Date: Feb. 24/14

Section 4.3 – THE PRODUCT RULE

Recall:

POWER RULE

If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$

Ex. 1. Differentiate each function.

a) $y = x^3 - 4x^2 + 8$

$\frac{dy}{dx} = 3x^2 - 8x$

b) $g(x) = \left(\frac{x}{3}\right)^4$

$g(x) = \frac{1}{81}x^4$

$g'(x) = \frac{4}{81}x^3$

c) $f(x) = (x^2 + 3x - 1)(2x + 7)$

$f(x) = 2x^3 + 7x^2 + 6x^2 + 21x - 2x - 7$

$f(x) = 2x^3 + 13x^2 + 19x - 7$

$f'(x) = 6x^2 + 26x + 19$

Ex. 2. Develop the ***PRODUCT RULE*** for differentiation from ***first principles***.

ie. If $f(x) = p(x) \cdot q(x)$, find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{p(x+h)\cdot q(x+h) - p(x)\cdot q(x)}{h}$$

$$= \lim_{h\to 0} \frac{\overbrace{p(x+h)\cdot q(x+h) - p(x)\cdot q(x+h)} + \overbrace{p(x)\cdot q(x+h) - p(x)\cdot q(x)}}{h}$$

$$= \lim_{h\to 0} \frac{q(x+h)[p(x+h) - p(x)] + p(x)[q(x+h) - q(x)]}{h}$$

$$= \lim_{h\to 0} \left[\frac{q(x+h)[p(x+h) - p(x)]}{h} + \frac{p(x)[q(x+h) - q(x)]}{h}\right]$$

$$= \lim_{h\to 0} \left[\frac{q(x+h)[p(x+h) - p(x)]}{h}\right] + \lim_{h\to 0} \left[\frac{p(x)[q(x+h) - q(x)]}{h}\right]$$

$$= \lim_{h\to 0} \left[q(x+h) \cdot \frac{p(x+h) - p(x)}{h}\right] + \lim_{h\to 0} \left[p(x) \cdot \frac{q(x+h) - q(x)}{h}\right]$$

$$= \lim_{h\to 0} q(x+h) \cdot \boxed{\lim_{h\to 0} \frac{p(x+h) - p(x)}{h}} + \underbrace{\lim_{h\to 0} p(x)} \cdot \boxed{\lim_{h\to 0} \frac{q(x+h) - q(x)}{h}}$$

$$= q(x) \cdot p'(x) + p(x) \cdot q'(x)$$

$$= p'(x) \cdot q(x) + q'(x) \cdot p(x)$$

$\therefore$ If $f(x) = p(x) \cdot q(x)$ then $f'(x) = p'(x) \cdot q(x) + q'(x) \cdot p(x)$

PRODUCT RULE

If $y = f(x)\cdot g(x)$ then $\dfrac{dy}{dx} = f'(x)\cdot g(x) + g'(x)\cdot f(x)$

"The derivative of a product is the derivative of the first times the second, plus the derivative of the second times the first."

Ex. 3. Use the ***Product Rule*** to differentiate each function. Simplify your answers.

a) $y = (x^2+3x-1)(2x+7)$

$\frac{dy}{dx} = (2x+3)\cdot(2x+7) + 2\cdot(x^2+3x-1)$

$= 4x^2+20x+21+2x^2+6x-2$

$\therefore \frac{dy}{dx} = 6x^2+26x+19$

b) $s(t) = t^4\cdot(2t-t^3)$

$s'(t) = 4t^3\cdot(2t-t^3) + (2-3t^2)\cdot t^4$

$s'(t) = 8t^4-4t^6+2t^4-3t^6$

$\therefore s'(t) = 10t^4-7t^6$

Ex. 4. Find $\dfrac{dy}{dx}$ at the given value of x for the function below.

(Note: There is no need to simplify the expression for $\dfrac{dy}{dx}$ before substituting the given value.**)**

$y = \underbrace{(3-4x^3)}\cdot\underbrace{(2x+x^2)}$ at $x=-1$

$\frac{dy}{dx} = -12x^2\cdot(2x+x^2) + (2+2x)\cdot(3-4x^3)$

at $x=-1$

$\frac{dy}{dx} = (-12)(-1) + (0)(7)$

$= 12$

$\therefore \frac{dy}{dx} = 12$ at $x=-1$

Ex. 5. Find the equation of the ***normal line*** to the curve $y = (x^3-3x+1)(2x^2-5x)$ at $x=1$

$y = (x^3-3x+1)\cdot(2x^2-5x)$

$\frac{dy}{dx} = (3x^2-3)\cdot(2x^2-5x) + (4x-5)\cdot(x^3-3x+1)$

at $x=1$

$\frac{dy}{dx} = (0)(-3) + (-1)(-1)$

$= 1$

$\therefore m_t = \frac{dy}{dx}$

$\therefore$ at $x=1$, $m_t = 1$ & $y = (-1)(-3)$
$= 3$

For the normal,

$m_n = -1$; $(1,3)$; $b=$___

$3 = -1(1)+b$

$3 = -1+b$

$4 = b$

$\therefore$ the equation of the normal is $y = -x+4$ or $x+y-4=0$

Ex. 6. Differentiate $y = f(x)\cdot g(x)\cdot h(x)$ with respect to x.

$$y = [\overbrace{f(x)\cdot g(x)}^{1st}]\cdot \overbrace{h(x)}^{2nd}$$

Using the ***Product Rule***,

$$\frac{dy}{dx} = [\, f'(x)\cdot g(x) + g'(x)\cdot f(x)\,]\cdot h(x) + h'(x)\cdot[\, f(x)\cdot g(x)\,]$$

$$= f'(x)\cdot g(x)\cdot h(x) + g'(x)\cdot f(x)\cdot h(x) + h'(x)\cdot f(x)\cdot g(x)$$

EXTENDED PRODUCT RULE
If $y = f(x)\cdot g(x)\cdot h(x)$ then
$\frac{dy}{dx} = f'(x)\cdot g(x)\cdot h(x) + g'(x)\cdot f(x)\cdot h(x) + h'(x)\cdot f(x)\cdot g(x)$

Ex. 7. Find the ***slope*** of the ***tangent*** to $y = (2x-3)(x^2+1)(x^3-x)$ at $x = 1$.

$$y = (2x-3)\cdot(x^2+1)\cdot(x^3-x)$$

$$m_t = \frac{dy}{dx}$$

$$= 2\cdot(x^2+1)(x^3-x) + 2x\cdot(2x-3)(x^3-x) + (3x^2-1)\cdot(2x-3)(x^2+1)$$

at $x=1$

$$m_t = 2(2)(0) + (2)(-1)(0) + (2)(-1)(2)$$

$$= -4$$

$\therefore$ at $x=1$, the slope of the tangent is -4.

HW: p. 145 #1, 4abcdgh, 5, 6, 10, 11ab, 12

Date: Feb.25/14

Section 4.4 – THE QUOTIENT RULE

Recall:

Power Rule: If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$

Product Rule: If $y = f(x) \cdot g(x)$ then $\frac{dy}{dx} = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

Extended Product Rule: If $y = f(x) \cdot g(x) \cdot h(x)$ then

$$\frac{dy}{dx} = f'(x) \cdot g(x) \cdot h(x) + g'(x) \cdot f(x) \cdot h(x) + h'(x) \cdot f(x) \cdot g(x)$$

Ex. 1. Use the ***limit definition of derivative*** to differentiate $f(x) = \frac{2x}{1-x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2(x+h)}{1-(x+h)} - \frac{2x}{1-x}}{h} \cdot \frac{(1-x-h)(1-x)}{(1-x-h)(1-x)}$$

$$= \lim_{h \to 0} \frac{2(x+h)(1-x) - 2x(1-x-h)}{h(1-x-h)(1-x)}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} - \cancel{2x^2} + 2h - \cancel{2hx} - \cancel{2x} + \cancel{2x^2} + \cancel{2hx}}{h(1-x-h)(1-x)}$$

$$= \lim_{h \to 0} \frac{2}{(1-x-h)(1-x)}$$

$$= \frac{2}{(1-x)^2}$$

$\therefore$ If $f(x) = \frac{2x}{1-x}$ then $f'(x) = \frac{2}{(1-x)^2}$

Ex. 2. Develop the ***QUOTIENT RULE*** for derivatives. **ie.** If $y = \frac{f(x)}{g(x)}$, find $\frac{dy}{dx}$.

$$y = \frac{f(x)}{g(x)}$$

$$\frac{y}{1} = \frac{f(x)}{g(x)}$$

$$f(x) = y \cdot g(x)$$

*Using the **product rule**,*

$$f'(x) = \frac{dy}{dx} \cdot g(x) + g'(x) \cdot y$$

$$\frac{dy}{dx} \cdot g(x) = f'(x) - g'(x) \cdot y$$

$$\frac{dy}{dx} = \frac{f'(x) - g'(x) \cdot y}{g(x)}$$

$\because y = \frac{f(x)}{g(x)}$,

$$\frac{dy}{dx} = \frac{f'(x) - g'(x) \cdot \frac{f(x)}{g(x)}}{g(x)} \cdot \frac{g(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$$

$\therefore$ If $y = \frac{f(x)}{g(x)}$ then

$$\frac{dy}{dx} = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$$

QUOTIENT RULE

If $y = \dfrac{f(x)}{g(x)}$ then

$$\frac{dy}{dx} = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{[g(x)]^2}$$

"The derivative of a quotient is the derivative of the numerator times the denominator, minus the derivative of the denominator times the numerator, all over the denominator squared."

Ex. 3. Using the ***Quotient Rule***, differentiate each function and simplify.

a) $f(x) = \dfrac{2x}{1-x}$

$$f'(x) = \frac{2\cdot(1-x) - (-1)\cdot 2x}{(1-x)^2}$$

$$f'(x) = \frac{2-2x+2x}{(1-x)^2}$$

$$\therefore f'(x) = \frac{2}{(1-x)^2}$$

b) $f(x) = \dfrac{3x-4}{x^2+5}$

$$f'(x) = \frac{3\cdot(x^2+5) - 2x\cdot(3x-4)}{(x^2+5)^2}$$

$$f'(x) = \frac{3x^2+15-6x^2+8x}{(x^2+5)^2}$$

$$\therefore f'(x) = \frac{-3x^2+8x+15}{(x^2+5)^2}$$

c) $y = \dfrac{1}{x^2-4}$

$$\frac{dy}{dx} = \frac{0\cdot(x^2-4) - 2x\cdot 1}{(x^2-4)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{(x^2-4)^2}$$

d) $y = \dfrac{(2x-1)(2x+1)}{2+x}$

$$y = \frac{4x^2-1}{2+x}$$

$$\frac{dy}{dx} = \frac{8x\cdot(2+x) - 1(4x^2-1)}{(2+x)^2}$$

$$\frac{dy}{dx} = \frac{16x+8x^2-4x^2+1}{(2+x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{4x^2+16x+1}{(x+2)^2}$$

Ex. 4. Find the ***slope*** of the ***tangent line*** to the curve $y = \dfrac{x^3}{x^2 - 6}$ at $x = 3$.

Illustrate graphically.

$$y = \frac{x^3}{x^2-6}$$

$$m_t = \frac{dy}{dx}$$

$$= \frac{3x^2(x^2-6) - 2x \cdot x^3}{(x^2-6)^2}$$

$$\therefore m_t = \frac{x^4 - 18x^2}{(x^2-6)^2}$$

at $x = 3$

$$m_t = \frac{81 - 18(9)}{9}$$

$$= 9 - 18$$

$$= -9$$

$\therefore$ at $x=3$ the slope of the tangent is -9.

$$y = \frac{x^3}{x^2-6}$$

For v.a.: $x^2 - 6 = 0$

$$x^2 = 6$$

$$x = \pm\sqrt{6}$$

$$\begin{array}{r} x \\ x^2+0x-6 \,\big)\, x^3 + 0x^2 + 0x + 0 \\ \underline{x^3 + 0x^2 - 6x} \\ 6x + 0 \end{array}$$

$$y = x + \frac{6x}{x^2-6}$$

For l.o.a.

as $x \to \pm\infty$, $y \to x$

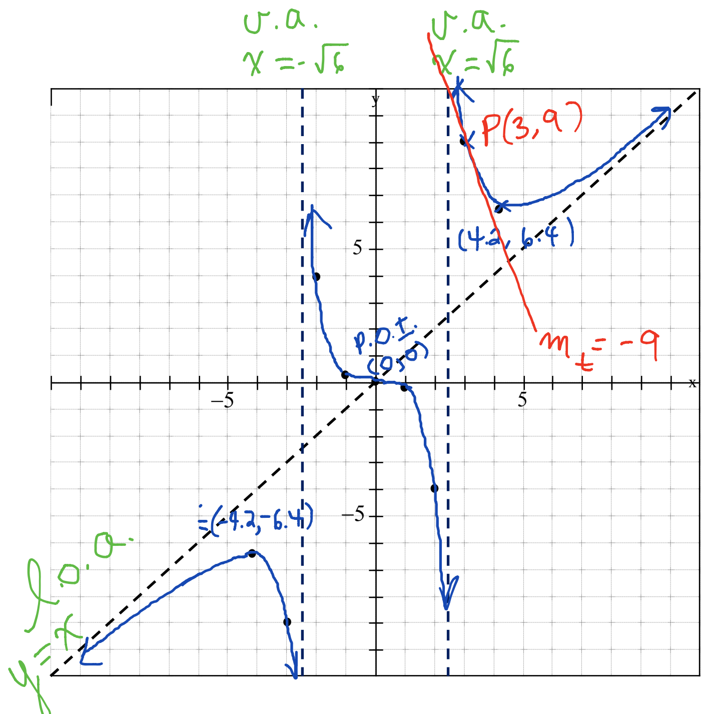
$\therefore$ l.o.a. is $y = x$

$$\frac{(\sqrt{18})^3}{12}$$

turning points

at $\doteq (-4.2, -6.4)$

& $\doteq (4.2, 6.4)$



For turning pts.

$$m_t = 0$$

$$\frac{x^4 - 18x^2}{(x^2-6)^2} = \frac{0}{1}$$

$$x^4 - 18x^2 = 0$$

$$x^2(x^2 - 18) = 0$$

$$x^2 = 0$$

$$x = 0$$

$$x^2 = 18$$

$$x = \pm\sqrt{18}$$

$$x = \pm 3\sqrt{2}$$

For v.a. $2x+3=0$

$x=-\frac{3}{2}$

Ex. 5. Find the ***equation*** of the ***normal line*** to the curve $y=\dfrac{4x-1}{2x+3}$ at $x=-1$.

Illustrate graphically.

$y=\dfrac{4x-1}{2x+3}$

$m_t=\dfrac{dy}{dx}$

$=\dfrac{4(2x+3)-2(4x-1)}{(2x+3)^2}$

$m_t=\dfrac{14}{(2x+3)^2}$

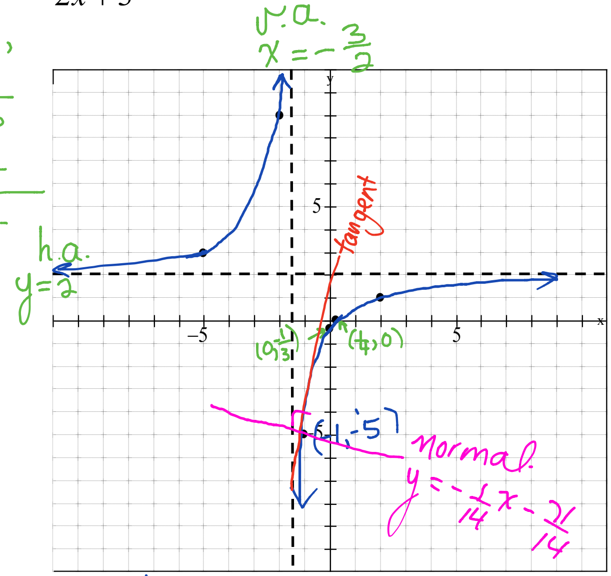
For h.a.,

$\lim\limits_{x\to\pm\infty}\dfrac{4x-1}{2x+3}$

$=\lim\limits_{x\to\pm\infty}\dfrac{4-\frac{1}{x}}{2+\frac{3}{x}}$

$=\dfrac{4-0}{2+0}$

$=2$



at $x=-1$,

$m_t=\dfrac{14}{(1)^2}$, $y=-5$

$=14$

For the normal,

$m_n=-\frac{1}{14}$; $(-1,-5)$; $b=$ ___

$-5=-\frac{1}{14}(-1)+b$

$-\frac{70}{14}=\frac{1}{14}+b$

$-\frac{71}{14}=b$

$\therefore$ the equation of the normal is $y=-\frac{1}{14}x-\frac{71}{14}$ or

$x+14y+71=0$.

Ex. 6. Find the ***point(s)*** where the ***tangent line*** to the curve $f(x)=\dfrac{x^2-1}{x^2+1}$ is ***horizontal***.

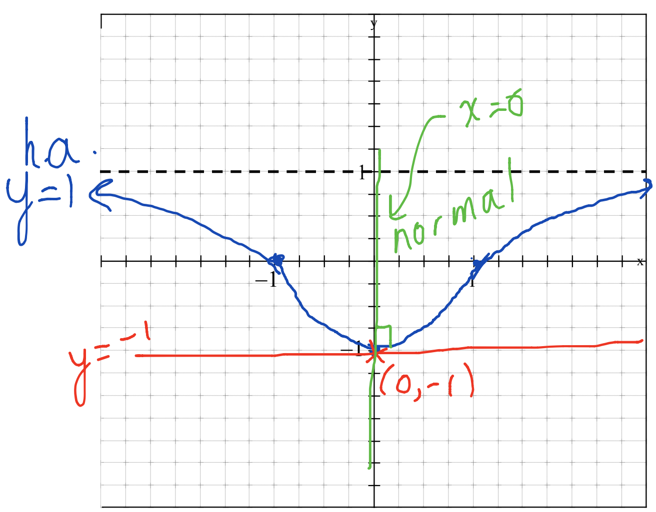
Illustrate graphically.

$f(x)=\dfrac{x^2-1}{x^2+1}$

$m_t=f'(x)$

$=\dfrac{2x\cdot(x^2+1)-2x\cdot(x^2-1)}{(x^2+1)^2}$

$\therefore m_t=\dfrac{4x}{(x^2+1)^2}$



Find x if

$m_t=0$

$\dfrac{4x}{(x^2+1)^2}=\dfrac{0}{1}$

$4x=0$

$x=0$

$f(0)=-1$

$\therefore$ the tangent is horizontal at the point $(0,-1)$

For h.a,

$\lim\limits_{x\to\pm\infty}\dfrac{x^2-1}{x^2+1}\div\dfrac{x^2}{x^2}$

$=\lim\limits_{x\to\pm\infty}\dfrac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}$

$=\dfrac{1-0}{1+0}$

$=1$

HW: p. 149 #4 odd parts, 5abd, 6 to 12