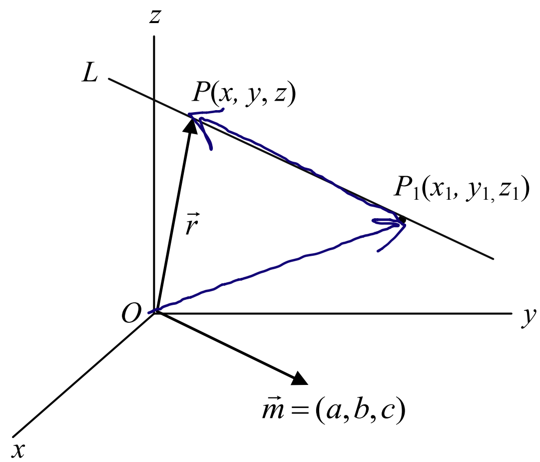
**Date:** May 28/2014 **Section 7.3 – Equations of Lines in 3-Space**

Find the ***vector equation*** of the line through $P_1(x_1, y_1\ z_1)$ and parallel to the direction vector $\vec{m}$.




Let $P(x, y, z)$ be any point on the line $L$.

$\vec{OP} = \vec{OP_1} + \vec{P_1P}$

$\vec{OP} = \vec{OP_1} + t\vec{m},\ t \in \mathbb{R}$

$\vec{r} = \vec{r_1} + t\vec{m},\ t \in \mathbb{R}$

or

$(x, y, z) = (x_1, y_1, z_1) + t(a, b, c)$

***SUMMARY OF EQUATIONS OF LINES IN 3-SPACE:***

| **Vector Equation** | **Parametric Equations** | **Symmetric Equation** |
|---|---|---|
| $\vec{r} = \vec{r_1} + t\,\vec{m}$ | $x = x_1 + at$ | $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ |
| or | $y = y_1 + bt$ | No scalar equation. |
| $(x, y, z) = (x_1, y_1, z_1) + t(a, b, c)$ | $z = z_1 + ct$ | |

where $(x, y, z)$ is the position vector of any point on the line
$(x_1, y_1, z_1)$ is the position vector of some particular point on the line
$(a, b, c)$ is a *direction vector* for the line
and $t \in \Re$ is the parameter.

**Ex. 1.** Find the ***vector***, ***parametric*** and ***symmetric equations*** of the following lines:

**a)** through the point $(5, 4, -1)$ with direction vector $(1, -3, 2)$

$\vec{r_1} = (5, 4, -1) \qquad \vec{m} = (1, -3, 2)$

Vector Equation:

$\vec{r} = (5, 4, -1) + t(1, -3, 2),\ t \in \mathbb{R}$

Parametric Equations:

$x = 5 + t$

$y = 4 - 3t$

$z = -1 + 2t$

Symmetric Equations:

$\frac{x - 5}{1} = \frac{y - 4}{-3} = \frac{z + 1}{2}$

**b)** through the point $(1,2,3)$ with direction vector $(3,0,2)$

$\vec{r_1} = (1,2,3)$ ; $\vec{m} = (3,0,2)$

Vector Equation:

$\vec{r} = (1,2,3) + t(3,0,2), t \in R$

Parametric Equations:

$x = 1+3t$

$y = 2$

$z = 3+2t$

Symmetric Equations:

$\frac{x-1}{3} = \frac{z-3}{2}$ ; $y=2$

Note: This line is parallel to the xz-plane.

**c)** through the points $\overset{A}{(4,5,-1)}$ and $\overset{B}{(7,5,-1)}$

$\vec{r_1} = \vec{OA}$

$\vec{r_1} = (4,5,-1)$

$\vec{m} = \vec{AB}$

$= \vec{OB} - \vec{OA}$

$= (7,5,-1) - (4,5,-1)$

$\therefore \vec{m} = (3,0,0)$

use $\vec{m} = (1,0,0)$

Vector Equation:

$\vec{r} = (4,5,-1) + t(1,0,0),$
$t \in R$

Parametric Equations:

$x = 4+t$

$y = 5$

$z = -1$

No Symmetric Equations

Note: This line is parallel to the x-axis.

**Ex. 2.** Show that the point $(2,-1,5)$ lies on the line with vector equation $\vec{r} = (1,2,3) + t(1,-3,2)$, $t \in \Re$.

Sub $(2,-1,5)$ in for $\vec{r}$ and solve for $t$.

$(2,-1,5) = (1,2,3) + \underbrace{t(1,-3,2)}$

$(2,-1,5) - (1,2,3) = t(1,-3,2)$

$(1,-3,2) = t(1,-3,2)$

$\because t = 1$

$\therefore (2,-1,5)$ is on the line.

**Ex. 3.** Write a ***vector equation*** for the line $x+1=-y=z-3$.

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

$$\frac{x-(-1)}{1}=\frac{y-0}{-1}=\frac{z-3}{1}$$

$\vec{r_1}=(-1,0,3)$ ; $\vec{m}=(1,-1,1)$

$\therefore$ $\vec{r}=(-1,0,3)+t(1,-1,1)$, $t\in\mathbb{R}$ is the vector equation.

**Ex. 4.** Do $\frac{x-5}{2}=\frac{y+4}{-5}=\frac{z+1}{3}$ and $\frac{x+1}{-4}=\frac{y-11}{10}=\frac{z+4}{-6}$ represent the same line? NO

$L_1$ $\qquad$ $L_2$

For $L_1$: $\vec{r}=(5,-4,-1)+t(2,-5,3)$, $t\in\mathbb{R}$

For $L_2$: $\vec{r}=(-1,11,-4)+s(-4,10,-6)$, $s\in\mathbb{R}$

$\vec{m_1}=(2,-5,3)$ & $\vec{m_2}=(-4,10,-6)$

$\because$ $\vec{m_2}=k\vec{m_1}$, $k=-2$

$\therefore$ the lines are parallel

ie. parallel and distinct or parallel and coincident

sub $(5,-4,-1)$ in for $\vec{r}$ in $L_2$ & solve for $s$

$$(5,-4,-1)=(-1,11,-4)+s(-4,10,-6)$$

$$(5,-4,-1)-(-1,11,-4)=s(-4,10,-6)$$

$$(6,-15,3)=s(-4,10,-6)$$

$6=-4s$ & $-15=10s$ & $3=-6s$

$s=-\frac{3}{2}$ $\qquad$ $s=-\frac{3}{2}$ $\qquad$ $s=-\frac{1}{2}$

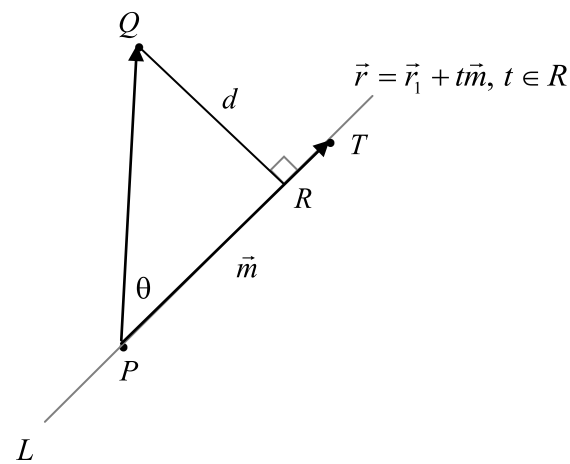
$\therefore s$ is undefined. or $s=\emptyset$ ← null set.

$\therefore$ the lines are parallel and distinct.

**Ex. 5. a)** Show that the ***shortest distance*** from a point $Q$ in space to a line with a vector equation $\vec{r} = \vec{r}_1 + t\,\vec{m},\ t \in \Re$, is given by the formula $d = \dfrac{\left|\vec{m} \times \overrightarrow{PQ}\right|}{|\vec{m}|}$.

**b)** Use the formula to find the distance from the point $Q(-1,1,6)$ to the line $\vec{r} = (1,2,-1) + t(0,1,1)$.

**a)** In the diagram, we would like to find *d*.




In triangle *PQR*,

$$\sin\theta = \frac{d}{\left|\overrightarrow{PQ}\right|}, \text{ so}$$

$$d = \left|\overrightarrow{PQ}\right| \sin\theta$$

From our earlier lessons on cross products, we know that

$$\left|\vec{m} \times \overrightarrow{PQ}\right| = |\vec{m}|\left|\overrightarrow{PQ}\right| \sin\theta .$$

If we substitute $d = \left|\overrightarrow{PQ}\right| \sin\theta$ into this formula,

$$\left|\vec{m} \times \overrightarrow{PQ}\right| = |\vec{m}|(d)$$

$$\therefore d = \frac{\left|\vec{m} \times \overrightarrow{PQ}\right|}{|\vec{m}|}$$

**b)**

$$d = \frac{|\vec{m} \times \overrightarrow{PQ}|}{|\vec{m}|}$$

$$= \frac{|(0,1,1) \times (-2,-1,7)|}{|(0,1,1)|}$$

$$= \frac{|(8,-2,2)|}{\sqrt{(0)^2+(1)^2+(1)^2}}$$

$$= \frac{\sqrt{64+4+4}}{\sqrt{2}}$$

$$= \frac{\sqrt{72}}{\sqrt{2}}$$

$$= \sqrt{36}$$

$$= 6$$

$\vec{r} = (1,2,-1) + t(0,1,1)$

$P(1,2,-1)$ ; $Q(-1,1,6)$ ; $\vec{m} = (0,1,1)$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (-1,1,6) - (1,2,-1)$$

$$= (-2,-1,7)$$

$\vec{m} = (0,1,1)$ ; $\overrightarrow{PQ} = (-2,-1,7)$

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ -1 & 7 & -2 & -1 \end{array}$$

$$\vec{m} \times \overrightarrow{PQ} = (7-(-1), -2-0, 0-(-2))$$

$$= (8,-2,2)$$

$\therefore$ point Q is 6 units from the line.

**HW: pg. ~~245~~ 256 #1-4, 6-14, 15bc**

**Date:** May 29/14 **Section 7.4 – The Intersection of Two Lines in a Plane**

**Warm-up:**

**Ex. 1.** Given the scalar equation for a line is $5x-2y+2=0$, find the:

**a)** ***vector***, ***parametric*** and ***symmetric equations*** for the same line.

**b)** ***acute*** angle this line makes with the line $x=2-3s,\ y=-3+s,\ s\in\Re$.

**c)** ***point of intersection*** of this line with the line $x=2-3s,\ y=-3+s,\ s\in\Re$.

a) $\vec{r}=\vec{r}_1+t\vec{m},\ t\in R$

$\because \vec{r}_1=(0,1)$ ; $\vec{m}=(2,5)$

$\therefore$ the vector equation is

$\vec{r}=(0,1)+t(2,5),\ t\in R$

parametric equations:

$x=2t$

$y=1+5t$

if $x=0$, $-2y+2=0$

$-2y=-2$

$y=1$

$\vec{n}=(5,-2)$

$\vec{m}=(2,5)$

symmetric equation:

$$\frac{x}{2}=\frac{y-1}{5}$$

b) $L_1$: $x=2t$

$y=1+5t$

$\vec{m}_1=(2,5)$

$L_2$: $x=2-3s$

$y=-3+s$

$\vec{m}_2=(-3,1)$

$$\vec{m}_1\cdot\vec{m}_2=|\vec{m}_1||\vec{m}_2|\cos\theta$$

$$(2,5)\cdot(-3,1)=\sqrt{(2)^2+(5)^2}\cdot\sqrt{(-3)^2+(1)^2}\cos\theta$$

$$-6+5=\sqrt{29}\cdot\sqrt{10}\cos\theta$$

$$\cos\theta=\frac{-1}{\sqrt{29}\cdot\sqrt{10}}$$

$$\theta\doteq 93^\circ$$

$\therefore$ the acute angle between the lines is about $87^\circ$

c) Find the scalar equation for $x=2-3s,\ y=-3+s$

$$\frac{x-2}{-3}=\frac{y+3}{1}$$

$x-2=-3y-9$

$x+3y+7=0$

Solve: $5x-2y=-2$ ①

$x+3y=-7$ ②

Eliminate $x$

①×1 $5x-2y=-2$

②×5 $5x+15y=-35$

Subtract

$-17y=33$

$y=-\frac{33}{17}$

Sub $y=-\frac{33}{17}$ in ②

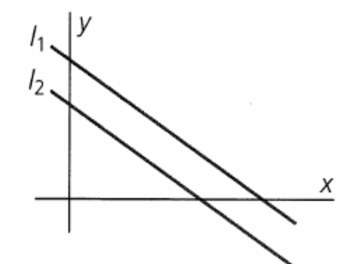
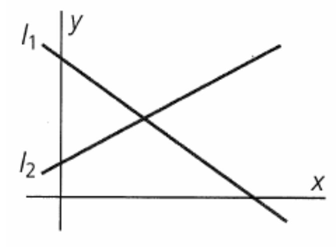
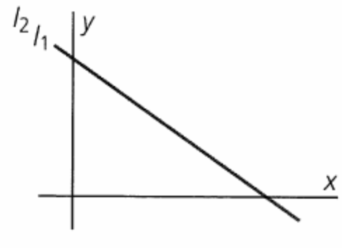
$x-\frac{99}{17}=-7$

$x=-\frac{119}{17}+\frac{99}{17}$

$x=-\frac{20}{17}$

$\therefore$ the pt. of intersection is $\left(-\frac{20}{17},-\frac{33}{17}\right)$.

**SUMMARY:** Two lines in a plane can intersect in one of three possible ways.

| **parallel and distinct** | **intersecting** | **parallel and coincident** |
|---|---|---|
|  |  |  |
| There is no intersection. | The intersection is a single point. | The intersection is the line. |



**Ex. 2.** Determine whether the following pairs of lines are parallel and coincident, parallel and distinct, or neither. Find the intersection if possible.

a) $\vec{r} = (6,-2) + t(-3,2)$ and $\vec{r} = (-12,10) + s(6,-4)$ ($L_1$ and $L_2$)

$\vec{m}_1 = (-3,2)$ & $\vec{m}_2 = (6,-4)$

$\because \vec{m}_2 = k\vec{m}_1$, $k = -2$

$\therefore$ the lines are parallel

ie. parallel & coincident or parallel & distinct.

Sub $(6,-2)$ for $\vec{r}$ in $L_2$

$(6,-2) = (-12,10) + s(6,-4)$

$(6,-2) - (-12,10) = s(6,-4)$

$(18,-12) = s(6,-4)$

$\therefore s = 3$

$\therefore$ the lines are coincident & the intersection is the line $\vec{r} = (6,-2) + t(-3,2)$.

b) $x = 3 - 2t,\ y = 5 + t$ and $x = 5 + 2s,\ y = 3 - s$ ($L_1$: and $L_2$:)

$\vec{m}_1 = (-2,1)$ & $\vec{m}_2 = (2,-1)$

$\because \vec{m}_2 = k\vec{m}_1$, $k = -1$

$\therefore$ the lines are parallel.

sub $(3,5)$ ($x$, $y$) in $L_2$

$3 = 5 + 2s$ & $5 = 3 - s$

$-2 = 2s$    $s = -2$

$s = -1$

$\because$ the values for $s$ are different,

$\therefore$ the lines are parallel & distinct.

There is no intersection.

c) $\dfrac{x-6}{-3} = \dfrac{y+1}{2}$ and $\dfrac{x-5}{2} = \dfrac{y-3}{2}$ ($L_1$ and $L_2$)

$\vec{m}_1 = (-3,2)$ & $\vec{m}_2 = (2,2)$

$\because \vec{m}_2 \neq k\vec{m}_1$, $\therefore$ the lines intersect at a point.

$L_1$: $2x - 12 = -3y - 3$

$2x + 3y = 9$

$L_2$: $2x - 10 = 2y - 6$

$2x - 2y = 4$

Solve.

$2x + 3y = 9$ ①

$2x - 2y = 4$ ②

① − ②

$5y = 5$

$y = 1$

sub $y = 1$ in ①

$2x + 3 = 9$

$2x = 6$

$x = 3$

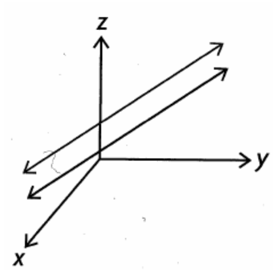
$\therefore$ the lines intersect at the point $(3,1)$.

**HW: pg. 263 #2, 3ab, 8; Worksheet on Equations of Lines**

**Date:** May 30/14

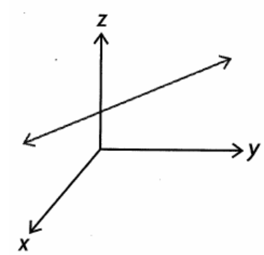
## Section 7.4 – The Intersection of Two Lines in 3-Space

**Parallel & Distinct**



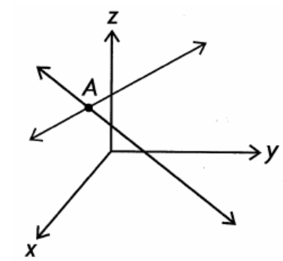

There is no intersection.

**Parallel & Coincident**



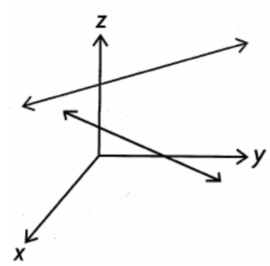

The intersection is the line.

**Intersecting**




The intersection is a point.

**Skew**




There is no intersection.

**Examples**

Find the intersection of the following pairs of lines, if any exist:

**a)** $L_1 : \vec{r} = (-3, 2, 1) + t(-4, -6, -2)$

$L_2 : \vec{r} = (5, -4, 3) + s(2, 3, 1)$

$\vec{m}_1 = (-4, -6, -2)$

$\vec{m}_2 = (2, 3, 1)$

$\because \vec{m}_1 = k\vec{m}_2, k = -2$

$\therefore$ the lines are parallel.

sub $(-3, 2, 1)$ in for $\vec{r}$ in $L_2$.

$(-3, 2, 1) = (5, -4, 3) + s(2, 3, 1)$

$(-3, 2, 1) - (5, -4, 3) = s(2, 3, 1)$

$(-8, 6, -2) = s(2, 3, 1)$

$s = \emptyset$

$\therefore$ the lines are parallel and distinct and there is no intersection.

**b)** $L_1 : \vec{r} = (4, 3, 7) + t(3, -1, 2)$

$L_2 : \vec{r} = (-5, 6, 1) + u(-6, 2, -4)$

$\vec{m}_1 = (3, -1, 2)$

$\vec{m}_2 = (-6, 2, -4)$

$\because \vec{m}_2 = k\vec{m}_1, k = -2$

$\therefore$ the lines are parallel.

Sub $(4, 3, 7)$ in for $\vec{r}$ in $L_2$

$(4, 3, 7) = (-5, 6, 1) + u(-6, 2, -4)$

$(4, 3, 7) - (-5, 6, 1) = u(-6, 2, -4)$

$(9, -3, 6) = u(-6, 2, -4)$

$-6u = 9$ ; $2u = -3$ ; $-4u = 6$

$u = -\frac{3}{2}$ $\quad u = -\frac{3}{2}$ $\quad u = -\frac{3}{2}$

$\therefore u = -\frac{3}{2}$

$\therefore$ the lines are parallel and coincident. The intersection is the line $\vec{r} = (4, 3, 7) + t(3, -1, 2)$.

$L_1$ $L_2$

**c)** $\dfrac{x-2}{4}=\dfrac{y-7}{3}=\dfrac{z-15}{2}$ and $\dfrac{x+3}{3}=\dfrac{y-13}{-1}=\dfrac{z-2}{5}$

$\vec{m}_1=(4,3,2)$ & $\vec{m}_2=(3,-1,5)$

$\because \vec{m}_2 \neq k\vec{m}_1$ $\therefore$ the lines interesect at a pt. or are skew.

For $L_1$: $x=2+4t$, $y=7+3t$, $z=15+2t$

For $L_2$: $x=-3+3s$, $y=13-s$, $z=2+5s$

Set parametric equations

$2+4t=-3+3s \rightarrow 4t-3s=-5$ ①

$7+3t=13-s \rightarrow 3t+s=6$ ②

$15+2t=2+5s \rightarrow 2t-5s=-13$ ③

Solve ① & ③

Eliminate $t$

①×1 $4t-3s=-5$

③×2 $4t-10s=-26$

subtract $7s=21$

$\boxed{s=3}$

sub $s=3$ in ①

$4t-9=-5$

$4t=4$

$\boxed{t=1}$

For $L_1$ if $t=1$

$x=6$

$y=10$

$z=17$

Check $s=3, t=1$ in ②

L.S. $=3t+s$ $=3(1)+3$ $=6$

R.S. $=6$

$\because$ L.S.=R.S.

$\therefore$ the lines intersect at the pt. $(6,10,17)$.

**d)** $\vec{r}=(1,2,5)+t(4,3,-1)$ : $L_1$

$\vec{r}=(-3,-1,3)+s(1,-2,4)$ : $L_2$

$\vec{m}_1=(4,3,-1)$ & $\vec{m}_2=(1,-2,4)$

$\because \vec{m}_2 \neq k\vec{m}_1$ $\therefore$ the lines intersect or are skew.

For $L_1$: $x=1+4t$, $y=2+3t$, $z=5-t$

For $L_2$: $x=-3+s$, $y=-1-2s$, $z=3+4s$

Set parametric equations equal

$1+4t=-3+s \rightarrow 4t-s=-4$ ① Solve

$2+3t=-1-2s \rightarrow 3t+2s=-3$ ② Solve

$5-t=3+4s \rightarrow -t-4s=-2$ ③ ← check

Solve ① & ②

Eliminate $s$

①×2 $8t-2s=-8$

②×1 $3t+2s=-3$

Add $11t=-11$

$\boxed{t=-1}$

sub $t=-1$ in ①

$4(-1)-s=-4$

$-4-s=-4$

$-s=0$

$\boxed{s=0}$

Check $s=0$ & $t=-1$ in ③

L.S. $=-t-4s$ $=-(-1)-4(0)$ $=1$

R.S. $=-2$

$\because$ L.S. $\neq$ R.S.

$\therefore$ there is no intersection since the lines are skew.

**HW: pg. 263 #1, 3cd, 4-7, 11;** $\boxed{\text{Review pg. 266 \#1-15}}$