Date:______________ **UNIT 3 – APPLICATIONS OF DERIVATIVES – Part A**

Section 5.1 – Implicit Differentiation

So far we have dealt with curves that are defined by an equation of the form $y = f(x)$ *in which y is defined* ***"explicitly"*** *in terms of x.* ***Examples*** *are* $y = x^2 + 2x$ *and* $y = \frac{1}{1-\sqrt{x}}$.

However, curves may also be defined ***"implicitly"*** *by a relation where y is not given explicitly in terms of* x*.* ***Examples*** *are* $x^2 + y^2 = 25$ *and* $xy - 1 = 0$.

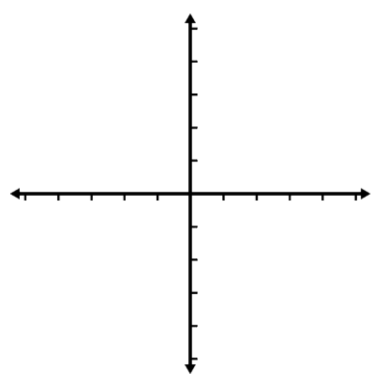
Ex. 1. If y is a function of x, find an expression for each derivative in terms of *x, y* and $\frac{dy}{dx}$.

a) $\frac{d}{dx}(x^2 + y^2)$

b) $\frac{d}{dx}(xy^3)$

Ex. 2. Find the ***equation*** of the ***tangent*** line to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the point $\left(4, \frac{12}{5}\right)$.

Leave your answer in $y = mx + b$ form and illustrate your solution graphically.



Ex. 3. Find $\frac{dy}{dx}$ in terms of *x* and *y*, using ***implicit*** differentiation.

a) $x^2 - xy + 2y^2 = 4$

b) $2xy + x^2 = 4 - y^2$

Ex. 4. For each of the following, find the indicated derivative using ***implicit*** differentiation.

a) $4h^2 + r^2 = 25$; find $D_r h$

b) $u^2 - \frac{u}{x} = \frac{12}{x}$; find $\frac{du}{dx}$

Ex. 5. Show that, if $15x = 15y + 5y^3 + 3y^5$, then $\frac{dy}{dx} = \left(1 + y^2 + y^4\right)^{-1}$.

Ex. 6. Let $y = -\left(65 - x^6\right)^{\frac{1}{6}}$. Find the ***equation*** of the ***normal*** line at $(1, -2)$ by first expressing the curve in a simple ***implicit*** form, and then using ***implicit*** differentiation. Leave your answer in standard form.

Ex. 7. Find all ***points*** (x, y) on the curve $x + xy + 2y^2 = 6$ at which the ***slope*** of the ***tangent*** is $-\frac{1}{3}$.

HW: p. 178 #2 odd parts, 3d, 4 to 8, 9a; p. 219 #1 to 3, 12

Date:____________ **Section 5.2 – Higher-Order Derivatives, Velocity, and Acceleration**

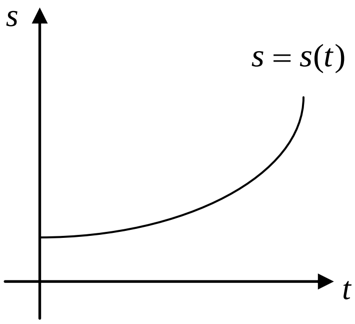
Higher-Order Derivatives

Ex. 1. Calculate the ***first*** and ***second derivatives*** of the following functions.

a) $f(x) = 2x^3 - 5x^2$ **b)** $y = \sqrt{1-x}$ **c)** $y = \dfrac{1}{(2x-3)^2}$

Velocity and Acceleration

Distance vs Time Graph



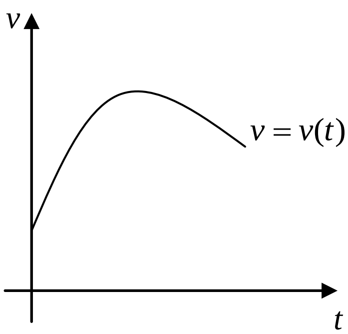
Average Velocity

$$v_{avg} = m_{secant}$$
$$= \frac{s_2 - s_1}{t_2 - t_1}$$
$$\therefore v_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Velocity

$$v = m_t$$
$$= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$
$$= \frac{ds}{dt} \text{ or } s'$$
$$\therefore v(t) = s'(t)$$

Velocity vs Time Graph



Average Acceleration

$$a_{avg} = m_{secant}$$
$$= \frac{v_2 - v_1}{t_2 - t_1}$$
$$\therefore a_{avg} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Acceleration

$$a = m_t$$
$$= \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$
$$= \frac{dv}{dt} \text{ or } v'$$
$$= \frac{d^2 s}{dt^2} \text{ or } s''$$
$$\therefore a(t) = v'(t) \text{ or } s''(t)$$

Note: Concerning the velocity:

If $v(t) > 0$, the object is moving to the right or up (in the +ve direction).
If $v(t) < 0$, the object is moving to the left or down (in the –ve direction).
If $v(t) = 0$, the object is stationary, or at rest.

Concerning acceleration: If $a(t) = 0$, braking begins.
The object is *accelerating* if the product of $a(t)$ and $v(t)$ is positive.
The object is *decelerating* if the product of $a(t)$ and $v(t)$ is negative.

Ex. 2. Starting at $t = 0$, a dragster accelerates down a strip and then brakes and comes to a stop.

The distance in metres at t seconds is given by $s(t) = 6t^2 - \frac{1}{5}t^3$.

a) Find the dragster's velocity and acceleration after 2 seconds.
b) After how many seconds does the dragster stop?
c) What distance does the dragster travel?
d) At what time does braking commence?

Ex. 3. The position function of an object moving in a straight line is $s(t) = 3t^2 - \frac{1}{4}t^4$.

Is the object moving towards or away from its starting position at:

a) $t = 2$?

b) $t = 4$?

c) $t = 3$?

______________________ s ______________________ s ______________________ s

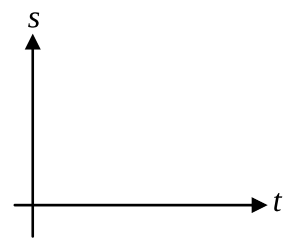
$\therefore$ at $t = 2$ the object is moving ______________________ its starting position.

$\therefore$ at $t = 4$ the object is moving ______________________ its starting position.

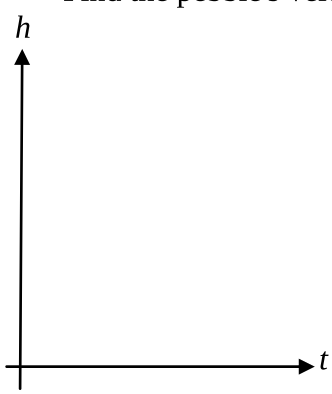
$\therefore$ at $t = 3$ the object is moving ______________________ its starting position.

Ex. 4. A fly ball is hit vertically upward. The position function of the ball is $s(t) = -5t^2 + 30t$ where s is its height, in metres, above the ground after t seconds.

a) Find the maximum height reached by the ball.
b) Find the velocity of the ball as it strikes the ground.



Ex. 5. A pebble is thrown vertically upwards from the top of a cliff of height 30 m. The pebble's height, h, in metres, in relation to the bottom of the cliff t seconds later is $h(t) = -5t^2 + 25t + 30$. Find the pebble's velocity on impact at the base of the cliff.



HW: p. 185 #2 to 9, 12, 14, 15; p. 219 #4 to 9, 11ab, 17

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Section 5.3 – RELATED RATES

Ex. 1. ***Differentiate each of the following formulas with respect to time,*** t.

CIRCLE

$C = 2\pi\ r$

TRIANGLE

$A = \frac{1}{2}bh$

SPHERE

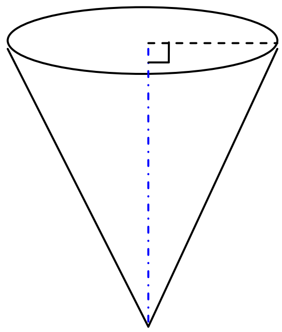
$S.A. = 4\pi\ r^2$

CYLINDER

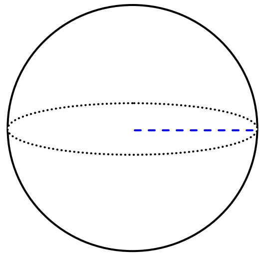
$V = \pi\ r^2 h$

Ex. 2. A raindrop falls in a puddle and the ripples spread in circles. If the radius is growing at a rate of 2 *cm/s*, find the rate of increase of the area of such a circle when its area is $36\pi\ \ cm^2$.

Ex. 3. A conical flower vase is 30 *cm* high with a radius of 6 *cm* at the top. If it is being filled with water at a rate of 10 cm^3/s, find the rate at which the water level is rising when the depth is 20 *cm*.



Ex. 4. A spherical weather balloon with radius 9 *m* springs a leak losing air at the rate of $171\pi\ m^3/\text{min}$. Find the rate of decrease of the radius after 4 minutes.



HW: p. 193 #1ab, 3 to 7, 12, 13, 15, 16, 21

Strategies for Related Rate Problems

1. Identify the primary equation from information in the question.

Circle: $C = 2\pi r$, $A = \pi r^2$ ***Square:*** $P = 4s$, $A = s^2$ ***Rectangle:*** $P = 2l + 2w$, $A = lw$ ***Triangle:*** $A = \frac{1}{2}bh$

Cube: $S.A. = 6s^2$, $V = s^3$ ***Sphere:*** $S.A. = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$ ***Cylinder:*** $S.A. = 2\pi r^2 + 2\pi rh$, $V = \pi r^2 h$

Cone: $V = \frac{1}{3}\pi r^2 h$ ***Prism:*** $V = A_{base} \cdot h$ ***Pythagorean Relationship:*** $x^2 + y^2 = s^2$

2. Identify what you are asked to find. $\frac{dA}{dt}, \frac{dr}{dt}, \frac{dV}{dt}, \frac{dh}{dt}$, *etc.*

3. Identify the information you are given. $\frac{dA}{dt}, \frac{dr}{dt}, \frac{dV}{dt}, \frac{dh}{dt}$, *etc.*

Note: Units like cm^3/s, m/s, m^2/h give excellent clues for steps #**2** and **3**.

4. Identify what is ***constant*** and what is changing, ie. *variable*.
Does the radius/height change in the question or is it always the same?
If there is a constant and you know its value, you can plug it in anytime.

5. If there is more than one variable (ie. radius and height in a cone question), look for a relationship using similar triangles. Express one variable in terms of the other, then substitute into the primary equation, making certain that you <u>keep the variable that you need</u> to solve the problem.

6. ***NEVER***, ***EVER***, ***EVER*** substitute a value into the equation for a **<u>variable</u>** (changing dimension) until ***<u>AFTER</u>*** you have taken the derivative!!!

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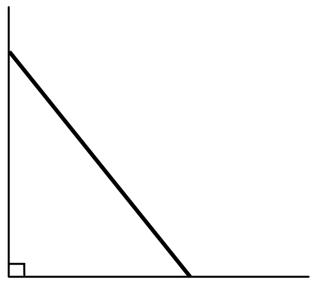
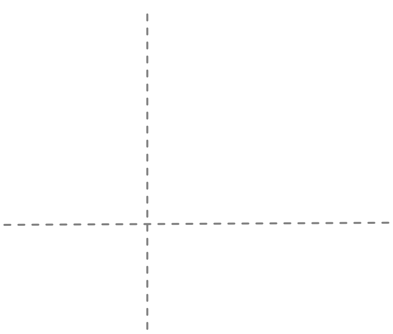
RELATED RATES – I

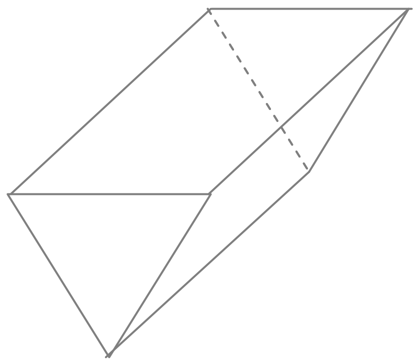
Ex. 1. A ladder 5 m long rests against a vertical wall. The base of the ladder begins to slide outwards at a rate of 1 m/s. How fast is the top of the ladder descending when the base is 3 m away from the wall?



Ex. 2. Car A approaches an intersection from the east at a rate of 12 m/s and Car B approaches from the north at a rate of 15 m/s. How fast is the distance between the cars decreasing at the instant Car A is 30 m east of the intersection and Car B is 40 m north of the intersection?



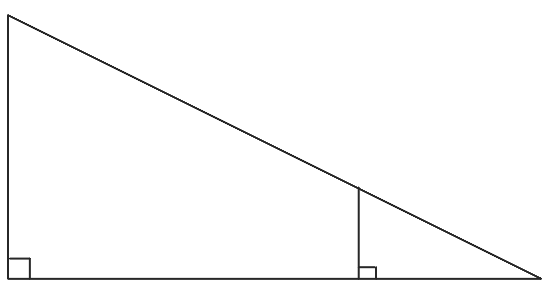
Ex. 3. A horizontal eavestrough 3 m long has a triangular cross-section 10 cm across the top and 10 cm deep. During a rainstorm, the water in the trough is rising at a rate of 1 cm/min. How fast is the volume of water in the trough increasing when the depth of water is 5 cm?



Ex. 4. A woman 2 m tall walks away from a streetlight that is 6 m high at a rate of 1.5 m/s.

a) At what rate is her shadow lengthening when she is 3 m from the base of the light?

b) At what rate is the tip of her shadow moving when she is 3 m from the base of the light?



HW: p. 193 #1cd, 2a, 8 to 10, 18, 19; Related Rates – Sheet I

Date:______________

Related Rates – Sheet I

1. A kite is flying at a height of 40 m. The boy who is flying this kite is running horizontally at 3 m/s and is letting out the string so that the length of the string is increasing. Find this rate when the length of the string is 50 m. [9/5 m/s]

2. A spherical balloon is being inflated so that its volume is increasing at the rate of 5 m^3/min. At what rate is the diameter increasing when the radius is 6 m? [5/72л m/min]

3. A spherical snowball is being made so that its volume is increasing at the rate of 8 m^3/min. Find the rate at which the radius is increasing when the snowball is 4 m in diameter. [1/2л m/min]

4. Sand is being dropped at the rate of 10 ft^3/min into a conical pile. If the height of the pile is always twice the base radius, at what rate is the height increasing when the pile is 8 ft high? [5/8л ft/min]

5. A light is hung 15 m above a straight horizontal path. If a man 2 m tall is walking away from the light at a rate of 2 m/s, how fast is his shadow lengthening? [4/13 m/s]

6. A man 6 ft tall is walking towards a building at the rate of 5 ft/s. If there is a light on the ground 50 ft from the building, how fast is the man's shadow on the building growing shorter at the moment he is 30 ft from the building? [15/4 ft/s]

7. A water tank in the shape of an inverted cone is being emptied at the rate of 6 m^3/min. If the height of the cone is 24 m and the base radius is 12 m, how fast is the water level going down when the depth of the water is 10 m? [6/25л m/min]

8. A trough is 12 m long and its ends are in the form of inverted isosceles triangles with height 3 m and base 3 m. If water is flowing into the trough at the rate of 2 m^3/min, how fast is the water level rising when the water is 1 m deep? [1/6 m/min]

9. A man on a dock is pulling in a small boat at the rate of 50 ft/min by means of a rope attached to the boat. If the man's hands are 16 ft above the point where the rope is attached to the boat, determine how fast the boat is approaching the dock when it is 10 ft from the dock. [94.3 ft/min]

10. If the length of the side of an equilateral triangle is increasing at the rate of 3 cm/min, how fast is the area of the triangle increasing when the length of the side is 8 cm? [$12\sqrt{3}$ cm^2/min]

11. A funnel in the shape of a right circular cone is 10 cm across the top and has a height of 8 cm. Water is flowing into this funnel at the rate of 12 cm^3/s and out at a rate of 4 cm^3/s. How fast is the water level rising when the depth of the water is 5 cm? [512/625л cm/s]

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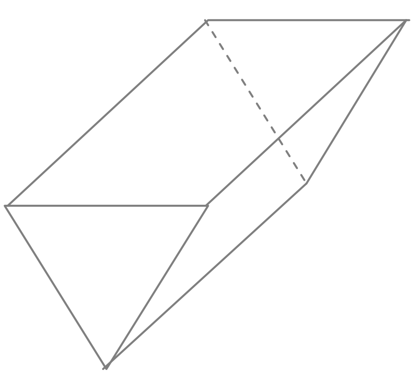
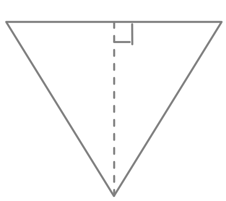
RELATED RATES – II

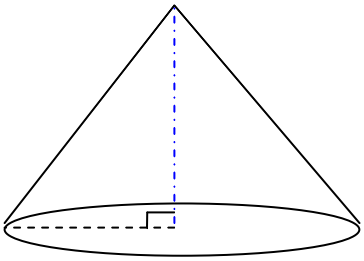
Ex. 1. A ship K is sailing due north at 16 km/h, and a second ship R, which is 44 km north of K, is sailing due east at 10 km/h. At what rate is the distance between ships K and R changing 90 minutes later? Are they approaching one another or separating at this time?

Ex. 2. The cross-section of a water trough is an equilateral triangle with a horizontal top edge. If the trough is 5 m long and 25 cm deep, and water is flowing in at a rate of 0.25 m^3/min, how fast is the water level rising when the water is 10 cm deep at the deepest point?

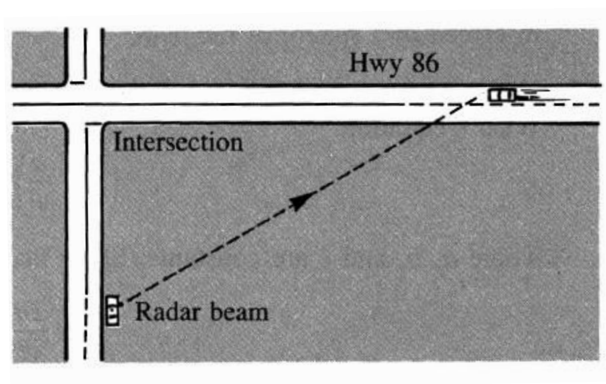


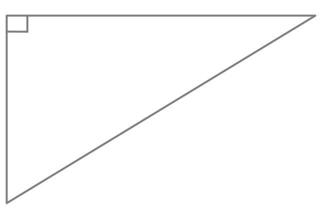


Ex. 3. A conveyor belt system at a gravel pit pours washed sand onto the ground at the rate of 180 m^3/h. The sand forms a conical pile with height one-third the diameter of the base. Determine how fast the height of the pile is increasing at the instant the radius of the base is 6 m.



Ex. 4. An OPP officer is operating a radar speed trap on a sideroad 100 m from Highway 86, near Listowel. When a car is 200 m from the intersection, its velocity of approach is measured as 70 km/h. Is the car exceeding the speed limit of 80 km/h?





HW: Related Rates – Sheet II

REVIEW for TEST: p. 219 #1-9, 11, 12, 17-22, 23a; p. 223 #1-7, 9; p. 291 #1-6, 11, 12

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Related Rates – Sheet II

1. A kite flying 100 m high is blown horizontally by the wind at a speed of 4 m/s. If the string is played out from a fixed position, how fast is the length of the string increasing when it is 125 m long? [2.4 m/s]

2. Ship A approaches a harbour entrance from the north at 15 knots, while Ship B approaches from the west at 18 knots. How rapidly are the ships approaching each other at the instant both are 5 nautical miles from the entrance? [$33/\sqrt{2}$ knots]

3. A jet flying north at 600 mph passes over a town at 12 noon exactly. A second jet flying east at 540 mph passes over the town at 12:01 pm. If the altitudes of the two planes are the same, how fast are they moving apart at 12:06 pm? [804 mph]

4. Northbound Ship A leaves harbour at 10 am with a speed of 12 knots. If the Westbound Ship B leaves at 10:30 am with a speed of 16 knots, how fast are the ships separating at 11:30 am? [19.6 knots]

5. A ladder 10 ft long leans against a vertical wall. If the bottom slides out at the rate of 1 ft/min, how fast is the top descending when the bottom is 6 ft away from the wall? [3/4 ft/min]

6. A ladder 20 ft long is propped against a vertical wall. As the ladder slides down, the foot of the ladder is moving out at a rate of 1/12 ft/s. Find the rate of descent of the top of the ladder at the following times:

a) When the bottom of the ladder is 12 ft from the wall. [1/16 ft/s]

b) When the ladder makes an angle of 60º with the ground. [$1/12\sqrt{3}$ ft/s]

7. A pedestrian 6 ft tall walks directly away from a street light 18 ft above the ground at a rate of 4 ft/s. Determine the following at the instant he is 24 ft from the base of the light post:

a) the rate of increase in the length of his shadow. [2 ft/s]

b) the speed of the end of his shadow. [6 ft/s]

8. A dog chases a squirrel at a speed of 12 ft/s. The squirrel dashes up a tree trunk at the rate of 6 ft/s. Find the rate of change of distance between the squirrel and dog at the instant the dog is 12 ft from the tree trunk and the squirrel is 5 ft up the trunk. [8.77 ft/s]