Date: Feb. 19/14

UNIT 2 – DERIVATIVES

Section 4.1 – The Derivative Function

Ex. 1. Find the slope of the tangent to $y = \frac{1}{\sqrt{x}}$ at any point (x, y).

Let $f(x) = \frac{1}{\sqrt{x}}$

$$m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\sqrt{x}\,(\sqrt{x+h})}{\sqrt{x}\,(\sqrt{x+h})}$$

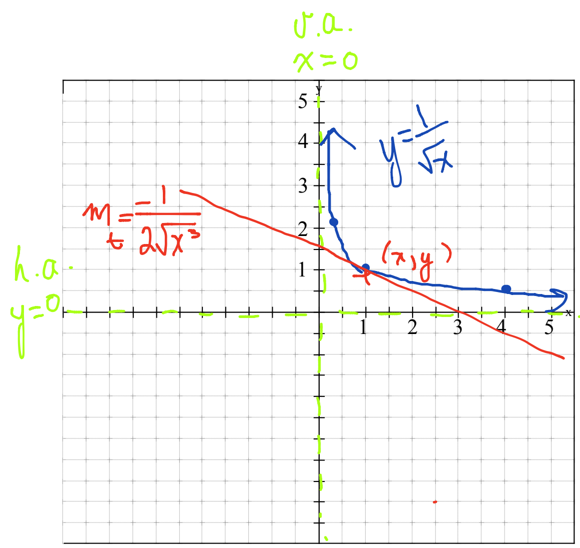
$$= \lim_{h \to 0} \frac{(\sqrt{x} - \sqrt{x+h})}{h \cdot \sqrt{x} \cdot \sqrt{x+h}} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{h \cdot \sqrt{x} \cdot \sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x} \cdot \sqrt{x+h} \cdot (\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x} \cdot \sqrt{x} \cdot 2\sqrt{x}}$$

$$= \frac{-1}{2x\sqrt{x}} \quad \text{or} \quad = \frac{-1}{2(\sqrt{x})^3} \quad \text{or} \quad \frac{-1}{2\sqrt{x^3}}$$



We say that $\frac{-1}{2\sqrt{x^3}}$ is the "***derivative***" of $y = \frac{1}{\sqrt{x}}$.

$\therefore$ The ***derivative*** is an expression for the ***slope of the tangent*** to a curve.

Notation:

✱ If $f(x) = \frac{1}{\sqrt{x}}$ then $f'(x) = \frac{-1}{2\sqrt{x^3}}$ ***"f prime at x"***

✱ If $y = \frac{1}{\sqrt{x}}$ then $\frac{dy}{dx} = \frac{-1}{2\sqrt{x^3}}$ ***"dy by dx" or "the derivative of y with respect to x"***

If $y = \frac{1}{\sqrt{x}}$ then $y' = \frac{-1}{2\sqrt{x^3}}$ ***"y prime"***

Note: $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2\sqrt{x^3}}$ reads as ***"the derivative of*** $\frac{1}{\sqrt{x}}$ ***w.r.t. x"***

The ***derivative*** of $y = f(x)$ is: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Ex. 2. Find the derivative of the following functions from ***first principles***.

a) $f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$=\lim_{h\to 0}(2x+h)$$
$$=2x$$

$\therefore$ If $f(x)=x^2$ then $f'(x)=2x$

b) $y=\frac{1}{x^2}$, Let $f(x)=\frac{1}{x^2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\cdot\frac{x^2(x+h)^2}{x^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{hx^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}\div\frac{h}{h}$$
$$=\lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}$$
$$=\frac{-2x}{x^4}$$
$$=-\frac{2}{x^3}$$

$\therefore$ If $y=\frac{1}{x^2}$ then $\frac{dy}{dx}=-\frac{2}{x^3}$

Ex. 3. An object moves in a straight line with its position at time t seconds given by $s(t)=8t-t^2$ where s is measured in metres.

a) Find the ***initial*** velocity. Find v if $t=0$

b) Determine when the object is at rest. Find t if $v=0$

c) Find the ***average*** velocity during the third second. Find m_{secant} between $t=2$ & $t=3$

a) $v(t)=m_t$
$$=s'(t)$$
$$=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}$$
$$=\lim_{h\to 0}\frac{8(t+h)-(t+h)^2-(8t-t^2)}{h}$$
$$=\lim_{h\to 0}\frac{\cancel{8t}+8h-\cancel{t^2}-2th-h^2-\cancel{8t}+\cancel{t^2}}{h}$$
$$=\lim_{h\to 0}(8-2t-h)$$
$$=8-2t$$

$\boxed{\therefore v(t)=8-2t}$

$v(0)=8$

$\therefore$ the initial velocity is 8 m/s.

b) Find t if $v(t)=0$
$$8-2t=0$$
$$-2t=-8$$
$$t=4$$

$\therefore$ At $t=4$ s the object is at rest.

c) $$v_{avg}=\frac{s(3)-s(2)}{3-2}$$
$$=\frac{15-12}{1}$$
$$=3$$

$\therefore$ the average velocity is 3 m/s during the third second.

Ex. 4. Show that the derivative of the absolute value function $f(x) = |x|$ does not exist at $x = 0$. Illustrate your solution graphically.

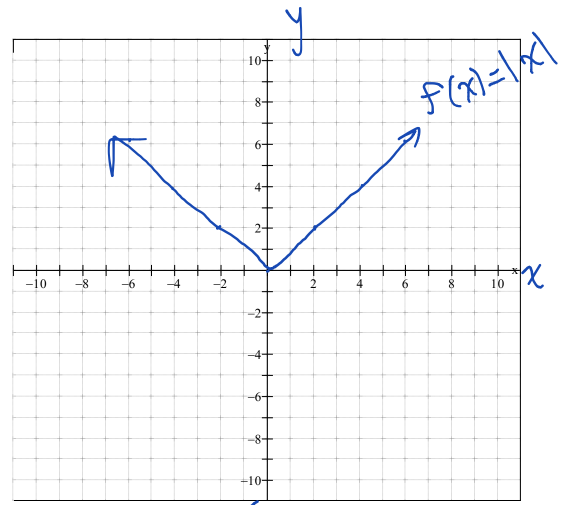
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$



$$f(x) = \begin{cases} -x &, \text{if } x < 0 \\ x &, \text{if } x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -1 &, \text{if } x < 0 \\ 1 &, \text{if } x > 0 \end{cases}$$

L.S.L.

$$\lim_{h \to 0^-} \frac{-h}{h}$$

$$= \lim_{h \to 0^-} (-1)$$

$$= -1$$

R.S.L.

$$\lim_{h \to 0^+} \frac{+h}{h}$$

$$= \lim_{h \to 0^+} 1$$

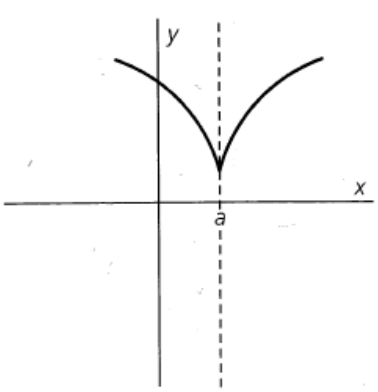
$$= 1$$

$\because$ L.S.L. $\neq$ R.S.L.

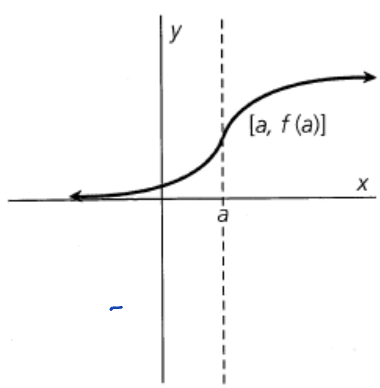
$\therefore \lim_{h \to 0} \frac{|h|}{h}$ does not exist.

The Existence of Derivatives

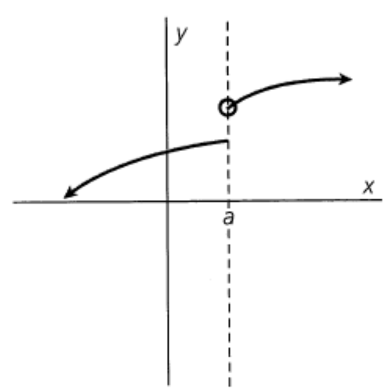
A function f is said to be **differentiable** at a if $f'(a)$ exists. At points where f is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.



Cusp



Vertical Tangent



Discontinuity

HW: p. 130 #1, 4bc, 5abc, 6bc, 7, 8, 9, 10d, 11, 12

Date: Feb. 20/14

Section 4.2 – The Derivatives of Polynomial Functions

Ex. 1. From ***first principles*** find the ***derivative*** of $y = x^2 - 5x + 1$.

Let $f(x) = x^2 - 5x + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) + 1 - (x^2 - 5x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-5x} - 5h \cancel{+1} \cancel{-x^2} \cancel{+5x} \cancel{-1}}{h}$$

$$= \lim_{h \to 0} (2x + h - 5)$$

$$= 2x - 5$$

$\therefore$ If $y = x^2 - 5x + 1$ then $\frac{dy}{dx} = 2x - 5$

$y = x^2 - 5x + 1$

$y = 1x^2 - 5x^1 + 1x^0$

Using the Power Rule on each term,

$\frac{dy}{dx} = 2x^1 - 5x^0 + 0x^{-1}$

$\therefore \frac{dy}{dx} = 2x - 5$

Recall: **i)** if $f(x) = x^2$ then $f'(x) = 2x$ **ii)** if $f(x) = x^3$ then $f'(x) = 3x^2$ **iii)** if $f(x) = x^4$ then $f'(x) = 4x^3$ **iv)** if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Recall:

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3)$

✱ $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$

Ex. 2. From ***first principles*** find the ***derivative*** of $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h) - x][(x+h)^{n-1} + (x+h)^{n-2} \cdot x^1 + (x+h)^{n-3} \cdot x^2 + \dots + (x+h)^2 \cdot x^{n-3} + (x+h)^1 \cdot x^{n-2} + x^{n-1}]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}^{1}[(x+h)^{n-1} + (x+h)^{n-2} \cdot x^1 + (x+h)^{n-3} \cdot x^2 + \dots + (x+h)^2 \cdot x^{n-3} + (x+h)^1 \cdot x^{n-2} + x^{n-1}]}{\cancel{h}_{1}}$$

$$= x^{n-1} + \underbrace{x^{n-2} \cdot x^1} + \underbrace{x^{n-3} \cdot x^2} + \dots + \underbrace{x^2 \cdot x^{n-3}} + \underbrace{x^1 \cdot x^{n-2}} + x^{n-1}$$

$$= \left. x^{n-1} + x^{n-1} + x^{n-1} + \dots + x^{n-1} + x^{n-1} + x^{n-1} \right\} n \text{ terms}$$

$$= n \cdot x^{n-1}$$

$\therefore$ If $f(x) = x^n$ then $f'(x) = n \cdot x^{n-1}$

POWER RULE

If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$

Note: If $y = k$, where k is a constant, $\frac{dy}{dx} = 0$

$y = kx^0$

$\frac{dy}{dx} = 0 \cdot kx^{0-1}$

$= 0$

Ex. 3. Differentiate each function using the **Power Rule**. Use either the ***Leibniz notation*** $\frac{dy}{dx}$ or the ***prime notation*** $f'(x)$, depending on which is appropriate.

a) $y = 4$

$\frac{dy}{dx} = 0$

b) $g(x) = 3x^4 + 12x^3 - 8x^1 + 1$

$g'(x) = 12x^3 + 36x^2 - 8x^0 + 0$

$g'(x) = 12x^3 + 36x^2 - 8$

c) $f(x) = \frac{1}{x}$

$f(x) = x^{-1}$

$f'(x) = -1x^{-2}$

$\therefore f'(x) = -\frac{1}{x^2}$

d) $s = \sqrt{t}$

$s = t^{\frac{1}{2}}$

$\frac{ds}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$

$\frac{ds}{dt} = \frac{1}{2\sqrt{t}}$

e) $y = \left(\frac{x}{2}\right)^2$

$y = \frac{x^2}{4}$

$y = \frac{1}{4}x^2$

$\frac{dy}{dx} = \frac{1}{2}x$

f) $h(t) = -2(t^2 - 3)^2$

$h(t) = -2(t^2-3)(t^2-3)$

$h(t) = -2t^4 + 12t^2 - 18$

$h'(t) = -8t^3 + 24t$

Ex. 4. Find the ***equations*** of the ***tangent*** and ***normal*** to the curve $y = (x-3)^2 - 2$ at $x = 5$.
(Note: The ***normal line*** is ***perpendicular*** to the ***tangent line*** at the point of tangency.)

$y = (x-3)^2 - 2$

$y = x^2 - 6x + 7$

$m_t = \frac{dy}{dx}$

$= 2x - 6$

at $x = 5$

$m_t = 4$; $y = 2$

For tangent,

$m_t = 4$; $(5, 2)$; $b = \_\_$

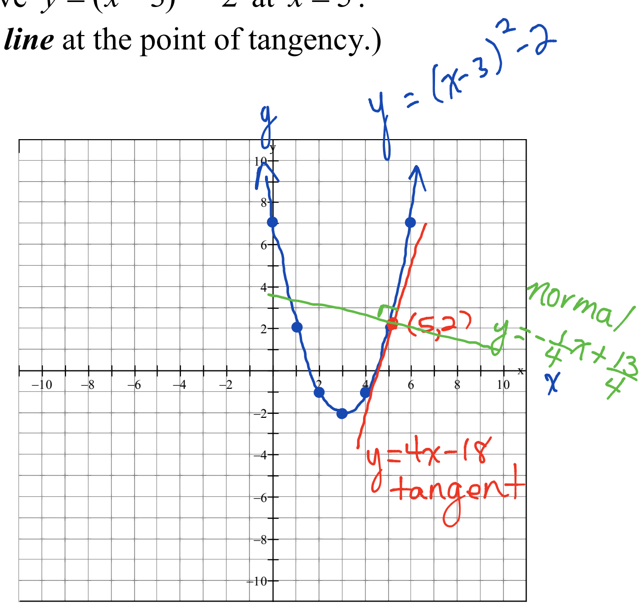
Find b

$2 = 4(5) + b$

$2 = 20 + b$

$-18 = b$

$\therefore$ the equation of the tangent is $y = 4x - 18$.



For the normal,

$m_n = -\frac{1}{4}$; $(5, 2)$; $b = \_\_$

Find b

$2 = -\frac{1}{4}(5) + b$

$\frac{8}{4} = -\frac{5}{4} + b$

$\frac{13}{4} = b$

$\therefore$ the equation of the normal is $y = -\frac{1}{4}x + \frac{13}{4}$

Ex. 5. Find $f'(a)$ for the given function $f(x)$ at the given value of a.

a) $f(x)=\left(1-\frac{2}{x}\right)\left(3-\frac{4}{x}\right)$; $a=1$

$f(x)=3-\frac{10}{x}+\frac{8}{x^2}$

$f(x)=3-10x^{-1}+8x^{-2}$

$f'(x)=10x^{-2}-16x^{-3}$

$f'(x)=\frac{10}{x^2}-\frac{16}{x^3}$

$f'(1)=10-16$
$=-6$

$\therefore f'(1)=-6$

b) $f(x)=\frac{2+\sqrt{x^3}}{\sqrt{4x}}$; $a=4$

$f(x)=\frac{2+x^{\frac{3}{2}}}{\sqrt{4}\cdot\sqrt{x}}$

$f(x)=\frac{2+x^{\frac{3}{2}}}{2x^{\frac{1}{2}}}$

$f(x)=\frac{2x^0}{2x^{\frac{1}{2}}}+\frac{1x^{\frac{3}{2}}}{2x^{\frac{1}{2}}}$

$f(x)=1x^{-\frac{1}{2}}+\frac{1}{2}x^1$

$f'(x)=-\frac{1}{2}x^{-\frac{1}{2}-1}+\frac{1}{2}x^{1-1}$

$f'(x)=-\frac{1}{2}x^{-\frac{3}{2}}+\frac{1}{2}$

$f'(4)=-\frac{1}{2}(4)^{-\frac{3}{2}}+\frac{1}{2}$

$f'(4)=-\frac{1}{2}\times\frac{1}{(\sqrt{4})^3}+\frac{1}{2}$

$f'(4)=-\frac{1}{2}\times\frac{1}{8}+\frac{1}{2}$

$f'(4)=-\frac{1}{16}+\frac{8}{16}$

$\therefore f'(4)=\frac{7}{16}$

HW: p. 138 #2 to 8, 9abd

Date: Feb. 21/14 **Section 4.2 – The Derivatives of Polynomial Functions Continued**

Ex. 1. Find the ***slope*** of the ***tangent*** to the curve $y = \sqrt{3x^3}$ at the point $P\ (3, 9)$.

$y = \sqrt{3x^3}$

$y = \sqrt{3} \cdot \sqrt{x^3}$

$y = \sqrt{3}\, x^{\frac{3}{2}}$

$m_t = \frac{dy}{dx}$

$= \frac{3\sqrt{3}}{2} x^{\frac{1}{2}}$

$m_t = \frac{3\sqrt{3}}{2} \cdot \sqrt{x}$

at $x = 3$

$m_t = \frac{3\sqrt{3}}{2} \cdot \sqrt{3}$

$= \frac{9}{2}$

$\therefore$ the slope of the tangent is $\frac{9}{2}$.

Ex. 2. Find the ***slope*** of the ***normal*** to the curve $y = \left(\sqrt{x} - 2\right)\left(3\sqrt{x} + 8\right)$ at the $x = 4$.

$y = (\sqrt{x} - 2)(3\sqrt{x} + 8)$

$y = 3x + 2\sqrt{x} - 16$

$y = 3x^1 + 2x^{\frac{1}{2}} - 16$

$m_t = \frac{dy}{dx}$

$= 3 + x^{-\frac{1}{2}}$

$m_t = 3 + \frac{1}{\sqrt{x}}$

at $x = 4$

$m_t = 3 + \frac{1}{2}$

$m_t = \frac{7}{2}$

$\therefore m_n = -\frac{2}{7}$

$\therefore$ at $x = 4$ the slope of the normal is $-\frac{2}{7}$.

Ex. 3. Find the values of x so that the tangent to the function $y = \frac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

$x + 16y + 3 = 0$

$\frac{16y}{16} = \frac{-1x}{16} - \frac{3}{16}$

$y = -\frac{1}{16}x - \frac{3}{16}$

slope $= -\frac{1}{16}$

$y = \frac{3}{\sqrt[3]{x}}$

$y = 3x^{-\frac{1}{3}}$

$m_t = \frac{dy}{dx}$

$= -x^{-\frac{4}{3}}$

$= -\frac{1}{\sqrt[3]{x^4}}$

Find x if

$m_t = -\frac{1}{16}$

$-\frac{1}{\sqrt[3]{x^4}} = -\frac{1}{16}$

$\left(\sqrt[3]{x^4}\right)^3 = (16)^3$

$x^4 = 4096$

$x = \pm \sqrt[4]{4096}$

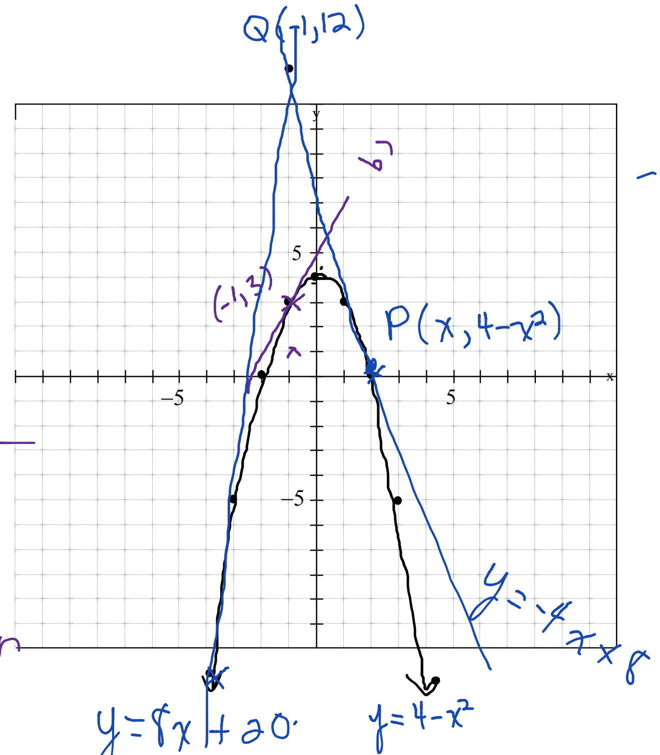
$x = \pm 8$

$\therefore x = \pm 8$.

Ex. 4. Find the equation(s) of the tangent(s) to the curve $y = 4 - x^2$ that pass through the following points and illustrate graphically.

a) $(-1, 1)$ **b)** $(-1, 3)$ **c)** $(-1, 12)$

a) no tangent can be drawn to the curve $y = 4 - x^2$ through $(-1, 1)$



b) $y = 4 - x^2$

$m_t = \frac{dy}{dx}$

$\boxed{m_t = -2x}$

at $x = -1$

$m_t = -2(-1)$
$= 2$

For the tangent, $m_t = 2$; $(-1, 3)$; $b = \_\_$

$3 = 2(-1) + b$

$3 = -2 + b$

$5 = b$

$\therefore$ the equation of the tangent is $y = 2x + 5$

c) Let $P(x, 4 - x^2)$ be the point of tangency through $Q(-1, 12)$.

x_1, y_1 ; x_2, y_2

$m_t = m_{PQ}$

$-2x = \frac{12 - (4 - x^2)}{-1 - x}$

$\frac{-2x}{1} = \frac{8 + x^2}{-1 - x}$

$-2x(-1 - x) = 1(8 + x^2)$

$2x + 2x^2 = 8 + x^2$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$\therefore x = -4$ or $x = 2$

For tangent at $x = -4$

$m_t = -2(-4)$; $(-4, -12)$; $b = \_\_$
$= 8$

$-12 = 8(-4) + b$

$-12 = -32 + b$

$20 = b$

$\therefore$ the equation of the tangent at $x = -4$ is $y = 8x + 20$

at $x = 2$,

$m_t = -4$; $(2, 0)$; $b = \_\_$

$0 = -4(2) + b$

$8 = b$

$\therefore$ the equation of the tangent at $x = 2$ is $y = -4x + 8$.

Follow Ex. 4.

HW: p. 139 #9df, 12, 14, 16, 17, 20, 22, 23