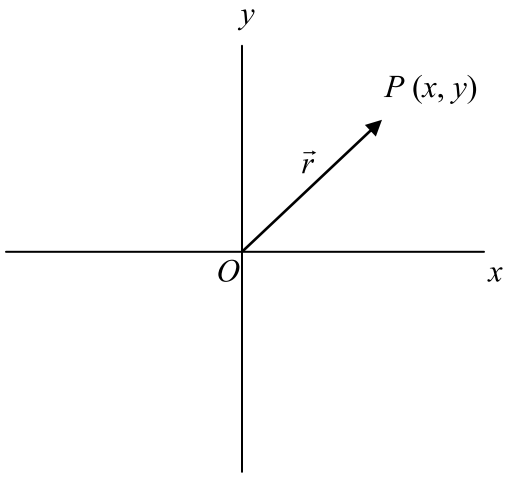
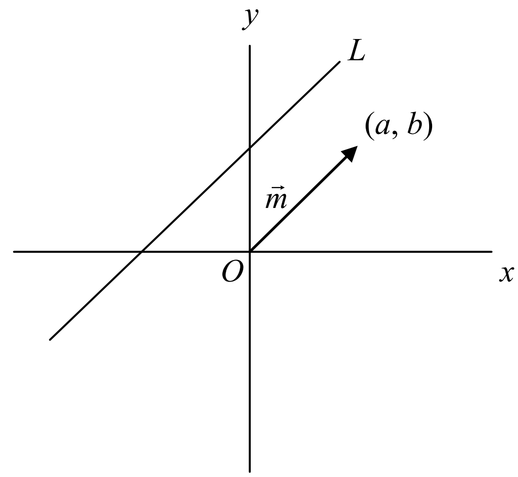
Date: May 23/14

UNIT 9 – LINES IN A PLANE

Section 7.1 – Parametric and Vector Equations of a Line in a Plane





The ***position vector*** of a point *P* is a vector from the origin to *P*.

$\vec{r} = \overrightarrow{OP}$

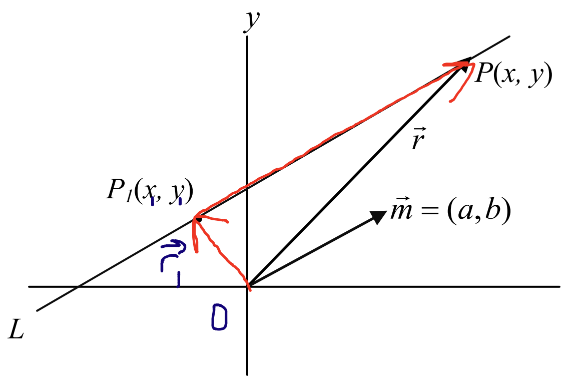
$\vec{r} = (x, y)$

A ***direction vector***, $\vec{m}$, of a line, *L*, is a non-zero vector that is *parallel* to L.

$\vec{m} = (a, b)$

Note: The slope of the line is $\frac{b}{a}$.

Find the ***vector equation*** of the line through $P_1(x_1, y_1)$ and parallel to the direction vector $\vec{m}$.



Let *P*(*x*, *y*) be any point on the line *L*.

$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P}$

$\vec{r} = \vec{r}_1 + \underbrace{t\vec{m}}_{}, t \in \mathbb{R}$

vector which is a scalar multiple of $\vec{m}$, $\because \overrightarrow{P_1P}$ is parallel to $\vec{m}$.

$\therefore$ the vector equation of the line through $P_1(x_1, y_1)$ in the direction of $\vec{m}$ is

$$\vec{r} = \vec{r}_1 + t\vec{m}, t \in \mathbb{R}$$

or

$$(x, y) = (x_1, y_1) + t(a, b), t \in \mathbb{R}.$$

SUMMARY OF EQUATIONS OF LINES IN A PLANE:

Vector Equation	**Parametric Equations**	**Symmetric Equation**
$\vec{r} = \vec{r}_1 + t\,\vec{m}$	$x = x_1 + at$	$\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b}$
or	$y = y_1 + bt$	**Scalar or Cartesian Equation**
$(x, y) = (x_1, y_1) + t(a, b)$		$Ax + By + C = 0$

where (x, y) is the position vector of any point on the line
(x_1, y_1) is the position vector of some particular point on the line
(a, b) is a *direction vector* for the line
and $t \in \Re$ is the parameter.

Ex. 1. Given a line passes through the point $(-1, 2)$ and has $(4, -5)$ as a direction vector find:

a) the ***vector equation*** and use it to find three other points on the line.

b) the ***parametric***, ***symmetric*** and ***scalar equations***.

a) Vector equation

$\vec{r} = \vec{r}_1 + t\vec{m}, \ t \in R$

$(x, y) = (x_1, y_1) + t(a, b)$

$\vec{r}_1 = (-1, 2)$ & $\vec{m} = (4, -5)$

$\therefore \vec{r} = (-1, 2) + t(4, -5)$

or

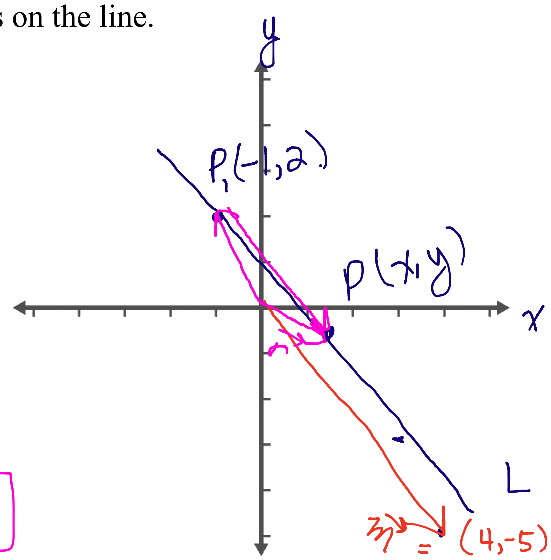
$\boxed{(x, y) = (-1, 2) + t(4, -5), \ t \in \mathbb{R}}$

if $t = 2$, $(x, y) = (-1, 2) + 2(4, -5)$
$= (7, -8)$

if $t = 3$, $(x, y) = (-1, 2) + 3(4, -5)$
$= (11, -13)$

If $t = -3$, $(x, y) = (-1, 2) - 3(4, -5)$
$= (-13, 17)$

$\therefore$ three other points on the line are $(7, -8)$, $(11, -13)$ & $(-13, 17)$



parametric equations

$x = -1 + 4t$

$y = 2 - 5t$

symmetric equation

$\dfrac{x + 1}{4} = \dfrac{y - 2}{-5}$

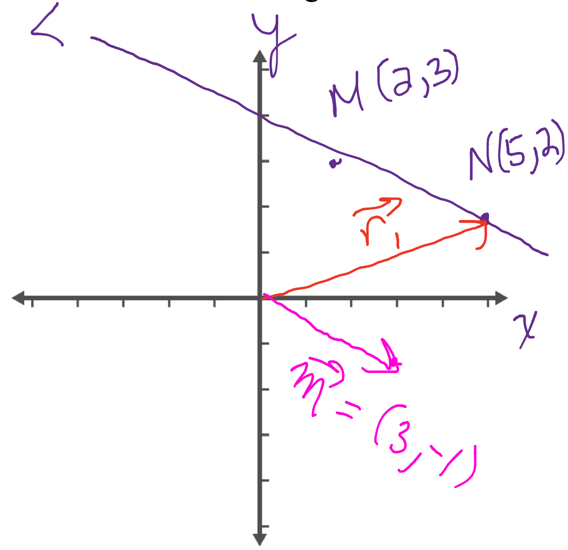
Scalar equation.

$-5x - 5 = 4y - 8$

$-5x - 4y + 3 = 0$

$\therefore 5x + 4y - 3 = 0$

Ex. 2. Find the ***vector***, ***parametric***, ***symmetric*** and ***Cartesian equations*** of the line through the points $M(2,3)$ and $N(5,2)$.



$\vec{r} = \vec{r}_1 + t\vec{m}, t \in \mathbb{R}$

$\vec{r}_1 = \overrightarrow{ON}$ $\quad$ $\vec{m} = \overrightarrow{MN}$

$= (5,2)$ $\quad$ $= \overrightarrow{ON} - \overrightarrow{OM}$

$\;\; x_1, y_1$ $\quad$ $= (5,2) - (2,3)$

$= (3,-1)$

$\;\; a \;\; b$

$\therefore$ the vector equation is

$\vec{r} = (5,2) + t(3,-1), t \in \mathbb{R}$

or

$(x,y) = (5,2) + t(3,-1), t \in \mathbb{R}$

Parametric Equations:

$x = 5 + 3t$

$y = 2 - t$

Symmetric Equation

$\dfrac{x-5}{3} = \dfrac{y-2}{-1}$

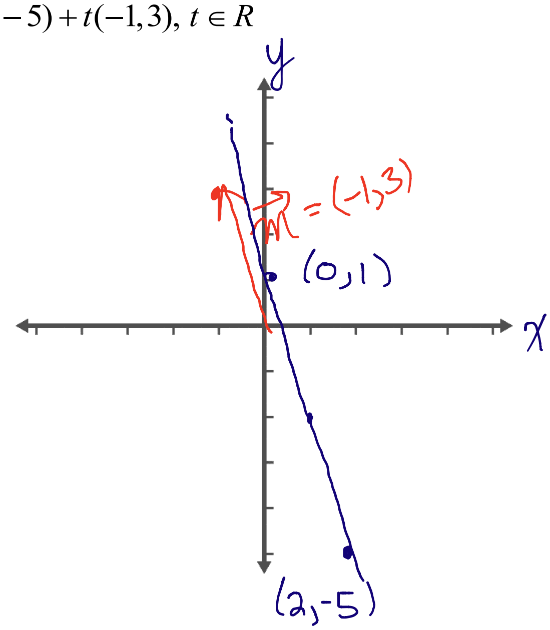
Scalar/Cartesian Equation

$-x + 5 = 3y - 6$

$-x - 3y + 11 = 0$

$x + 3y - 11 = 0$

Ex. 3. Find the coordinates of the point at which the line $\vec{r} = (2,-5) + t(-1,3),\ t \in R$ meets the y-axis.



$\vec{r} = (2,-5) + t(-1,3)$

$(x,y) = (2,-5) + t(-1,3)$

$x = 2 - t$

$y = -5 + 3t$

For y-int,

$x = 0$

$2 - t = 0$

$t = 2$

If $t = 2$

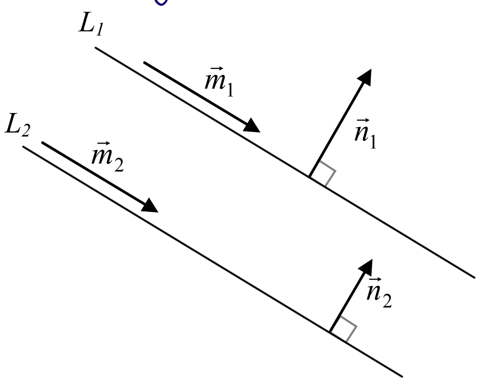
$y = -5 + 3(2)$

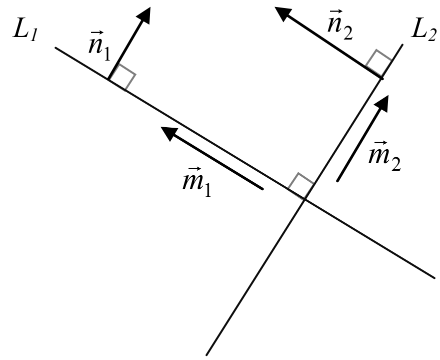
$= 1$

$\therefore$ the line meets the y-axis at the point $(0,1)$.

HW: pg. 245 #1-7, 8bd, 9ab, 10ab, 11, 12b, 13, 14a, 16bc, 17

Date: May 27/14 **Section 7.2 – The Scalar Equation of a Line in a Plane**





If $L_1 \parallel L_2$ (L_1 is ***parallel*** to L_2) then

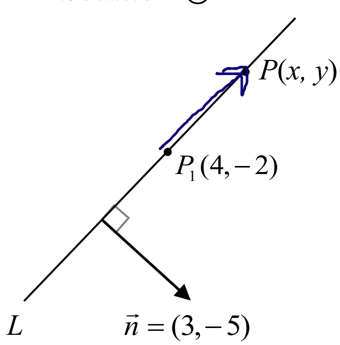
- $\vec{m}_1 \parallel \vec{m}_2$ where $\vec{m}_1 = k\vec{m}_2$
- $\vec{n}_1 \parallel \vec{n}_2$ where $\vec{n}_1 = k\vec{n}_2$

If $L_1 \perp L_2$ (L_1 is ***perpendicular*** to L_2) then

- $\vec{m}_1 \perp \vec{m}_2$ where $\vec{m}_1 \cdot \vec{m}_2 = 0$
- $\vec{n}_1 \perp \vec{n}_2$ where $\vec{n}_1 \cdot \vec{n}_2 = 0$

Ex. 1. Find the ***scalar*** or ***Cartesian equation***, of the form $Ax + By + C = 0$ for the line through $P_1(4,-2)$ and perpendicular to $\vec{n} = (3,-5)$.

Solution ①



Let $P(x, y)$ be a point on the line.

$\overrightarrow{P_1P} \perp \vec{n}$

$\overrightarrow{P_1P} \cdot \vec{n} = 0$

$(x-4, y+2) \cdot (3,-5) = 0$

$3(x-4) - 5(y+2) = 0$

$3x - 12 - 5y - 10 = 0$

$\therefore 3x - 5y - 22 = 0$

$\overrightarrow{P_1P} = \overrightarrow{OP} - \overrightarrow{OP_1}$
$= (x,y) - (4,-2)$
$= (x-4, y+2)$

How does $\vec{n}$ relate to the equation $Ax + By + C = 0$ in this example?

$\vec{n} = (3,-5)$
$\vec{n} = (A,B)$

We can conclude that the ***scalar*** or ***Cartesian equation*** of a straight line in a plane has the form $Ax + By + C = 0$, where $\vec{n} = (A, B)$ is a vector perpendicular to the line.

Solution ②

$\vec{n} = (\underset{A}{3}, \underset{B}{-5})$; $P_1(\underset{x}{4}, \underset{y}{-2})$

Let $Ax + By + C = 0$

$\because \vec{n} = (3,-5)$

$\therefore 3x - 5y + C = 0$

Find C if
$x = 4$, $y = -2$
$3(4) - 5(-2) + C = 0$
$22 + C = 0$
$\therefore C = -22$

$\therefore$ the scalar equation is $3x - 5y - 22 = 0$.

> ***SUMMARY:*** The **scalar** or **Cartesian equation** of a line in a plane has the form
> $$Ax + By + C = 0$$
> where (A, B) is a *normal* to the line.

Ex. 2. Find the ***scalar equation*** of the line through $P_1(-2,3)$ and perpendicular to the line $5x - 2y + 3 = 0$.

For required line: $\vec{n} = (2,5)$; $P_1(-2,3)$

For given line:
$5x - 2y + 3 = 0$
$\vec{n} = (5,-2)$
$\because$ required line $\perp$ given line
$\vec{n}_r \perp \vec{n}_g$
$\therefore \vec{n}_r = (2,5)$

let $Ax + By + C = 0$
$\because \vec{n} = (2,5)$
$\therefore 2x + 5y + C = 0$
Find C if $x = -2$ & $y = 3$
$2(-2) + 5(3) + C = 0$
$11 + C = 0$
$C = -11$

$\therefore$ the required scalar equation is $2x + 5y - 11 = 0$.

Ex. 3. a) Show that the ***shortest distance*** from a point $Q(x_1, y_1)$ to a line with a scalar equation $Ax + By + C = 0$ is given by the formula $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

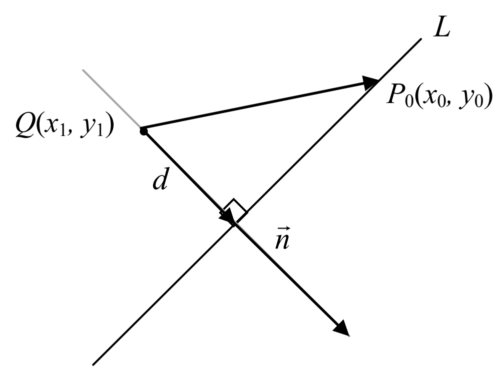
b) Use the formula to find the distance from the point $Q(5,8)$ to the line $7x + y - 23 = 0$.

a) Let $P_o(x_o, y_o)$ be a specific point on the line.

$$\overrightarrow{QP_0} = \overrightarrow{OP_0} - \overrightarrow{OQ}$$
$$\overrightarrow{QP_0} = (x_0, y_0) - (x_1, y_1)$$
$$\therefore \overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1) \text{ and } \vec{n} = (A, B)$$
$$d = \left| \underset{\overrightarrow{QP_0}\, on\, \vec{n}}{VP} \right|$$
$$= \frac{|\overrightarrow{QP_0} \cdot \vec{n}|}{|\vec{n}|}$$
$$= \frac{|(x_0 - x_1, y_0 - y_1) \cdot (A, B)|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|Ax_0 - Ax_1 + By_0 - By_1|}{\sqrt{A^2 + B^2}}$$

*** see Note above**

$$= \frac{|-Ax_1 - By_1 - C|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|-(Ax_1 + By_1 + C)|}{\sqrt{A^2 + B^2}}$$
$$\therefore d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$



Note: Since the point $P_0(x_0, y_0)$ is on the line $Ax + By + C = 0$, it satisfies the equation.
ie. $Ax_0 + By_0 + C = 0$, so $Ax_0 + By_0 = -C$

b) $Q(5,8)$; $7x + y - 23 = 0$
$A = 7$, $B = 1$, $C = -23$, $x_1 = 5$, $y_1 = 8$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|(7)(5) + (1)(8) + (-23)|}{\sqrt{(7)^2 + (1)^2}}$$
$$= \frac{|20|}{\sqrt{50}}$$
$$= \frac{20}{5\sqrt{2}}$$
$$= \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$
$$= 2\sqrt{2}$$

$\therefore$ the pt. is $2\sqrt{2}$ units from the line.

HW: pg. 251 #1, 2, 3ad, 4, 5, 6bd, 7-9, 11, 12bd