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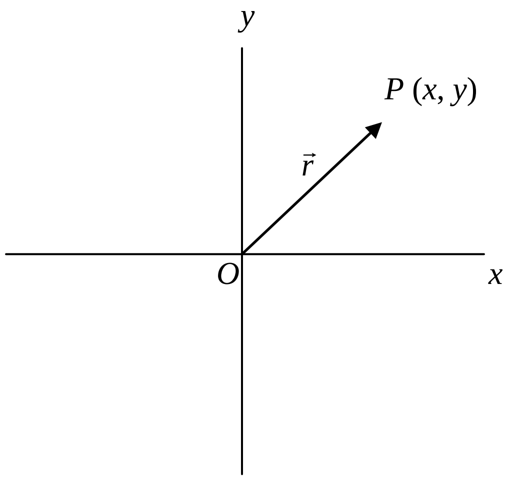
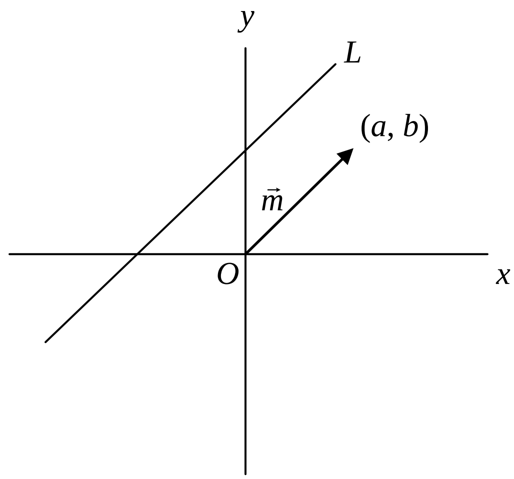
UNIT 9 – LINES IN A PLANE

Section 7.1 – Parametric and Vector Equations of a Line in a Plane





The ***position vector*** of a point *P* is a vector from the origin to *P*.

$\vec{r} = \overrightarrow{OP}$

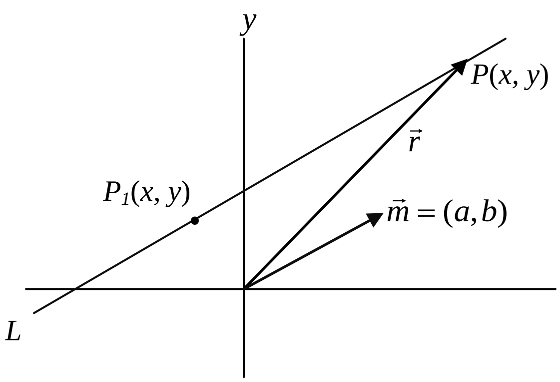
$\vec{r} = (x, y)$

A ***direction vector***, $\vec{m}$, of a line, *L*, is a non-zero vector that is *parallel* to L.

$\vec{m} = (a, b)$

Note: The slope of the line is $\frac{b}{a}$.

Find the ***vector equation*** of the line through $P_1(x_1, y_1)$ and parallel to the direction vector $\vec{m}$.



Let $P(x, y)$ be any point on the line *L*.

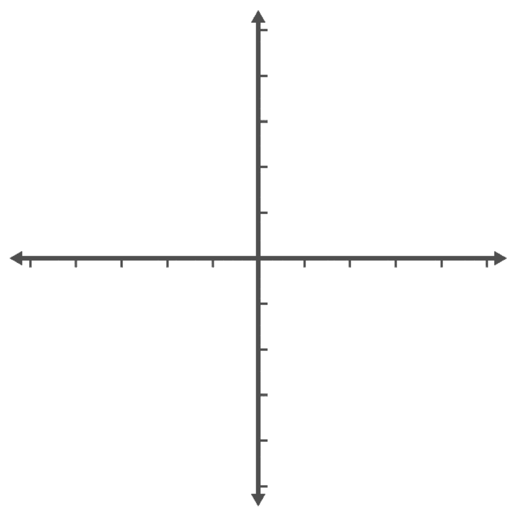
SUMMARY OF EQUATIONS OF LINES IN A PLANE:

Vector Equation	**Parametric Equations**	**Symmetric Equation**
$\vec{r} = \vec{r}_1 + t\,\vec{m}$	$x = x_1 + at$	$\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b}$
or	$y = y_1 + bt$	**Scalar or Cartesian Equation**
$(x, y) = (x_1, y_1) + t\,(a, b)$		$Ax + By + C = 0$

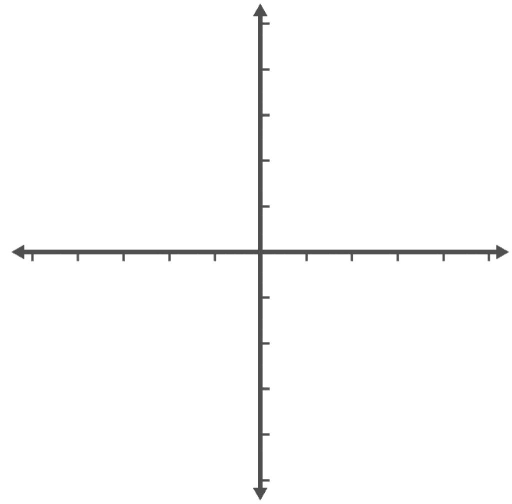
where (x, y) is the position vector of any point on the line
(x_1, y_1) is the position vector of some particular point on the line
(a, b) is a *direction vector* for the line
and $t \in \Re$ is the parameter.

Ex. 1. Given a line passes through the point $(-1, 2)$ and has $(4, -5)$ as a direction vector find:

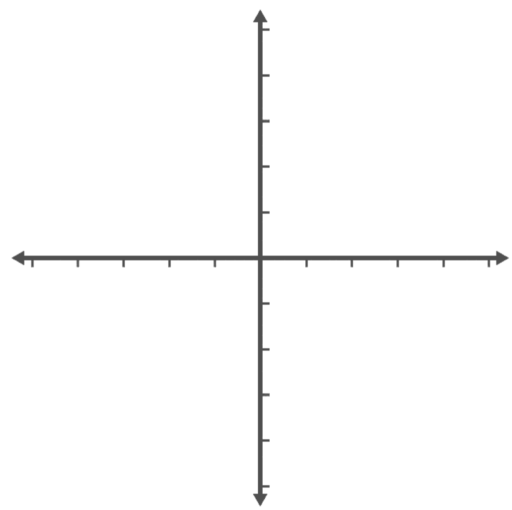
a) the ***vector equation*** and use it to find three other points on the line.
b) the ***parametric***, ***symmetric*** and ***scalar equations***.



Ex. 2. Find the ***vector***, ***parametric***, ***symmetric*** and ***Cartesian equations*** of the line through the points $M(2,3)$ and $N(5,2)$.

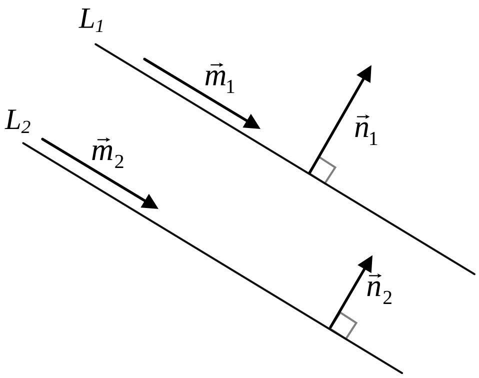


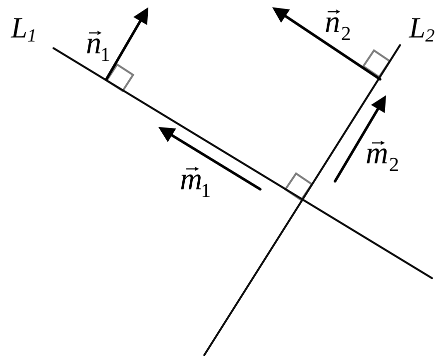
Ex. 3. Find the coordinates of the point at which the line $\vec{r}=(2,-5)+t(-1,3),\ t\in R$ meets the y-axis.



HW: p. 245 #1-7, 8bd, 9ab, 10ab, 11, 12b, 13, 14a, 16bc, 17

Date:_____________ **Section 7.2 – The Scalar Equation of a Line in a Plane**





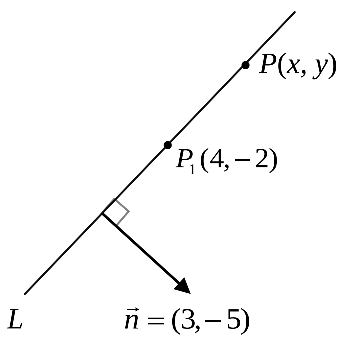
If $L_1 \parallel L_2$ (L_1 is ***parallel*** to L_2) then
- $\vec{m}_1 \parallel \vec{m}_2$ where $\vec{m}_1 = k\,\vec{m}_2$
- $\vec{n}_1 \parallel \vec{n}_2$ where $\vec{n}_1 = k\,\vec{n}_2$

If $L_1 \perp L_2$ (L_1 is ***perpendicular*** to L_2) then
- $\vec{m}_1 \perp \vec{m}_2$ where $\vec{m}_1 \cdot \vec{m}_2 = 0$
- $\vec{n}_1 \perp \vec{n}_2$ where $\vec{n}_1 \cdot \vec{n}_2 = 0$

Ex. 1. Find the ***scalar*** or ***Cartesian equation***, of the form $Ax + By + C = 0$ for the line through $P_1(4,-2)$ and perpendicular to $\vec{n} = (3,-5)$.

Solution ①

Let $P(x, y)$ be a point on the line.



How does $\vec{n}$ relate to the equation $Ax + By + C = 0$ in this example?

We can conclude that the ***scalar*** or ***Cartesian equation*** of a straight line in a plane has the form $Ax + By + C = 0$, where $\vec{n} = (A, B)$ is a vector perpendicular to the line.

Solution ②

SUMMARY: The **scalar** or **Cartesian equation** of a line in a plane has the form

$$Ax + By + C = 0$$

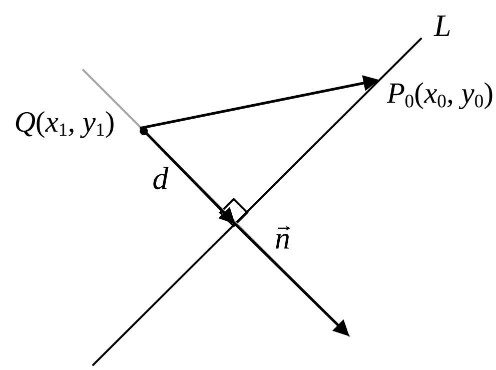
where (A, B) is a *normal* to the line.

Ex. 2. Find the ***scalar equation*** of the line through $P_1(-2,3)$ and perpendicular to the line $5x - 2y + 3 = 0$.

Ex. 3. a) Show that the ***shortest distance*** from a point $Q(x_1, y_1)$ to a line with a scalar equation $Ax + By + C = 0$ is given by the formula $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

b) Use the formula to find the distance from the point $Q(5,8)$ to the line $7x + y - 23 = 0$.

a) Let $P_o(x_o, y_o)$ be a specific point on the line.



$\overrightarrow{QP_0} = \overrightarrow{OP_0} - \overrightarrow{OQ}$

$\overrightarrow{QP_0} = (x_0, y_0) - (x_1, y_1)$

$\therefore \overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1)$ and $\vec{n} = (A, B)$

$d = \left| \underset{\overrightarrow{QP_0}\, on\, \vec{n}}{VP} \right|$

$= \dfrac{|\overrightarrow{QP_0} \cdot \vec{n}|}{|\vec{n}|}$

$= \dfrac{|(x_0 - x_1, y_0 - y_1) \cdot (A,\ B)|}{\sqrt{A^2 + B^2}}$

$= \dfrac{|Ax_0 - Ax_1 + By_0 - By_1|}{\sqrt{A^2 + B^2}}$

**** see Note above***

$= \dfrac{|-Ax_1 - By_1 - C|}{\sqrt{A^2 + B^2}}$

$= \dfrac{|-(Ax_1 + By_1 + C)|}{\sqrt{A^2 + B^2}}$

$\therefore d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

Note: Since the point $P_0(x_0, y_0)$ is on the line $Ax + By + C = 0$, it satisfies the equation.
ie. $Ax_0 + By_0 + C = 0$, so $Ax_0 + By_0 = -C$

HW: p. 251 #1, 2, 3ad, 4, 5, 6bd, 7-9, 11, 12bd

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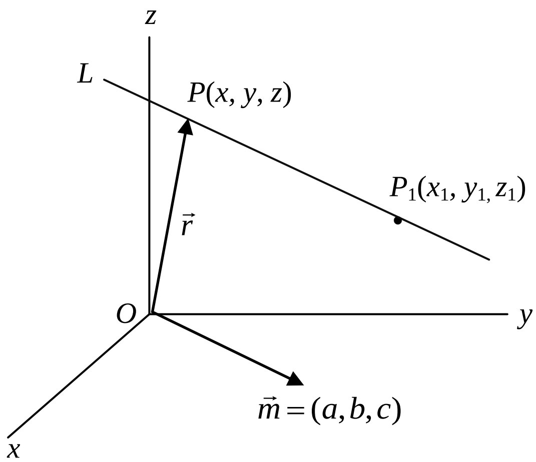
Section 7.3 – Equations of Lines in 3-Space

Find the ***vector equation*** of the line through $P_1(x_1, y_1\ z_1)$ and parallel to the direction vector $\vec{m}$.



Let $P(x, y, z)$ be any point on the line L.

SUMMARY OF EQUATIONS OF LINES IN 3-SPACE:

Vector Equation	**Parametric Equations**	**Symmetric Equation**
$\vec{r} = \vec{r}_1 + t\vec{m}$	$x = x_1 + at$	$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
or	$y = y_1 + bt$	
$(x, y, z) = (x_1, y_1, z_1) + t(a, b, c)$	$z = z_1 + ct$	

where (x, y, z) is the position vector of any point on the line
(x_1, y_1, z_1) is the position vector of some particular point on the line
(a, b, c) is a *direction vector* for the line
and $t \in \Re$ is the parameter.

Ex. 1. Find the ***vector***, ***parametric*** and ***symmetric equations*** of the following lines:

a) through the point $(5, 4, -1)$ with direction vector $(1, -3, 2)$

b) through the point $(1,2,3)$ with direction vector $(3,0,2)$

c) through the points $(4,5,-1)$ and $(7,5,-1)$

Ex. 2. Show that the point $(2,-1,5)$ lies on the line with vector equation $\vec{r}=(1,2,3)+t(1,-3,2)$, $t\in\Re$.

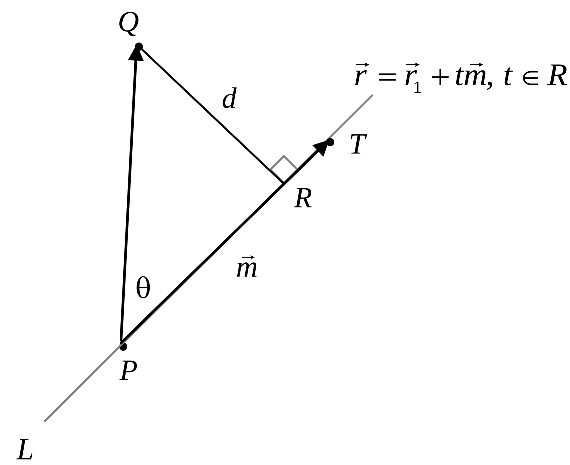
Ex. 3. Write a ***vector equation*** for the line $x+1=-y=z-3$.

Ex. 4. Do $\frac{x-5}{2}=\frac{y+4}{-5}=\frac{z+1}{3}$ and $\frac{x+1}{-4}=\frac{y-11}{10}=\frac{z+4}{-6}$ represent the same line? ________

Ex. 5. a) Show that the ***shortest distance*** from a point Q in space to a line with a vector equation $\vec{r} = \vec{r}_1 + t\,\vec{m},\ t \in \Re$, is given by the formula $d = \dfrac{\left|\vec{m} \times \overrightarrow{PQ}\right|}{\left|\vec{m}\right|}$.

b) Use the formula to find the distance from the point $Q(-1,1,6)$ to the line $\vec{r} = (1,2,-1) + t(0,1,1)$.

a) In the diagram, we would like to find *d*.



In triangle *PQR*,

$\sin\theta = \dfrac{d}{\left|\overrightarrow{PQ}\right|}$, so

$d = \left|\overrightarrow{PQ}\right|\sin\theta$

From our earlier lessons on cross products, we know that

$\left|\vec{m} \times \overrightarrow{PQ}\right| = \left|\vec{m}\right|\left|\overrightarrow{PQ}\right|\sin\theta$.

If we substitute $d = \left|\overrightarrow{PQ}\right|\sin\theta$ into this formula,

$\left|\vec{m} \times \overrightarrow{PQ}\right| = \left|\vec{m}\right|(d)$

$\therefore d = \dfrac{\left|\vec{m} \times \overrightarrow{PQ}\right|}{\left|\vec{m}\right|}$

b)

HW: p. 256 #1-4, 6-15

Date:_____________ **Section 7.4 – The Intersection of Two Lines in a Plane**

Warm-up:

Ex. 1. Given the scalar equation for a line is $5x-2y+2=0$, find the:

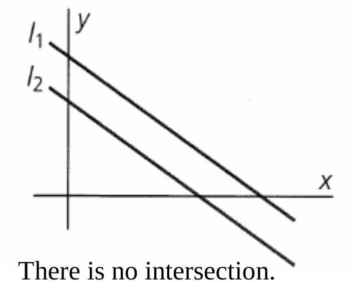
a) ***vector***, ***parametric*** and ***symmetric equations*** for the same line.

b) ***acute*** angle this line makes with the line $x=2-3s,\ \ y=-3+s,\ \ s\in\Re$.

c) ***point of intersection*** of this line with the line $x=2-3s,\ \ y=-3+s,\ \ s\in\Re$.

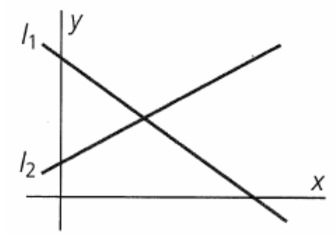
SUMMARY: Two lines in a plane can intersect in one of three possible ways.

parallel and distinct



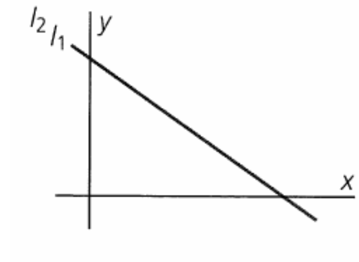
There is no intersection.

intersecting



The intersection is a single point.

parallel and coincident



The intersection is the line.

Ex. 2. Determine whether the following pairs of lines are parallel and coincident, parallel and distinct, or neither. Find the intersection if possible.

a) $\vec{r} = (6,-2) + t(-3,2)$ and $\vec{r} = (-12,10) + s(6,-4)$

b) $x = 3 - 2t,\ \ y = 5 + t$ and $x = 5 + 2s,\ \ y = 3 - s$

c) $\dfrac{x-6}{-3} = \dfrac{y+1}{2}$ and $\dfrac{x-5}{2} = \dfrac{y-3}{2}$

HW: p. 263 #2, 3ab, 8; Worksheet on Equations of Lines

Date:______________

WORKSHEET on Equations of Lines

1. Find vector and scalar equations of the line through the point $A(4,1)$ and parallel to the line $x = 3 + 2t,\ \ y = 7t, t \in \Re$.

2. Find parametric equations of the line through the points $R(3,-1,0)$ and $S(2,11,-5)$.

3. Develop a scalar equation for the line perpendicular to the line $2x - 3y - 5 = 0$ with the same x-intercept as the line $\vec{r} = (0,1) + t(-3,4), t \in \Re$.

4. Find the scalar equation for the line parallel to the y-axis and through the point $P(5,-1)$.

5. Find the vector equation for the line perpendicular to the xz-plane through the point $Q(3,2,-1)$.

6. Find a vector equation for the line parallel to the z-axis through the point $R(3,2,-1)$.

7. Determine the vector, parametric and symmetric equations of the line that passes through the points $A(2,-4)$ and $B(6,1)$.

8. Where does the line with the vector equation $\vec{r} = (1,-3) + t(6,4), t \in \Re$ intersect the x and y axes?

9. Find the Cartesian equation of the line with the vector equation $\vec{r} = (1,-3) + t(6,4), t \in \Re$.

10. Find the symmetric equations of the line passing through the point $R(3,0,-2)$ and perpendicular to both $x = 2 + s,\ \ y = -3s,\ \ z = 2s - 1$ and $\dfrac{x-3}{4} = \dfrac{y+2}{-2} = z - 1$.

Answers

1. $\vec{r} = (4,1) + t(2,7), t \in \Re,\ 7x - 2y - 26 = 0$ **2.** $x = 3 + t,\ \ y = -1 - 12t,\ \ z = 5t, t \in \Re$ **3.** $12x + 8y - 9 = 0$

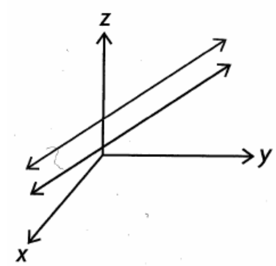
4. $x - 5 = 0$ **5.** $\vec{r} = (3,2,-1) + t(0,1,0), t \in \Re$ **6.** $\vec{r} = (3,2,-1) + t(0,0,1), t \in \Re$

7. $\vec{r} = (6,1) + t(4,5), t \in \Re;\ \ x = 6 + 4t,\ \ y = 1 + 5t;\ \ \dfrac{x-6}{4} = \dfrac{y-1}{5}$ **8.** $\left(\dfrac{11}{2}, 0\right); \left(0, -\dfrac{11}{3}\right)$

9. $2x - 3y - 11 = 0$ **10.** $\dfrac{x-3}{1} = \dfrac{y}{7} = \dfrac{z+2}{10}$

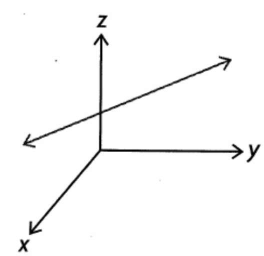
Date:______________ **Section 7.4 – The Intersection of Two Lines in 3-Space**

Parallel & Distinct



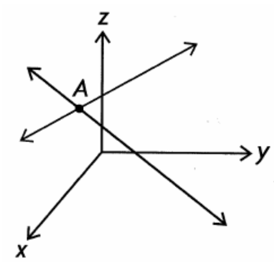
There is no intersection.

Parallel & Coincident



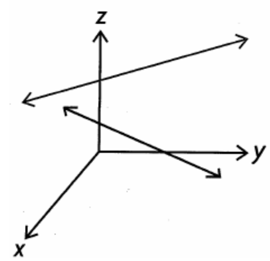
The intersection is the line.

Intersecting



The intersection is a point.

Skew



There is no intersection.

Examples

Find the intersection of the following pairs of lines, if any exist:

a) $L_1 : \vec{r} = (-3,2,1) + t(-4,-6,-2)$

$L_2 : \vec{r} = (5,-4,3) + s(2,3,1)$

b) $L_1 : \vec{r} = (4,3,7) + t(3,-1,2)$

$L_2 : \vec{r} = (-5,6,1) + u(-6,2,-4)$

c) $\frac{x-2}{4}=\frac{y-7}{3}=\frac{z-15}{2}$ and $\frac{x+3}{3}=\frac{y-13}{-1}=\frac{z-2}{5}$

d) $\vec{r}=(1,2,5)+t(4,3,-1)$
$\vec{r}=(-3,-1,3)+s(1,-2,4)$

HW: p. 263 #1, 3cd, 4-7, 11; REVIEW for TEST: p. 266 #1-15