\vec{a}|^2|\vec{b}|^2} - \underbrace{|\vec{a}|^2|\vec{b}|^2}\cos^2\theta$$

$$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta)$$

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

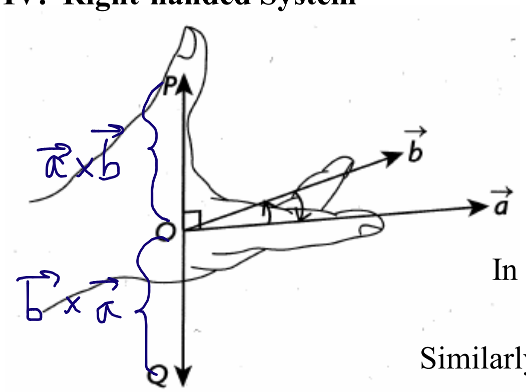
**III. Properties of Cross Product**

1. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ ***Anti-commutative Law***
2. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ ***Distributive Law***
3. $k(\vec{a}\times\vec{b}) = (k\vec{a})\times\vec{b} = \vec{a}\times(k\vec{b}),\ k\in\Re$ ***Associative Law***

*SUMMARY:*

- The ***cross product*** $\vec{a}\times\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ in $\Re^3$ is the vector that is perpendicular to both $\vec{a}$ and $\vec{b}$.
- $\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$
- $\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$
- Vectors $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ form a ***right-handed system*** where $\vec{a}\times\vec{b}$ points in the opposite direction of $\vec{b}\times\vec{a}$.

**IV. Right-handed System**




$\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are two vectors perpendicular to $\vec{a}$ and $\vec{b}$.

The direction of $\vec{a}\times\vec{b}$ can be found by placing the extended fingers of your **right hand** on $\vec{a}$ and curling them towards $\vec{b}$ through an angle less than 180°. Your thumb points in the direction of $\vec{a}\times\vec{b}$.

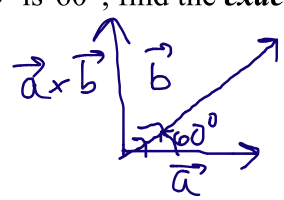
In this case $\vec{a}\times\vec{b} =$ $\overrightarrow{OP}$ and is directed out of the page.

Similarly, the direction of $\vec{b}\times\vec{a}$ can be found by placing the extended fingers of your **right hand** on $\vec{b}$ and curling them towards $\vec{a}$ through an angle less than 180°. Your thumb points in the direction of $\vec{b}\times\vec{a}$.

In this case $\vec{b}\times\vec{a} =$ $\overrightarrow{OQ}$ and is directed into the page.

**Ex. 3.** If $|\vec{a}| = 4$, $\left|\vec{b}\right| = 10$ and the angle between $\vec{a}$ and $\vec{b}$ is 60°, find the ***exact*** value of $\left|\vec{a}\times\vec{b}\right|$.

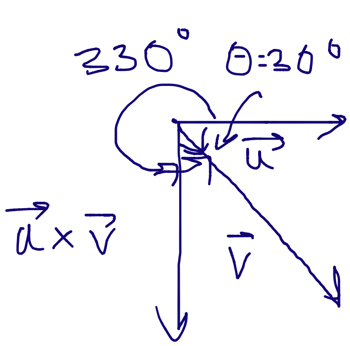
$$\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$$
$$= (4)(10)\sin 60^\circ$$
$$= 40\left(\frac{\sqrt{3}}{2}\right)$$
$$= 20\sqrt{3}$$
$$\therefore \left|\vec{a}\times\vec{b}\right| = 20\sqrt{3} \text{ units}$$




$\vec{a}\times\vec{b}$ is directed out of the page.

**Ex. 4.** If $|\vec{u}| = 3$, $|\vec{v}| = 6$ and the angle between $\vec{u}$ and $\vec{v}$ is 330°, find the ***exact*** value of $|\vec{u}\times\vec{v}|$.

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$
$$= (3)(6)\sin 30^\circ$$
$$= 18\left(\frac{1}{2}\right)$$
$$= 9$$
$$\therefore |\vec{u}\times\vec{v}| = 9 \text{ units}$$




$\vec{u}\times\vec{v}$ is directed into the page.

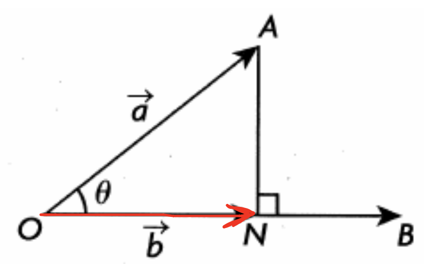
**HW. pg. 185 #2 to 7, 15b**

**Date:** May 21/14

**Section 5.5 – Applications of Dot and Cross Products**

**I. Projections:** A *projection* is formed by dropping a perpendicular from an object onto a line or plane. The shadow of an object is a physical example of a projection.

The projection of one vector onto another can be pictured below.




$\overrightarrow{OA}$ $\overrightarrow{OB}$

The **Vector Projection** of $\vec a$ onto $\vec b$ is the vector $\overrightarrow{ON}$

and

the **Scalar Projection** of $\vec a$ onto $\vec b$ is the *signed magnitude* of the vector projection $\overrightarrow{ON}$.

We will develop formulas for each type of projection.

**i) Scalar Projection** of $\vec a$ onto $\vec b$

Find $|\overrightarrow{ON}|$

$$\frac{|\overrightarrow{ON}|}{|\vec a|} = \cos\theta$$

$$|\overrightarrow{ON}| = |\vec a|\cos\theta \cdot \frac{|\vec b|}{|\vec b|}$$

$$|\overrightarrow{ON}| = \frac{|\vec a||\vec b|\cos\theta}{|\vec b|}$$

$$\therefore SP_{\vec a \text{ on } \vec b} = \frac{\vec a \cdot \vec b}{|\vec b|}$$

**ii) Vector Projection** of $\vec a$ onto $\vec b$

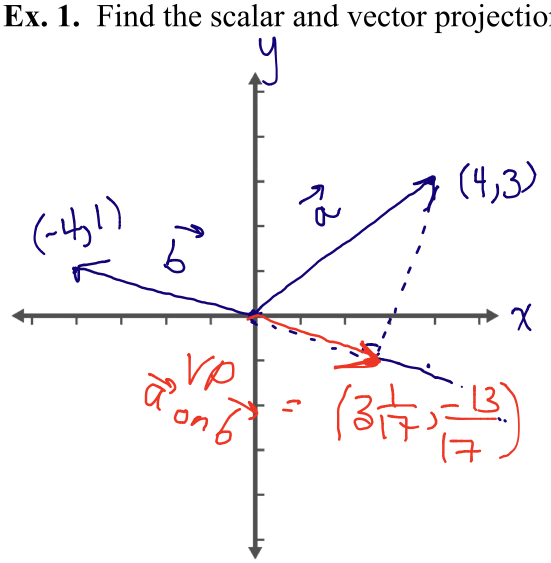
$$\overrightarrow{ON} = |\overrightarrow{ON}|\,\hat b$$

$$= \frac{\vec a \cdot \vec b}{|\vec b|}\,\hat b$$

$$= \underbrace{\left(\frac{\vec a \cdot \vec b}{|\vec b|}\right)\left(\frac{1}{|\vec b|}\right.}\left.\vec b\right)$$

$$\therefore VP_{\vec a \text{ on } \vec b} = \frac{\vec a \cdot \vec b}{|\vec b|^2}\,\vec b$$

**Ex. 1.** Find the scalar and vector projections of $\vec a = (4,3)$ onto $\vec b = (-4,1)$.




$$SP_{\vec a \text{ on } \vec b} = \frac{\vec a \cdot \vec b}{|\vec b|}$$

$$= \frac{(4,3)\cdot(-4,1)}{\sqrt{(-4)^2+(1)^2}}$$

$$= \frac{-16+3}{\sqrt{17}}$$

$$= \frac{-13}{\sqrt{17}}$$

$$VP_{\vec a \text{ on } \vec b} = \frac{\vec a \cdot \vec b}{|\vec b|^2}\,\vec b$$

$$= \frac{-13}{17}(-4,1)$$

$$= \left(\frac{52}{17}, \frac{-13}{17}\right)$$

$$= \left(3\frac{1}{17}, \frac{-13}{17}\right)$$

Note: the magnitude is $+\frac{13}{\sqrt{17}}$.

***SUMMARY OF PROJECTIONS:***

| **Scalar Projections** | **Vector Projections** | **Magnitudes** |
|---|---|---|
| $\underset{\vec{a}\,on\,\vec{b}}{SP} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{b}\rvert}$ | $\underset{\vec{a}\,on\,\vec{b}}{VP} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{b}\rvert}\hat{b}$ or $\underset{\vec{a}\,on\,\vec{b}}{VP} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{b}\rvert^2}\vec{b}$ | $\left\lvert\underset{\vec{a}\,on\,\vec{b}}{SP}\right\rvert = \dfrac{\lvert\vec{a}\cdot\vec{b}\rvert}{\lvert\vec{b}\rvert}$ |
| $\underset{\vec{b}\,on\,\vec{a}}{SP} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert}$ | $\underset{\vec{b}\,on\,\vec{a}}{VP} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert}\hat{a}$ or $\boxed{\underset{\vec{b}\,on\,\vec{a}}{VP} = \dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert^2}\vec{a}}$ * | $\left\lvert\underset{\vec{b}\,on\,\vec{a}}{SP}\right\rvert = \dfrac{\lvert\vec{a}\cdot\vec{b}\rvert}{\lvert\vec{a}\rvert}$ |

absolute value of a scalar

magnitude of a vector.

**Ex. 2.** Given $\vec{a} = (1,6,3)$ and $\vec{b} = (1,4,5)$, find the vector projection of $\vec{b}$ onto $\vec{a}$, and its magnitude.

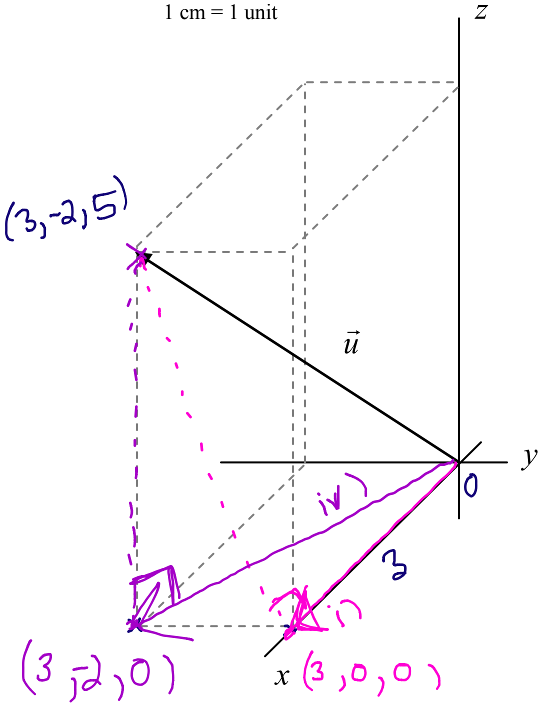
$$\underset{\vec{b}\,on\,\vec{a}}{VP} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$
$$= \frac{1+24+15}{1+36+9}(1,6,3)$$
$$= \frac{40}{46}(1,6,3)$$
$$= \frac{20}{23}(1,6,3)$$
$$= \left(\frac{20}{23}, \frac{120}{23}, \frac{60}{23}\right)$$

$$\left|\underset{\vec{b}\,on\,\vec{a}}{SP}\right| = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}|}$$
$$= \frac{|40|}{\sqrt{46}}$$
$$= \frac{40}{\sqrt{46}}\cdot\frac{\sqrt{46}}{\sqrt{46}}$$
$$= \frac{40\sqrt{46}}{46}$$
$$= \frac{20\sqrt{46}}{23}$$

$\therefore$ the vector projection of $\vec{b}$ onto $\vec{a}$ is $\left(\frac{20}{23}, \frac{120}{23}, \frac{60}{23}\right)$ and its magnitude is $\frac{20\sqrt{46}}{23}$ units.

**Ex. 3.** Graph $\vec{u} = (3,-2,5)$ and find the vector projections of $\vec{u}$ onto each of the coordinate axes and coordinate planes.

1 cm = 1 unit




i) $\underset{\vec{u}\text{ on }x\text{-axis}}{VP} = (3,0,0)$

ii) $\underset{\vec{u}\text{ on }y\text{-axis}}{VP} = (0,-2,0)$

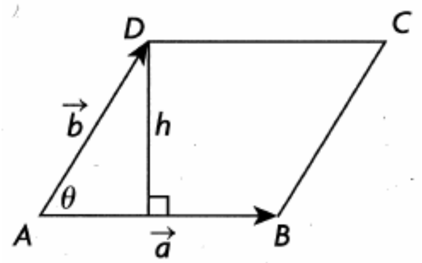
iii) $\underset{\vec{u}\text{ on }z\text{-axis}}{VP} = (0,0,5)$

iv) $\underset{\vec{u}\text{ on }xy\text{-plane}}{VP} = (3,-2,0)$

v) $\underset{\vec{u}\text{ on }xz\text{-plane}}{VP} = (3,0,5)$

vi) $\underset{\vec{u}\text{ on }yz\text{-plane}}{VP} = (0,-2,5)$

## II. Area of a Parallelogram




$Area = base \times height$

$= |\vec{a}|h$

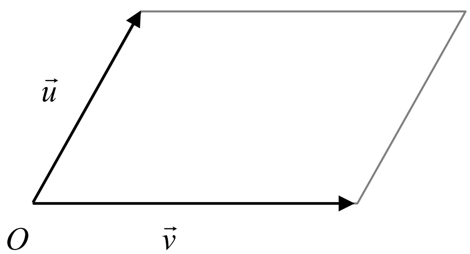
Find $h$.

$\dfrac{h}{|\vec{b}|} = \sin\theta$, so $h = |\vec{b}|\sin\theta$

magnitude of vector $\vec{a} \times \vec{b}$

$\therefore A = |\vec{a}||\vec{b}|\sin\theta$ or $A = |\vec{a} \times \vec{b}|$

**Ex. 4.** Calculate the *exact* area of the parallelogram with sides $\vec{u} = (-6,4,5)$ and $\vec{v} = (9,-6,2)$.




$$\begin{array}{cccccccc} 4 & & 5 & & -6 & & 4 \\ & \times & & \times & & \times & \\ -6 & & 2 & & 9 & & -6 \end{array}$$

$A = |\vec{u} \times \vec{v}|$

$= |(8+30, 45+12, 36-36)|$

$= |(38, 57, 0)|$

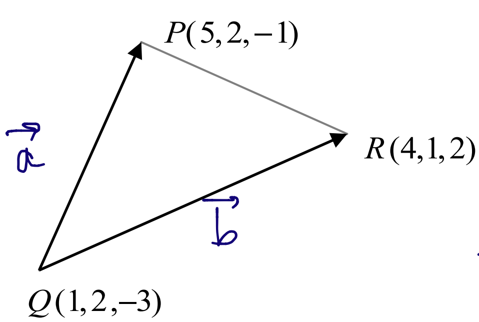
$= \sqrt{38^2 + 57^2 + 0^2}$

$= \sqrt{4693}$

$= 19\sqrt{13}$

$\therefore$ the exact area is $19\sqrt{13}$ units$^2$

**Ex. 5.** Find the *exact* area of the triangle with vertices $P(5,2,-1)$, $Q(1,2,-3)$ and $R(4,1,2)$,




Let $\vec{a} = \vec{QP}$

$= \vec{OP} - \vec{OQ}$

$= (5,2,-1) - (1,2,-3)$

$\therefore \vec{a} = (4,0,2)$

Let $\vec{b} = \vec{QR}$

$= \vec{OR} - \vec{OQ}$

$= (4,1,2) - (1,2,-3)$

$\therefore \vec{b} = (3,-1,5)$

$A_{triangle} = \frac{1}{2} A_{parallelogram}$

$A = \frac{1}{2}|\vec{a} \times \vec{b}|$

$$\begin{array}{cccccccc} 0 & & 2 & & 4 & & 0 \\ & \times & & \times & & \times & \\ -1 & & 5 & & 3 & & -1 \end{array}$$

$A = \frac{1}{2}|(0+2, 6-20, -4-0)|$

$A = \frac{1}{2}|(2,-14,-4)|$

$A = \frac{1}{2}\sqrt{(2)^2 + (-14)^2 + (-4)^2}$

$A = \frac{1}{2}\sqrt{4+196+16}$

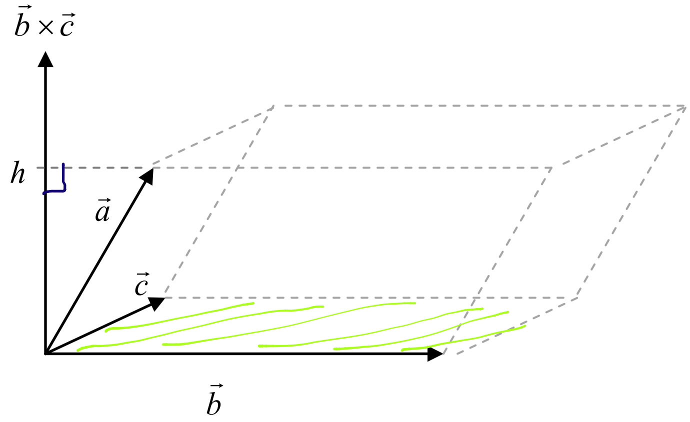
$A = \frac{1}{2}\sqrt{216}$

$= \frac{1}{2} \times 6\sqrt{6}$

$= 3\sqrt{6}$

$\therefore$ the exact area is $3\sqrt{6}$ units$^2$.

## III. Volume of a Parallelepiped




$$\begin{aligned} Volume &= Area_{base} \times height \\ &= A_{parallelogram} \times h \\ &= |\vec{b}\times\vec{c}|\,h \end{aligned}$$

Find $h$, where $h$ is the ***magnitude*** of the ***vector projection*** of $\vec{a}$ onto $\vec{b}\times\vec{c}$.

So, $h = \dfrac{|\vec{a}\cdot(\vec{b}\times\vec{c})|}{|\vec{b}\times\vec{c}|}$

$\therefore V = \cancel{|\vec{b}\times\vec{c}|}\dfrac{|\vec{a}\cdot(\vec{b}\times\vec{c})|}{\cancel{|\vec{b}\times\vec{c}|}}$ *or* $V = |\overbrace{\vec{a}\cdot(\underbrace{\vec{b}\times\vec{c}}_{\text{vector}})}^{\text{scalar}}|$

***SUMMARY OF GEOMETRIC APPLICATIONS:***

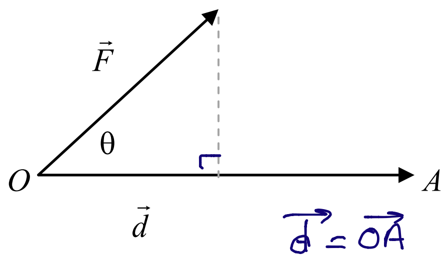
| Parallelogram | Triangle | Parallelepiped |
|---|---|---|
| $A = \lvert\vec{a}\times\vec{b}\rvert$ | $A = \frac{1}{2}\lvert\vec{a}\times\vec{b}\rvert$ | $V = \lvert\vec{a}\cdot(\vec{b}\times\vec{c})\rvert$<br>$V = \lvert\vec{b}\cdot(\vec{a}\times\vec{c})\rvert$<br>$V = \lvert\vec{c}\cdot(\vec{a}\times\vec{b})\rvert$ |

Note: vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ must be drawn tail to tail.

**HW. pg. 192 #1-8; pg. 194 #4, 12-14**

**Date:** May 22/14 **Section 5.5 – More Applications of Dot and Cross Products**

**IV. Work:** In everyday life, the word *work* is applied to any form of activity that requires physical exertion or mental effort.
In physics, **work** is done whenever a force acting on an object causes a displacement of the object from one position to another.




$\vec{d} = \vec{OA}$

Suppose a force $\vec{F}$ moves an object from $O$ to $A$.

$\vec{F}$ is the force acting on an object measured in newtons (N)
$\vec{d}$ is the displacement caused by the force, measured in metres (m)
$\theta$ is the angle between the force and the displacement
$W$ is the work done, measured in newton-metres, or joules (J)

**Work** is defined as the product of the distance an object has been displaced and the (signed) component of the force along the line of displacement. Work is a scalar quantity.

$$W = |\vec{d}|\, SP_{\vec{F} on \vec{d}}$$

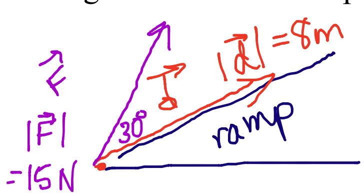
$$W = \cancel{|\vec{d}|}\frac{\vec{F}\cdot\vec{d}}{\cancel{|\vec{d}|}}$$

$$\therefore W = \vec{F}\cdot\vec{d} \quad or \quad W = |\vec{F}||\vec{d}|\cos\theta$$

***SUMMARY OF WORK:*** The **work** done by a force is defined as the dot product.

| **algebraic form** | *or* | **geometric form** |
|---|---|---|
| $W = \vec{F}\cdot\vec{d}$ | | $W = \lvert\vec{F}\rvert\lvert\vec{d}\rvert\cos\theta$ |

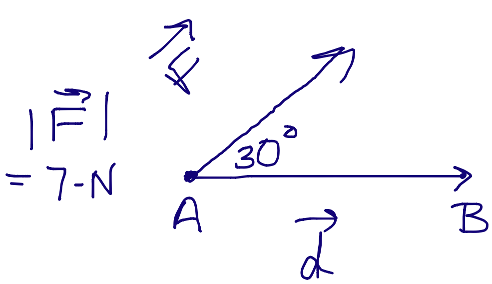
**Ex. 1.** A crate on a ramp is hauled 8 m up the ramp under a constant force of 15 N, applied at an angle of 30° to the ramp. Find the *exact* work done.




$W = |\vec{F}||\vec{d}|\cos\theta$
$= (15)(8)\cos 30^\circ$
$= 120\left(\frac{\sqrt{3}}{2}\right)$
$= 60\sqrt{3}$

$\therefore$ exact work done is $60\sqrt{3}$ N m or J.

**Ex. 2.** Find the *exact* work done by a 7-N force in moving an object from $A$ (3, 2) to $B$ (7, 5) when the force acts at an angle of 30° to $\overrightarrow{AB}$. The distance is in metres.




$\vec{d} = \vec{AB}$
$= \vec{OB} - \vec{OA}$
$= (7,5) - (3,2)$
$= (4,3)$
$|\vec{d}| = \sqrt{4^2+3^2}$
$|\vec{d}| = 5$

$W = |\vec{F}||\vec{d}|\cos\theta$
$= (7)(5)\cos 30^\circ$
$= 35\left(\frac{\sqrt{3}}{2}\right)$
$= \frac{35\sqrt{3}}{2}$

$\therefore$ the exact work done is $\frac{35\sqrt{3}}{2}$ N-m or J.

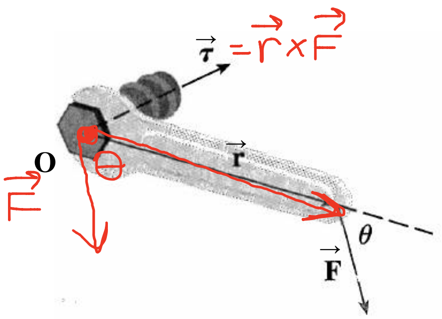
**Ex. 3.** Find the work done by a 24-N force in the direction of $\vec{v} = (1,2,2)$ when it moves an object from $A(2,-4,1)$ to $B(10,3,-1)$. The distance is in metres.

$\vec{d} = \vec{AB}$
$= \vec{OB} - \vec{OA}$
$= (10,3,-1) - (2,-4,1)$
$\therefore \vec{d} = (8,7,-2)$

$|\vec{F}| = 24\text{ N}$
$\vec{F} = 24\,\hat{v}$
$= 24\left(\frac{1}{|\vec{v}|}\vec{v}\right)$
$= \frac{24}{\sqrt{9}}(1,2,2)$
$= 8(1,2,2)$
$\therefore \vec{F} = (8,16,16)$

$W = \vec{F} \cdot \vec{d}$
$= (8,16,16)\cdot(8,7,-2)$
$= 64 + 112 - 32$
$= 144$
$\therefore$ the work done is 144 N·m or J.

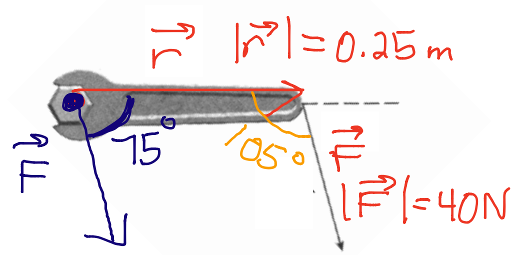
**V. Torque:** Sometimes instead of a force causing a change in position, a force causes an object to turn about a point or an axis. Examples are tightening a bolt using a wrench or applying a force to a bicycle pedal to make the crank arm rotate.
This turning effect of a force is called ***torque***. ***Torque*** is a ***vector*** quantity.




The ***torque*** caused by a force is defined as the cross product $\vec{T} = \vec{r} \times \vec{F}$ and its magnitude is $|\vec{T}| = |\vec{r} \times \vec{F}|$ or $|\vec{T}| = |\vec{r}||\vec{F}|\sin\theta$.

$\vec{F}$ is the applied force, $\vec{r}$ is the vector determined by the lever arm acting from the axis of rotation and $\theta$ is the angle between the force and lever arm.
**Note:** The magnitude of torque is measured in N-m or J.

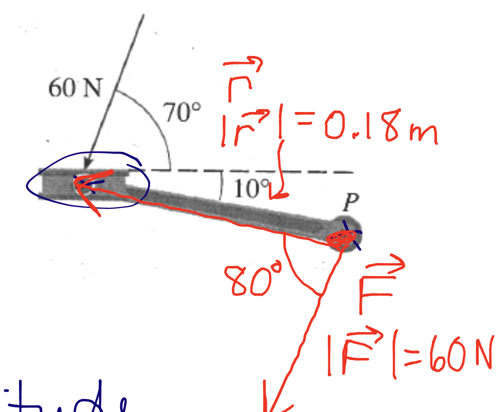
**Ex. 4.** A 40-N force is applied to the end of a 25 cm wrench with which it makes an angle of 105°. Calculate the magnitude of the torque about the centre of the bolt.




$|\vec{T}| = |\vec{r} \times \vec{F}|$
$= |\vec{r}||\vec{F}|\sin\theta$
$= (0.25)(40)\sin 75°$
$\doteq 9.7$ $\therefore$ the magnitude of the torque is about 9.7 Nm or J.

Note:
For maximum turning effect, $\theta = 90°$

**Ex. 5.** A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of torque about *P*.




$|\vec{T}| = |\vec{r} \times \vec{F}|$
$= |\vec{r}||\vec{F}|\sin\theta$
$= (0.18)(60)\sin 80°$
$\doteq 10.6$

$\therefore$ the magnitude of the torque is appoximately 10.6 J.

**HW. pg. 193 #9bc, 10 to 18 ; Pg. 197 #1 to 6**