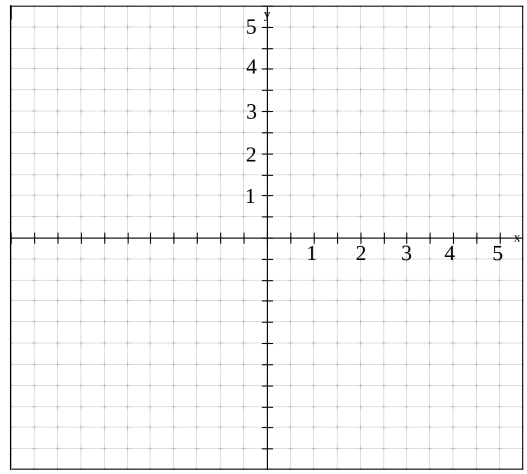
Date:______________

UNIT 2 – DERIVATIVES

Section 4.1 – The Derivative Function

Ex. 1. Find the slope of the tangent to $y = \frac{1}{\sqrt{x}}$ at any point (x, y).



We say that   is the **"*derivative*"** of $y = \frac{1}{\sqrt{x}}$.

$\therefore$ The ***derivative*** is an expression for the ***slope of the tangent*** to a curve.

<u>**Notation**</u>**:**

If $f(x) = \frac{1}{\sqrt{x}}$ then $= \frac{-1}{2\sqrt{x^3}}$ ***"f prime at x"***

If $y = \frac{1}{\sqrt{x}}$ then $= \frac{-1}{2\sqrt{x^3}}$ ***"dy by dx" or "the derivative of y with respect to x"***

If $y = \frac{1}{\sqrt{x}}$ then $= \frac{-1}{2\sqrt{x^3}}$ ***"y prime"***

Note: $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2\sqrt{x^3}}$ reads as ***"the derivative of*** $\frac{1}{\sqrt{x}}$ ***w.r.t. x"***

The ***derivative*** of $y = f(x)$ is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex. 2. Find the derivative of the following functions from ***first principles***.

a) $f(x) = x^2$

b) $y = \dfrac{1}{x^2}$

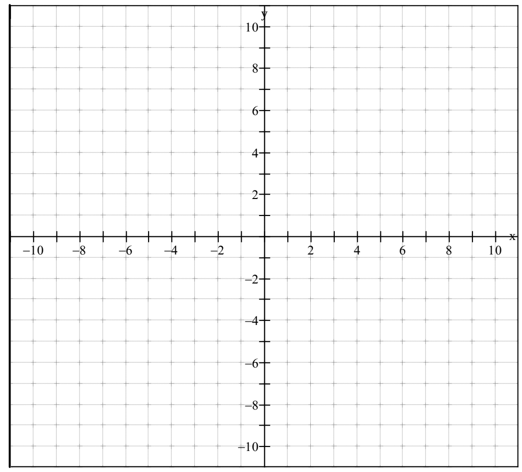
Ex. 3. An object moves in a straight line with its position at time t seconds given by $s(t) = 8t - t^2$ where s is measured in metres.

a) Find the ***initial*** velocity.

b) Determine when the object is at rest.

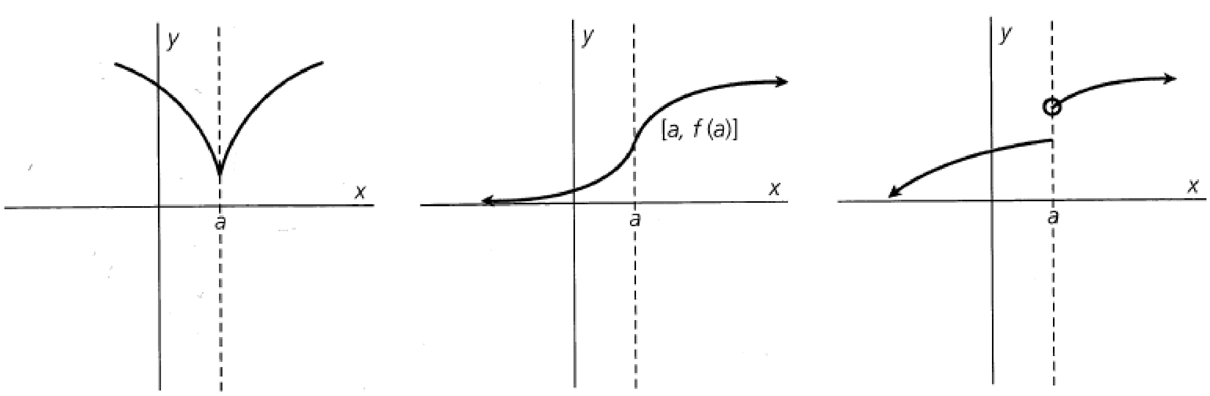
c) Find the ***average*** velocity during the third second.

Ex. 4. Show that the derivative of the absolute value function $f(x) = |x|$ does not exist at $x = 0$. Illustrate your solution graphically.



The Existence of Derivatives

A function f is said to be **differentiable** at a if $f'(a)$ exists. At points where f is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.



Cusp Vertical Tangent Discontinuity

HW: p. 130 #1, 4bc, 5abc, 6bc, 7, 8, 9, 10d, 11, 12

Date:______________ **Section 4.2 – The Derivatives of Polynomial Functions**

Ex. 1. From ***first principles*** find the ***derivative*** of $y = x^2 - 5x + 1$.

Recall: **i)** if $f(x) = x^2$ then $f'(x) =$ **ii)** if $f(x) = x^3$ then $f'(x) =$ **iii)** if $f(x) = x^4$ then $f'(x) =$ **iv)** if $f(x) = x^n$ then $f'(x) =$

Recall: $a^2 - b^2 =$

$a^3 - b^3 =$

$a^4 - b^4 =$

$a^n - b^n =$

Ex. 2. From ***first principles*** find the ***derivative*** of $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

$$= \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

$$= \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

=

=

=

POWER RULE

If $y = ax^n$ then $\dfrac{dy}{dx} =$

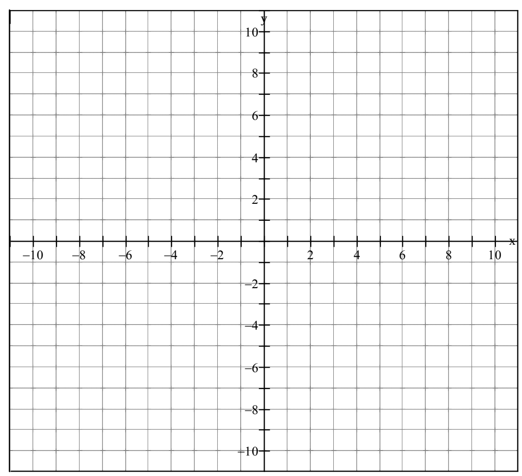
Note: If $y = k$, where k is a constant, $\dfrac{dy}{dx} =$

Ex. 3. Differentiate each function using the **Power Rule**. Use either the ***Leibniz notation*** $\dfrac{dy}{dx}$ or the ***prime notation*** $f'(x)$, depending on which is appropriate.

a) $y = 4$ **b)** $g(x) = 3x^4 + 12x^3 - 8x + 1$ **c)** $f(x) = \dfrac{1}{x}$

d) $s = \sqrt{t}$ **e)** $y = \left(\dfrac{x}{2}\right)^2$ **f)** $h(t) = -2(t^2 - 3)^2$

Ex. 4. Find the ***equations*** of the ***tangent*** and ***normal*** to the curve $y = (x-3)^2 - 2$ at $x = 5$.
(Note: The ***normal line*** is ***perpendicular*** to the ***tangent line*** at the point of tangency.)



Ex. 5. Find $f'(a)$ for the given function $f(x)$ at the given value of a.

a) $f(x)=\left(1-\frac{2}{x}\right)\left(3-\frac{4}{x}\right)$; $a=1$

b) $f(x)=\frac{2+\sqrt{x^3}}{\sqrt{4x}}$; $a=4$

HW: p. 138 #2 to 8, 9abd

Date:____________ **Section 4.2 – The Derivatives of Polynomial Functions Continued**

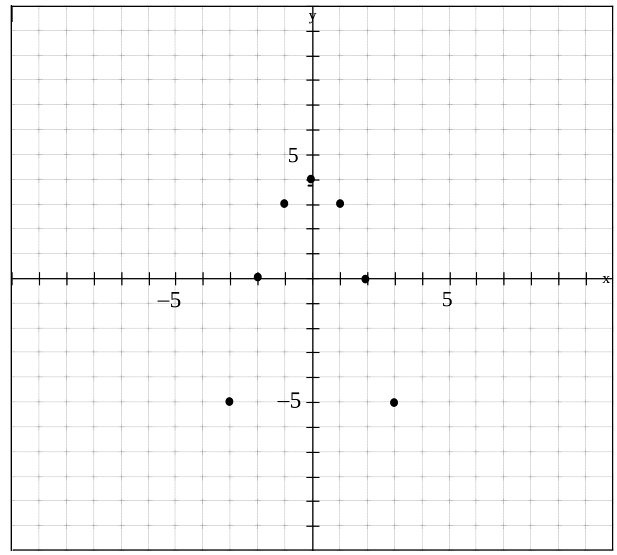
Ex. 1. Find the ***slope*** of the ***tangent*** to the curve $y=\sqrt{3x^3}$ at the point $P\ (3,\ 9)$.

Ex. 2. Find the ***slope*** of the ***normal*** to the curve $y=\left(\sqrt{x}-2\right)\left(3\sqrt{x}+8\right)$ at the $x=4$.

Ex. 3. Find the values of *x* so that the tangent to the function $y=\dfrac{3}{\sqrt[3]{x}}$ is parallel to the line $x+16y+3=0$.

Ex. 4. Find the equation(s) of the tangent(s) to the curve $y = 4 - x^2$ that pass through the following points and illustrate graphically.

a) $(-1, 1)$ **b)** $(-1, 3)$ **c)** $(-1, 12)$



HW: p. 139 #9df, 12, 14, 16, 17, 20, 22, 23

Date:_______________

Section 4.3 – THE PRODUCT RULE

Recall:

POWER RULE

If $y = ax^n$ then $\frac{dy}{dx} =$

Ex. 1. Differentiate each function.

a) $y = x^3 - 4x^2 + 8$

b) $g(x) = \left(\frac{x}{3}\right)^4$

c) $f(x) = (x^2 + 3x - 1)(2x + 7)$

Ex. 2. Develop the ***PRODUCT RULE*** for differentiation from ***first principles***.
ie. If $f(x) = p(x) \cdot q(x)$, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

$$= \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

$$= \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

$$= \lim_{h \to 0} \left[\frac{\qquad\qquad}{h} + \frac{\qquad\qquad}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\qquad\qquad}{h} \right] + \lim_{h \to 0} \left[\frac{\qquad\qquad}{h} \right]$$

$$= \lim_{h \to 0} \left[\qquad \cdot \frac{\qquad\qquad}{h} \right] + \lim_{h \to 0} \left[\qquad \cdot \frac{\qquad\qquad}{h} \right]$$

$$= \lim_{h \to 0} \qquad \cdot \lim_{h \to 0} \frac{\qquad\qquad}{h} + \lim_{h \to 0} \qquad \cdot \lim_{h \to 0} \frac{\qquad\qquad}{h}$$

$=$

$=$

PRODUCT RULE

If $y = f(x) \cdot g(x)$ then $\dfrac{dy}{dx} = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

"The derivative of a product is the derivative of the first times the second, plus the derivative of the second times the first."

Ex. 3. Use the ***Product Rule*** to differentiate each function. Simplify your answers.

a) $y = (x^2 + 3x - 1)(2x + 7)$

b) $s(t) = t^4 \cdot (2t - t^3)$

Ex. 4. Find $\dfrac{dy}{dx}$ at the given value of x for the function below.

(Note: There is no need to simplify the expression for $\dfrac{dy}{dx}$ before substituting the given value.**)**

$y = (3 - 4x^3)(2x + x^2)$ at $x = -1$

Ex. 5. Find the equation of the ***normal line*** to the curve $y = (x^3 - 3x + 1)(2x^2 - 5x)$ at $x = 1$

Ex. 6. Differentiate $y = f(x) \cdot g(x) \cdot h(x)$ with respect to *x*.

$$y = \left[f(x) \cdot g(x)\right] \cdot h(x)$$

Using the ***Product Rule***,

$$\frac{dy}{dx} = [\qquad\qquad\qquad\qquad\qquad\qquad] \cdot h(x) + h'(x) \cdot [\qquad\qquad\qquad\qquad]$$

EXTENDED PRODUCT RULE
If $y = f(x) \cdot g(x) \cdot h(x)$ then
$\frac{dy}{dx} =$

Ex. 7. Find the ***slope*** of the ***tangent*** to $y = (2x-3)(x^2+1)(x^3-x)$ at $x = 1$.

HW: p. 145 #1, 4abcdgh, 5, 6, 10, 11ab, 12

Date:_______________

Section 4.4 – THE QUOTIENT RULE

Recall:

Power Rule: If $y = ax^n$ then $\frac{dy}{dx} =$

Product Rule: If $y = f(x) \cdot g(x)$ then $\frac{dy}{dx} =$

Extended Product Rule: If $y = f(x) \cdot g(x) \cdot h(x)$ then

$\frac{dy}{dx} =$

Ex. 1. Use the ***limit definition of derivative*** to differentiate $f(x) = \frac{2x}{1-x}$.

Ex. 2. Develop the ***QUOTIENT RULE*** for derivatives. **ie.** If $y = \frac{f(x)}{g(x)}$, find $\frac{dy}{dx}$.

$y = \frac{f(x)}{g(x)}$

$\frac{y}{1} = \frac{f(x)}{g(x)}$

$f(x) = y \cdot g(x)$

*Using the **product rule**,*

$f'(x) =$

$\frac{dy}{dx} \cdot g(x) =$

$\frac{dy}{dx} =$ ______________________________

$\because y = \frac{f(x)}{g(x)}$,

$\frac{dy}{dx} =$ ______________________________

$\frac{dy}{dx} =$ ______________________________

QUOTIENT RULE

If $y = \dfrac{f(x)}{g(x)}$ then

$$\frac{dy}{dx} = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{\left[g(x)\right]^2}$$

"The derivative of a quotient is the derivative of the numerator times the denominator, minus the derivative of the denominator times the numerator, all over the denominator squared."

Ex. 3. Using the ***Quotient Rule***, differentiate each function and simplify.

a) $f(x) = \dfrac{2x}{1-x}$

b) $f(x) = \dfrac{3x-4}{x^2+5}$

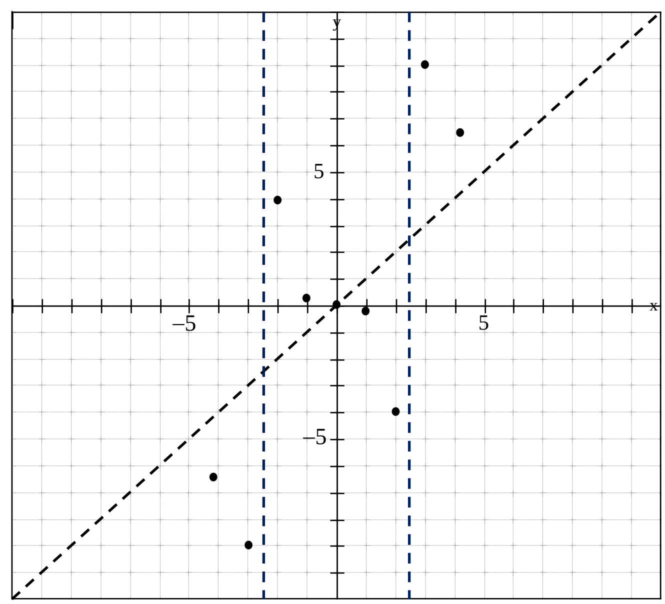
c) $y = \dfrac{1}{x^2-4}$

d) $y = \dfrac{(2x-1)(2x+1)}{2+x}$

Ex. 4. Find the ***slope*** of the ***tangent line*** to the curve $y = \frac{x^3}{x^2 - 6}$ at $x = 3$.

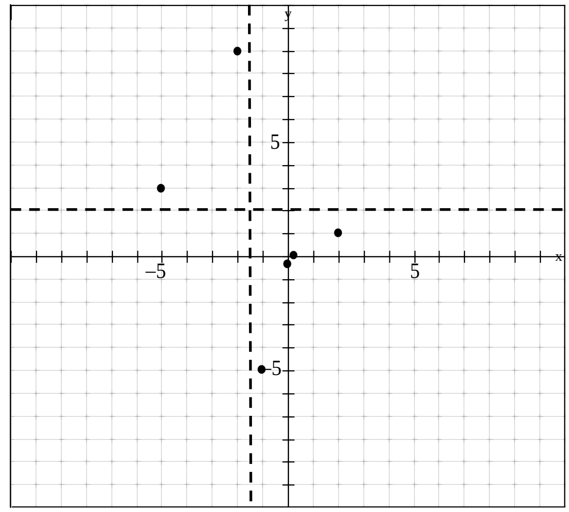
Illustrate graphically.

Graph $y = \frac{x^3}{x^2 - 6}$.



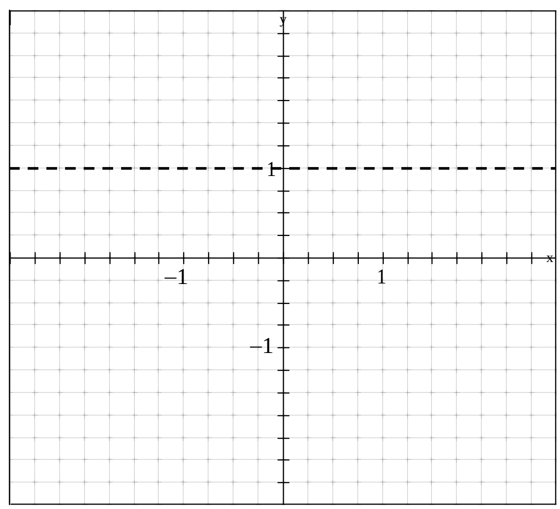
Ex. 5. Find the ***equation*** of the ***normal line*** to the curve $y = \dfrac{4x-1}{2x+3}$ at $x = -1$.

Illustrate graphically.



Ex. 6. Find the ***point(s)*** where the ***tangent line*** to the curve $f(x) = \dfrac{x^2-1}{x^2+1}$ is ***horizontal***.

Illustrate graphically.



HW: p. 149 #4 odd parts, 5abd, 6 to 12

Date:_____________ **Section 4. 4 – THE POWER OF A FUNCTION RULE**

Recall:

Power Rule: If $y = ax^n$ then $\frac{dy}{dx} =$

Product Rule: If $y = f(x) \cdot g(x)$ then $\frac{dy}{dx} =$

Extended Product Rule: If $y = f(x) \cdot g(x) \cdot h(x)$ then

$\frac{dy}{dx} =$

Ex. 1. Differentiate each of the following using the **Product Rules** and **Power Rule**.

a) $y = \left(3x^3 - 2x^2 - 1\right)^3$

Solution:

$$y = \left(3x^3 - 2x^2 - 1\right) \cdot \left(3x^3 - 2x^2 - 1\right) \cdot \left(3x^3 - 2x^2 - 1\right)$$

$$\frac{dy}{dx} = \left(9x^2 - 4x\right) \cdot \left(3x^3 - 2x^2 - 1\right)^2 + \left(9x^2 - 4x\right) \cdot \left(3x^3 - 2x^2 - 1\right)^2 + \left(9x^2 - 4x\right) \cdot \left(3x^3 - 2x^2 - 1\right)^2$$

$$\frac{dy}{dx} = 3\left(9x^2 - 4x\right)\left(3x^3 - 2x^2 - 1\right)^2$$

$$\therefore \frac{dy}{dx} = 3\left(3x^3 - 2x^2 - 1\right)^2 \left(9x^2 - 4x\right)$$

b) $y = \left[f(x)\right]^4$

Solution:

$$y = \left[f(x) \cdot f(x) \cdot f(x)\right] \cdot f(x)$$

$$\frac{dy}{dx} = \left[f'(x) \cdot \left[f(x)\right]^2 + f'(x) \cdot \left[f(x)\right]^2 + f'(x) \cdot \left[f(x)\right]^2\right] \cdot f(x) + f'(x) \cdot \left[f(x)\right]^3$$

$$\frac{dy}{dx} = 3f'(x) \cdot \left[f(x)\right]^2 \cdot f(x) + f'(x) \cdot \left[f(x)\right]^3$$

$$\frac{dy}{dx} = 3f'(x) \cdot \left[f(x)\right]^3 + f'(x) \cdot \left[f(x)\right]^3$$

$$\therefore \frac{dy}{dx} = 4\left[f(x)\right]^3 \cdot f'(x)$$

POWER OF A FUNCTION RULE: If $y = \left[f(x)\right]^n$ **then** $\frac{dy}{dx} =$

If $y = au^n$ **then** $\frac{dy}{dx} =$

Ex. 2. Differentiate each of the following using the **Power of a Function Rule**.

a) $y = \left[f(x)\right]^4$

b) $y = \left(3x^3 - 2x^2 - 1\right)^3$

POWER OF A FUNCTION RULE

If $y = a\left[f(x)\right]^n$ then $\frac{dy}{dx} =$

Ex. 2. Use the **Power Rule** or **Power of a Function Rule** to find the derivative.

a) $f(x) = \left(\frac{3}{2x}\right)^2 - \frac{1}{x^3} + \frac{1}{x}$

b) $y = (2x+1)^6$

c) $f(x) = \left(-x^4 - \pi^4\right)^4$

d) $y = \left[(3-x)^3 - x\right]^2$

e) $y = \frac{1}{\sqrt{2x-1}}$

f) $g(x) = \frac{-3}{x^2 - 4}$

Ex. 3. Use the **Power Rule, Power of a Function Rule** and **Product Rule** to differentiate each function. Express your answer in ***simplified factored form***.

a) $s = t^4\left(1-4t^2\right)^3$

b) $f(x) = (2x-1)^5(2-3x)^3$

c) $y = \dfrac{(3-x)^2}{(3x-2)^3}$

d) $y = \sqrt[3]{\dfrac{1+x^2}{1-2x^2}}$

HW: p. 158 #2 to 4 all parts

Date:______________

Section 4.6 – THE CHAIN RULE

Recall:

If $y = ax^n$ then $\frac{dy}{dx} =$

If $y = a\left[f(x)\right]^n$ then $\frac{dy}{dx} =$

If $y = au^n$ then $\frac{dy}{dx} =$

Ex. 1. Find the rate of change of $f(t) = \left(\frac{\pi - t}{6\pi + t}\right)^{\frac{1}{3}}$ at $t = 2\pi$.

Ex. 2. The radius of a circular oil slick on a body of water is increasing at a rate of 2 m/min. Find the ***rate of change of area*** of the oil slick when its radius is 50 m.

THE CHAIN RULE IN LEIBNIZ NOTATION

If y is a function of u and u is a function of x, then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$	If y is a function of u and u is a function of v and v is a function of x, then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Ex. 3. Use the ***chain rule*** to find $\frac{dy}{dx}$ at $x = 1$ if $y = 2u^3 + 3u^2$ and $u = x + \frac{1}{\sqrt{x}}$.

Ex. 4. Use the ***chain rule*** to find $\frac{dy}{dx}$ at $x = 4$ if $y = \frac{12}{(1+2u)^3}$, $u = 1 - \frac{2}{v^2}$ and $v = 2x - 3\sqrt{x}$.

THE CHAIN RULE FOR A COMPOSITE FUNCTION

If $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

"The derivative of a composite function is the derivative of the outer function with respect to the inner function times the derivative of the inner function with respect to x."

Ex. 5. Let $y = g(h(x))$ where $h(x) = \frac{x^2}{x+2}$. If $g'\left(\frac{9}{5}\right) = -2$, find $\frac{dy}{dx}$ when $x = 3$.

HW: p. 159 #5a, 6 to 10, 12 to 15;
REVIEW for TEST: p. 163 #2 to 20; p. 166 #3 to 11