Date: Feb. 5/14

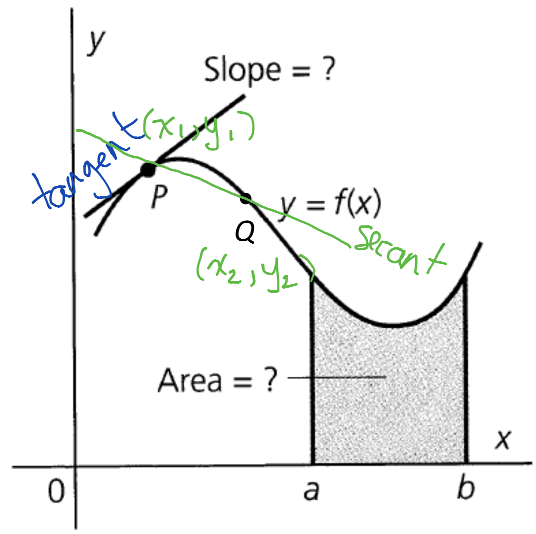
UNIT 1 – INTRODUCTION TO CALCULUS

Section 3.1 – The Slope of a Tangent

What Is Calculus?

Two simple geometric problems originally led to the development of what is now called calculus. Both problems can be stated in terms of the graph of a function $y = f(x)$.

- **The problem of tangents:** What is the value of the slope of the tangent to the graph of a function at a given point P?
- **The problem of areas:** What is the area under a graph of a function $y = f(x)$ between $x = a$ and $x = b$?



REVIEW: Secant Lines and Tangent Lines

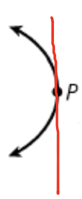
The ***tangent line*** touches the curve at one point, and is the straight line that most resembles the graph near that point. Its slope tells how steep the graph is at the point of tangency.

The ***secant line*** passes through more than one point on the curve.

$$m_{secant} = \frac{y_2 - y_1}{x_2 - x_1}$$

Ex. 1. Draw an approximate tangent for each curve at point P.

a.



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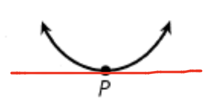
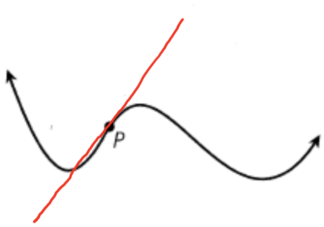


c.

d.



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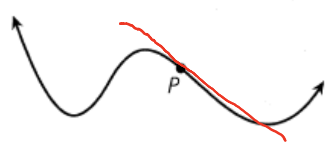

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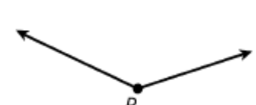
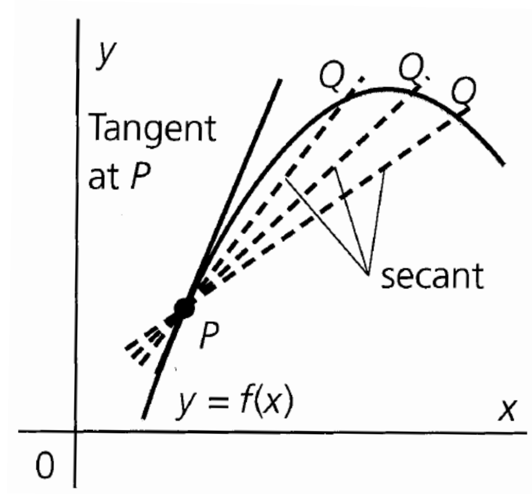


no tangent can be drawn at a corner or cusp

GOAL: Develop a method for determining the **slope of a tangent** at a given point on a curve.

To find the equation of a tangent to a curve at a given point, we first need to know the slope of the tangent. What can we do when we only have one point? We proceed as follows:

Consider a curve $y = f(x)$ and a point P that lies on the curve. Now consider another point Q on the curve. The line joining P to Q is called a **secant**. Think of Q as a moving point that slides along the curve towards P, so that the slope of the secant PQ becomes a progressively better estimate of the slope of the tangent at P.

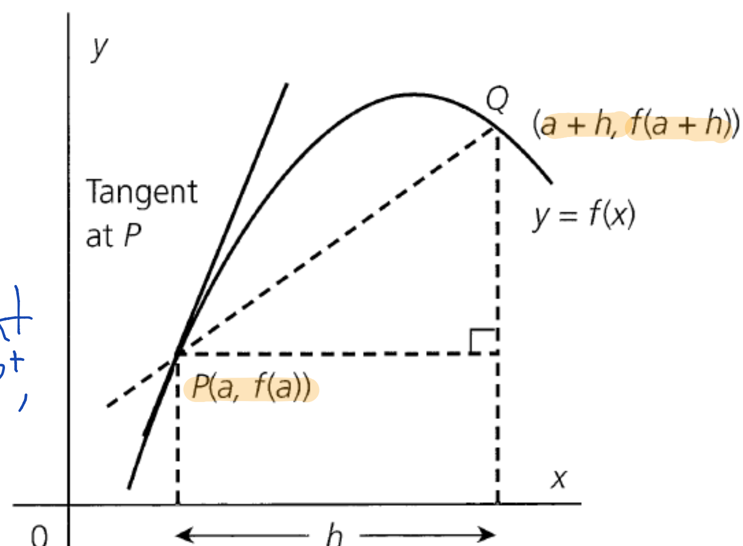


This suggests the following definition of the slope of the tangent:

> The **slope of the tangent** to a curve at a point P is the limiting slope of the secant PQ as the point Q slides along the curve towards P. In other words, the **slope of the tangent** is said to be the **limit** of the **slope of the secant** as Q approaches P along the curve

The Slope of a Tangent at an Arbitrary Point

We can now generalize the method used above to derive a formula for the slope of the tangent to the graph of any function $y = f(x)$.



$P\,(\underset{x_1}{a}, \underset{y_1}{f(a)})$; $Q(\underbrace{a+h}_{x_2}, \underbrace{f(a+h)}_{y_2})$

For secant PQ,

$$m_{PQ} = \frac{f(a+h) - f(a)}{a+h - a}$$

For tangent at P, $Q \to P^+$, so $h \to 0$

$$\therefore m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

$$\therefore m_t = \lim_{h \to 0} m_{PQ}$$

or

$$\boxed{m_t = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

"The limit as $h \to 0$ of..."

The **slope of the tangent** to the graph $y = f(x)$ at a specific point $P(a, f(a))$ is:

$$m_t = \lim_{h \to o} \frac{f(a+h) - f(a)}{h}$$

Ex. 2. Find the slope of the tangent to each curve at the point whose x-value is given. Illustrate your solution graphically.

a) $f(x) = -x^2$, *at* $(2, -4)$ ← a

$$m_t = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{-(2+h)^2 - (-4)}{h}$$

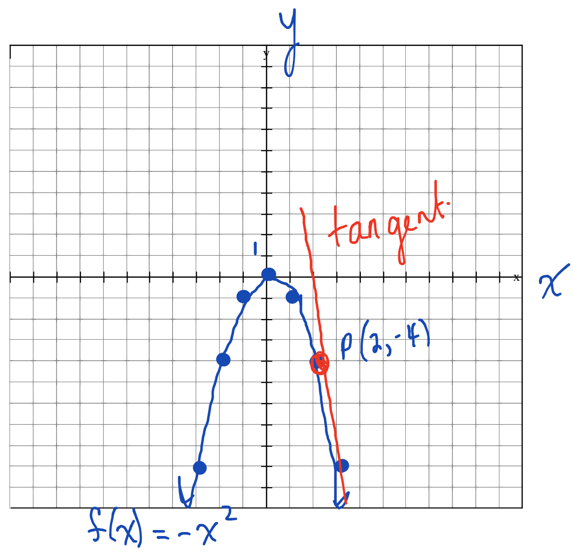
$$= \lim_{h \to 0} \frac{-(2+h)(2+h) + 4}{h}$$

$$= \lim_{h \to 0} \frac{-4 - 4h - h^2 + 4}{h}$$

$$= \lim_{h \to 0} \frac{-4h - h^2}{h}$$

$$= \lim_{h \to 0} (-4 - h)$$ ← brackets required

$$= -4$$



$\therefore m_t = -4$ at $(2, -4)$.

b) $y = 2\sqrt{x+4}$, *at* $x = -3$ ← a

Let $f(x) = 2\sqrt{x+4}$

$$m_t = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}$$

$$= \lim_{h \to 0} \frac{2\sqrt{(-3+h)+4} - (2)}{h}$$

$$= \lim_{h \to 0} \frac{2\sqrt{1+h} - 2}{h} \cdot \frac{2\sqrt{1+h} + 2}{2\sqrt{1+h} + 2}$$

$$= \lim_{h \to 0} \frac{4(1+h) - 4}{h(2\sqrt{1+h} + 2)}$$

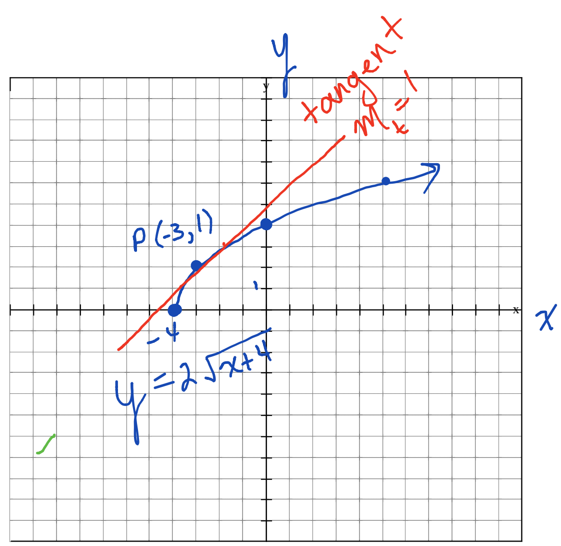
$$= \lim_{h \to 0} \frac{4 + 4h - 4}{h(2\sqrt{1+h} + 2)}$$

$$= \lim_{h \to 0} \frac{4h}{h(2\sqrt{1+h} + 2)}$$

$$= \lim_{h \to 0} \frac{4}{2\sqrt{1+h} + 2}$$

$$= \frac{4}{4}$$

$$= 1$$



$\therefore m_t = 1$ at $x = -3$.

c) $y=\frac{1}{x}$, *at* $x=1$

Let $f(x)=\frac{1}{x}$

$$m_t=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}\cdot\frac{(1+h)}{(1+h)}$$

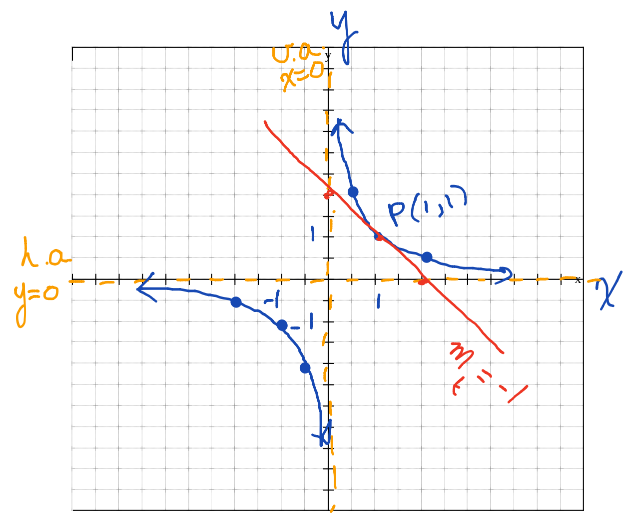
$$=\lim_{h\to 0}\frac{1-1-h}{h(1+h)}$$

$$=\lim_{h\to 0}\frac{-h}{h(1+h)}$$

$$=\lim_{h\to 0}\frac{-1}{1+h}$$

$$=-1$$

$\therefore m_t=-1$ at $x=1$.



$*\ (h-2)(h-2)(h-2)$
$=(h-2)(h^2-4h+4)$
$=h^3-6h^2+12h-8$

$y=x^3-4x$
$y=x(x^2-4)$
$y=x(x-2)(x+2)$
x-ints are 0, 2 & -2
all single roots

Ex. 3. Find the slope and equation of the tangent to $y=x^3-4x$, *at* $x=-2$. Illustrate your solution graphically.

Let $f(x)=x^3-4x$

$$m_t=\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$$

$$=\lim_{h\to 0}\frac{(h-2)^3-4(h-2)-(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^3-6h^2+12h-8-4h+8}{h}$$

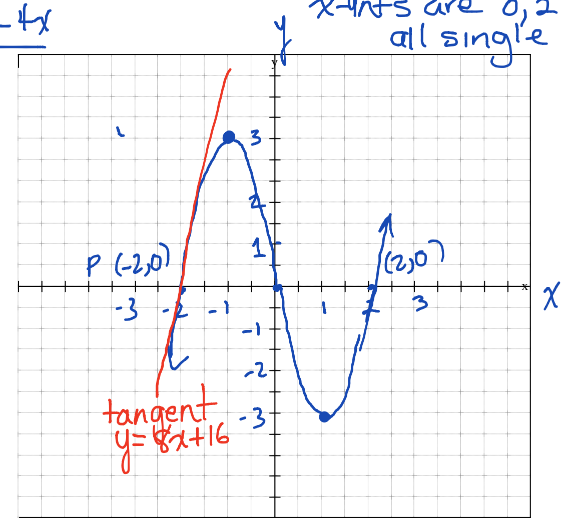
$$=\lim_{h\to 0}\frac{h^3-6h^2+8h}{h}$$

$$=\lim_{h\to 0}(h^2-6h+8)$$

$$=(0)^2-6(0)+8$$

$$=8$$

x	$y=x^3-4x$
-3	-15
* -2	0
-1	3
* 0	0
1	-3
* 2	0
3	15



For tangent
$m_t=8$; $P(-2,0)$; $b=\underline{16}$

Find b:
$y=mx+b$
$0=8(-2)+b$
$0=-16+b$
$16=b$

$\therefore$ the slope of the tangent is 8 and the equation is $y=8x+16$.

HW: p. 84 #8ac, 9ac, 10ac, *11abefgh (For #11, find the equation of the tangent & graph)

Date: Feb. 6/14

Warm-up

Recall:

The **slope of the tangent** to the graph $y = f(x)$ at a specific point $P(a, f(a))$ is:

$$m_t = \lim_{h \to o} \frac{f(a+h) - f(a)}{h}$$

Ex. 1. Find the equation of the tangent and normal to $g(x) = \dfrac{2}{3\sqrt{1-x}}$, *at* $x = -3$.

Illustrate graphically.

For the domain
$1 - x > 0$
$-x > -1$
$x < 1$

v.a.: $x = 1$
h.a.: $y = 0$

x	$g(x)$
1	X
0	$\frac{2}{3}$
-3	$\frac{1}{3}$
-8	$\frac{2}{9}$

$$m_t = \lim_{h \to 0} \frac{g(-3+h) - g(-3)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{3\sqrt{1-(-3+h)}} - \frac{2}{6}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{3\sqrt{4-h}} - \frac{1}{3}}{h} \cdot \frac{3\sqrt{4-h}}{3\sqrt{4-h}}$$

$$= \lim_{h \to 0} \frac{(2 - \sqrt{4-h})}{3h(\sqrt{4-h})} \cdot \frac{(2+\sqrt{4-h})}{(2+\sqrt{4-h})}$$

$$= \lim_{h \to 0} \frac{4 - (4-h)}{3h(\sqrt{4-h})(2+\sqrt{4-h})}$$

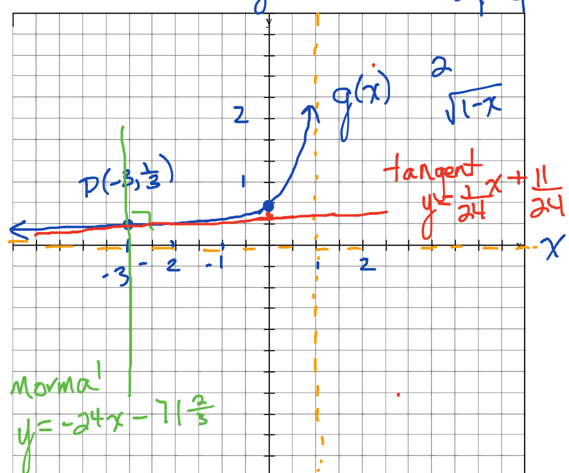
$$= \lim_{h \to 0} \frac{h}{3h(\sqrt{4-h})(2+\sqrt{4-h})}$$

$$= \lim_{h \to 0} \frac{1}{3(\sqrt{4-h})(2+\sqrt{4-h})}$$

$$= \frac{1}{3(2)(4)}$$

$$= \frac{1}{24}$$

h.a. $y = 0$



v.a. $x = 1$

For the tangent,
$m_t = \frac{1}{24}$; $(-3, \frac{1}{3})$; $b = $ ____

Find b: $\frac{1}{3} = \frac{1}{24}(-3) + b$

$\frac{1}{3} = -\frac{1}{8} + b$

$\frac{8}{24} + \frac{3}{24} = b$

$\frac{11}{24} = b$

$\therefore$ the equation of the tangent is $y = \frac{1}{24}x + \frac{11}{24}$ or $x - 24y + 11 = 0$

For the normal,
$m_n = -24$; $(-3, \frac{1}{3})$; $b = $ ____

Find b: $\frac{1}{3} = -24(-3) + b$

$\frac{1}{3} = 72 + b$

$\frac{1}{3} - 72 = b$

$-71\frac{2}{3} = b$

$-\frac{215}{3} = b$

$\therefore$ the equation of the normal at $x = -3$ is $y = -24x - \frac{215}{3}$ or $72x + 3y + 215 = 0$

We can also find an expression for the slope of the tangent to a curve at any point in terms of *x*.

The **slope of the tangent** to the graph $y = f(x)$ at any point $P(x, f(x))$ is:

$$m_t = \lim_{h \to o} \frac{f(x+h) - f(x)}{h}$$

Ex. 2. Graph the function $h(x) = -2x^2 - 8x - 6$, and determine the following:

a) the slopes of the tangents to the curve at the points whose *x*-coordinates are given

i) x **ii)** 0 **iii)** -2

b) the equation of the tangent with a slope of 4

$h(x) = -2x^2 - 8x - 6$

At the vertex: $x = \frac{-b}{2a}$ $= \frac{-(-8)}{2(-2)}$ $= -2$

$h(-2) = -2(-2)^2 - 8(-2) - 6$ $= 2$

$V(-2, 2)$ opens down, congruent to $y = 2x^2$

a) i) Let $f(x) = -2x^2 - 8x - 6$

Find m_t at $(x, f(x))$

$m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-2(x+h)^2 - 8(x+h) - 6 - (-2x^2 - 8x - 6)}{h}$

$= \lim_{h \to 0} \frac{\cancel{-2x^2} - 4xh - 2h^2 \cancel{- 8x} - 8h \cancel{- 6} + \cancel{2x^2} + \cancel{8x} + \cancel{6}}{h}$

$= \lim_{h \to 0} \frac{-4xh - 2h^2 - 8h}{h}$

$= \lim_{h \to 0} (-4x - 2h - 8)$

$= -4x - 8$

$\boxed{\therefore m_t = -4x - 8}$

ii) at $x = 0$

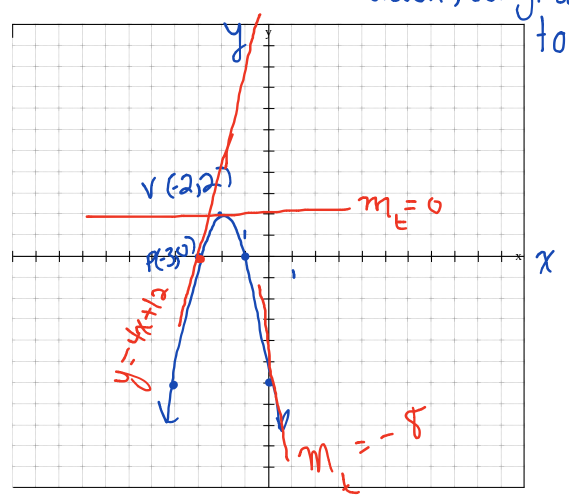
$m_t = -4(0) - 8$

$= -8$

iii) at $x = -2$

$m_t = -4(-2) - 8$

$= 0$



b) Find x if

$m_t = 4$

$-4x - 8 = 4$

$-4x = 12$

$x = -3$

$f(-3) = 0$

For tangent,

$m_t = 4$; $(-3, 0)$; $b = \_\_$

Find b: $0 = 4(-3) + b$

$0 = -12 + b$

$12 = b$

$\therefore$ the equation of the required tangent is $y = 4x + 12$.

HW: Worksheet on Tangents and Slopes of Tangents #1efgh, 2acdf, 5abd, *6bc, *7, *8a, *9

***Follow Ex. 2. from the note.**

Date: Feb. 7/14

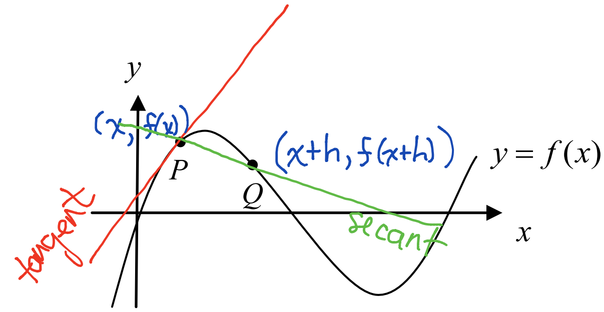
Section 3.2 – Rates of Change

Recall:

$m_t = \lim_{h\to 0} m_{PQ}$

$$\boxed{m_t = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

" rate of change of y $\underbrace{\text{with respect}}_{\text{w.r.t.}}$ to x"

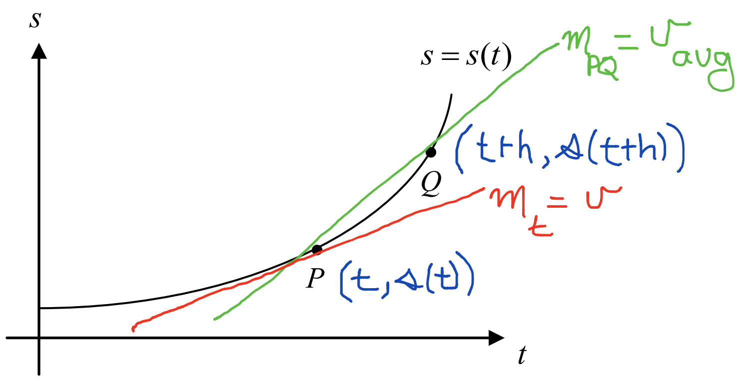


We begin by considering a familiar ***rate of change*** – the ***velocity*** of a moving object.
For example, a truck that travels a ***distance*** of 300 km in a ***time*** of 4 h has an ***average velocity*** of $\frac{300}{4}$ or 75 km/h

The ***velocity*** at an instant of time, called the ***instantaneous velocity*** will often vary.
This ***velocity*** would be the *speedometer reading* at an instant of time.

Note: The ***speed*** of an object is the absolute value of its velocity. It indicates how fast an object is moving, whereas ***velocity*** indicates both ***speed*** and ***direction*** relative to a given coordinate system.

Distance vs Time



Average Velocity

average velocity = m_{secant}

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s_2 - s_1}{t_2 - t_1}$$

$$\therefore \underbrace{v_{avg} = \frac{s(t_2)-s(t_1)}{t_2-t_1}}$$

" average rate of change of distance w.r.t. time "

Instantaneous Velocity

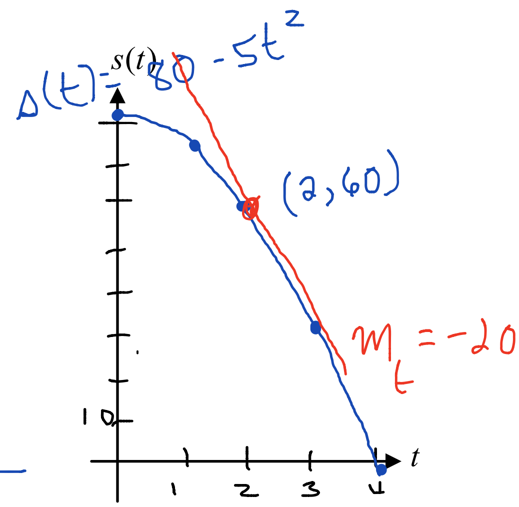
velocity = m_t

$$= \lim_{h\to 0} m_{PQ}$$

$$\therefore \underbrace{v = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h}}$$

" rate of change of distance w.r.t. time "

Ex. 1. A pebble is dropped from a cliff of height 80 m. After t seconds, it is s metres above the ground, where $s(t)=80-5t^2$, $0\le t\le 4$. Find the pebbles's:



a) ***average velocity***

i) for $t\in[2,4]$

ii) during the third second

a) i) $v_{aug} = \dfrac{s(4)-s(2)}{4-2}$

$= \dfrac{0-60}{2}$

$=-30$

$\therefore$ the average velocity of the pebble is -30 m/s

ii) between $t=2$ & $t=3$

$v_{aug} = \dfrac{s(3)-s(2)}{3-2}$

$= \dfrac{35-60}{1}$

$=-25$

$\therefore$ the average velocity is -25 m/s.

b) ***velocity*** at $t=2$

$v = m_t$

$= \lim\limits_{h\to 0} \dfrac{s(2+h)-s(2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{80-5(2+h)^2 - 60}{h}$

$= \lim\limits_{h\to 0} \dfrac{80-20-20h-5h^2-60}{h}$

$= \lim\limits_{h\to 0} \dfrac{-20h-5h^2}{h}$

$= \lim\limits_{h\to 0} (-20-5h)$

$=-20$

$\therefore$ at $t=2$ the velocity of the pebble is -20 m/s

1st second

t	$s(t)$
0	80
1	75
2	60
3	35
4	0

Ex. 2. A dragster races down a 400 m strip in 8 s. Its distance in metres from the starting line after t seconds is ~~$s(t)=80-5t^2$~~. $s(t)=6t^2+2t$

a) Find its ***average velocity*** over the first 4 seconds.

b) Find its ***velocity*** as it crosses the finish line. Find m_t at $t=8$

c) Determine when its ***velocity*** is 38 m/s. Find t if $m_t=38$

a) between $t=0$ & $t=4$

$v_{aug} = \dfrac{s(4)-s(0)}{4-0}$

$= \dfrac{104-0}{4}$

$=26$

$\therefore$ its average velocity is 26 m/s.

b) Find m_t at $(t, s(t))$

$v = m_t$

$= \lim\limits_{h\to 0} \dfrac{s(t+h)-s(t)}{h}$

$= \lim\limits_{h\to 0} \dfrac{6(t+h)^2+2(t+h)-(6t^2+2t)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{6t^2}+12th+6h^2+\cancel{2t}+2h-\cancel{6t^2}-\cancel{2t}}{h}$

$= \lim\limits_{h\to 0} \dfrac{12th+6h^2+2h}{h}$

$= \lim\limits_{h\to 0} (12t+6h+2)$

$=12t+2$

$\boxed{\therefore v(t)=12t+2}$

$v(8)=98$

$\therefore$ it crosses the finish line with a velocity of 98 m/s.

c) Find t if

$v(t)=38$

$12t+2=38$

$12t=36$

$t=3$

$\therefore$ the velocity is 38 m/s at $t=3$ seconds.

Ex. 3. When a certain drug is injected into a muscle, the muscle contracts an amount, $C(x)$ in millimetres, for an amount, x millitres, of the drug, where $C(x) = \dfrac{4}{12+2x}$.

Find the ***exact*** rate of change of the amount of muscle contraction if 50 ml of the drug is injected.

Find m_t at $x = 50$

rate of change $= m_t$ at $x = 50$

$4\overline{)112}$

$$= \lim_{h\to 0} \frac{C(50+h) - C(50)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{12+2(50+h)} - \frac{4}{112}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{112+2h} - \frac{1}{28}}{h} \cdot \frac{28(112+2h)}{28(112+2h)}$$

$$= \lim_{h\to 0} \frac{112 - (112+2h)}{28h(112+2h)}$$

$$= \lim_{h\to 0} \frac{-2h}{28h(112+2h)}$$

$$= \lim_{h\to 0} \frac{-1}{14(112+2h)}$$

$$= \frac{-1}{(14)(112)}$$

$$= \frac{-1}{1568}$$

$$\begin{array}{r} 112 \\ \times\ 14 \\ \hline 448 \\ 1120 \\ \hline 1568 \end{array}$$

$\therefore$ the rate of change of the amount of muscle contraction is $\dfrac{-1}{1568}$ mm/ml.

HW: p. 92 #1 to 3, 7 to 10, 12, 13