Date: Mar. 5 / 14

UNIT 3 – APPLICATIONS OF DERIVATIVES –Part A

Section 5.1 – IMPLICIT DIFFERENTIATION

So far we have dealt with curves that are defined by an equation of the form $y = f(x)$ *in which y is defined* ***"explicitly"*** *in terms of x.* ***Examples*** *are* $y = x^2 + 2x$ *and* $y = \frac{1}{1-\sqrt{x}}$.

However, curves may also be defined ***"implicitly"*** *by a relation where y is not given explicitly in terms of x.* ***Examples*** *are* $x^2 + y^2 = 25$ *and* $xy - 1 = 0$.

Ex. 1. If y is a function of x, find an expression for each derivative in terms of *x, y* and $\frac{dy}{dx}$.

a) $\frac{d}{dx}(x^2 + y^2)$

$= 2x + 2y\frac{dy}{dx}$

b) $\frac{d}{dx}(xy^3)$

$= \frac{d}{dx}(x \cdot y^3)$

$= 1 \cdot y^3 + 3y^2\frac{dy}{dx} \cdot x$

$= y^3 + 3xy^2\frac{dy}{dx}$

Ex. 2. Find the ***equation*** of the ***tangent*** line to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the point $\left(4, \frac{12}{5}\right)$.

Leave your answer in $y = mx + b$ form and illustrate your solution graphically.

$\frac{x^2}{25} + \frac{y^2}{16} = 1$

mult. by 400

$16x^2 + 25y^2 = 400$

differentiate w.r.t. "x"

$32x + 50y\frac{dy}{dx} = 0$

$50y\frac{dy}{dx} = -32x$

$\frac{dy}{dx} = \frac{-16x}{25y}$

$\therefore m_t = \frac{-16x}{25y}$

at $x = 4$, $y = \frac{12}{5}$

$m_t = \frac{-16(4)}{25\left(\frac{12}{5}\right)}$

$= -\frac{16}{15}$

For the tangent

$m_t = \frac{-16}{15}$; $\left(4, \frac{12}{5}\right)$

Find b

$\frac{12}{5} = \frac{-16}{15}(4) + b$

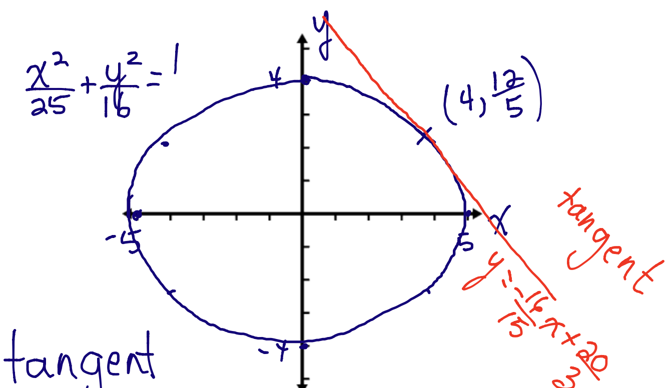
$\frac{36}{15} = \frac{-64}{15} + b$

$\frac{100}{15} = b$

$\frac{20}{3} = b$

$\therefore$ the equation of the required tangent is

$y = \frac{-16}{15}x + \frac{20}{3}$



For x int(s), $y = 0$

$\frac{x^2}{25} = 1$

$x^2 = 25$

$x = \pm 5$

For y-int(s), $x = 0$

$\frac{y^2}{16} = 1$

$y^2 = 16$

$y = \pm 4$

Ex. 3. Find $\frac{dy}{dx}$ in terms of x and y, using ***implicit*** differentiation.

a) $x^2 - xy + 2y^2 = 4$ use product rule

Diff. w.r.t. x

$2x' - 1\cdot y + 1\frac{dy}{dx}(-x) + 4y'\frac{dy}{dx} = 0$

$2x - y - x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$

$4y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$

$\frac{dy}{dx}(4y - x) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{4y-x} \cdot \frac{-1}{-1}$

or

$\frac{dy}{dx} = \frac{2x-y}{x-4y}$

b) $2xy + x^2 = 4 - y^2$ product rule

Diff. w.r.t. x

$2\cdot y + \frac{dy}{dx}\cdot 2x + 2x = -2y'\frac{dy}{dx}$

÷ by 2

$y + x\frac{dy}{dx} + x = -y\frac{dy}{dx}$

$x\frac{dy}{dx} + y\frac{dy}{dx} = -x - y$

$\frac{dy}{dx}(x + y) = -x - y$

$\frac{dy}{dx} = \frac{-x-y}{x+y}$

$\frac{dy}{dx} = \frac{-(x+y)}{(x+y)}$ or -1.

Ex. 4. For each of the following, find the indicated derivative using ***implicit*** differentiation.

a) $4h^2 + r^2 = 25$; find $D_r h$ or $\frac{dh}{dr}$

Diff. w.r.t. "r"

$8h'\frac{dh}{dr} + 2r' = 0$

$\frac{dh}{dr} = \frac{-2r}{8h}$

$\frac{dh}{dr} = \frac{-r}{4h}$

$\therefore D_r h = \frac{-r}{4h}$

b) $u^2 - \frac{u}{x} = \frac{12}{x}$; find $\frac{du}{dx}$

product rule

$u^2\cdot x - u = 12$

diff. w.r.t. x

$2u\frac{du}{dx}\cdot x + 1\cdot u^2 - 1\frac{du}{dx} = 0$

$2ux\frac{du}{dx} - \frac{du}{dx} = -u^2$

$\frac{du}{dx}(2ux - 1) = -u^2$

$\frac{du}{dx} = \frac{-u^2}{2ux-1}$

or

$\frac{du}{dx} = \frac{u^2}{1-2ux}$

Ex. 5. Show that, if $15x = 15y + 5y^3 + 3y^5$, then $\frac{dy}{dx} = (1 + y^2 + y^4)^{-1}$.

$15x = 15y + 5y^3 + 3y^5$

diff. w.r.t. x

$15 = 15\frac{dy}{dx} + 15y^2\frac{dy}{dx} + 15y^4\frac{dy}{dx}$

÷ by 15

$1 = \frac{dy}{dx} + y^2\frac{dy}{dx} + y^4\frac{dy}{dx}$

$1 = \frac{dy}{dx}(1 + y^2 + y^4)$

$\frac{dy}{dx} = \frac{1}{1+y^2+y^4}$

$\frac{dy}{dx} = (1 + y^2 + y^4)^{-1}$

$\therefore$ If $15x = 15y + 5y^3 + 3y^5$ then

$\frac{dy}{dx} = (1 + y^2 + y^4)^{-1}$

Ex. 6. Let $y = -\left(65 - x^6\right)^{\frac{1}{6}}$. Find the ***equation*** of the ***normal*** line at $(1, -2)$ by first expressing the curve in a simple ***implicit*** form, and then using ***implicit*** differentiation. Leave your answer in standard form.

$$y = -(65 - x^6)^{\frac{1}{6}}$$
$$(y)^6 = \left[-(65 - x^6)^{\frac{1}{6}}\right]^6$$
$$y^6 = +(65 - x^6)$$
$$y^6 = 65 - x^6$$

Diff. w.r.t. x

$$6y^5 \cdot \frac{dy}{dx} = -6x^5$$
$$\frac{dy}{dx} = \frac{-x^5}{y^5}$$

$$\therefore m_t = \frac{-x^5}{y^5}$$

at $x = 1$, $y = -2$

$$m_t = \frac{-(1)^5}{(-2)^5} = \frac{-1}{-32} = \frac{1}{32}$$

For normal,

$m_n = -32$; $(1, -2)$; $b = \_\_$

$$-2 = -32(1) + b$$
$$-2 = -32 + b$$
$$30 = b$$

$\therefore$ the equation of the required normal is $y = -32x + 30$ or

$$32x + y - 30 = 0.$$

ON TEST

Ex. 6. Find all ***points*** (x, y) on the curve $x + xy + 2y^2 = 6$ at which the ***slope*** of the ***tangent*** is $-\frac{1}{3}$.

$$x + x \cdot y + 2y^2 = 6$$

diff. w.r.t. x

$$1 + 1 \cdot y + 1\frac{dy}{dx} \cdot x + 4y\frac{dy}{dx} = 0$$
$$x\frac{dy}{dx} + 4y\frac{dy}{dx} = -y - 1$$
$$\frac{dy}{dx}(x + 4y) = -y - 1$$
$$\frac{dy}{dx} = \frac{-y-1}{x+4y}$$

$$\therefore m_t = \frac{-y-1}{x+4y}$$

Find x if $m_t = -\frac{1}{3}$

$$\frac{-y-1}{x+4y} = \frac{1}{-3}$$
$$x + 4y = 3y + 3$$
$$x + y = 3$$
$$x = 3 - y \quad ②$$

Solve:

$$\left.\begin{array}{ll} x + xy + 2y^2 = 6 & ① \\ x = 3 - y & ② \end{array}\right\}$$

Sub ② into ①

$$(3 - y) + y(3 - y) + 2y^2 = 6$$
$$3 - y + 3y - y^2 + 2y^2 - 6 = 0$$
$$y^2 + 2y - 3 = 0$$
$$(y - 1)(y + 3) = 0$$

$\therefore y = 1$ or $y = -3$

Sub in ②: $x = 3 - 1 = 2$

Sub in ②: $x = 3 - (-3) = 6$

$\therefore$ the slope of the tangent is $-\frac{1}{3}$ at the points $(2, 1)$ & $(6, -3)$.

HW: p. 178 #2 odd parts, 3d, 4 to 8, 9a; p. 219 #1 to 3, 12

Date: March 6/14 **Section 5.2 – Higher-Order Derivatives, Velocity, and Acceleration**

Higher-Order Derivatives

Ex. 1. Calculate the ***first*** and ***second derivatives*** of the following functions.

a) $f(x) = 2x^3 - 5x^2$

$f'(x) = 6x^2 - 10x$

$f''(x) = 12x - 10$

"f double prime"

"d two y by d x^2"

b) $y = \sqrt{1-x}$

$y = (1-x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$\frac{dy}{dx} = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = -\frac{1}{4}(1-x)^{-\frac{3}{2}}$

c) $y = \frac{1}{(2x-3)^2}$

$y = (2x-3)^{-2}$

$y' = -2(2x-3)^{-3}(2)$

$y' = -4(2x-3)^{-3}$

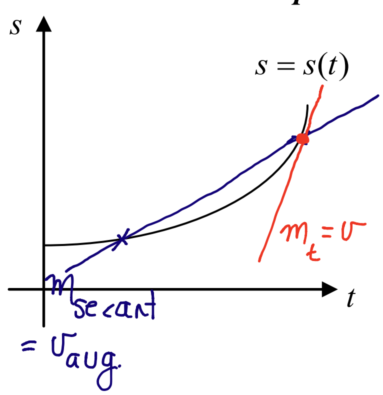
$y'' = 12(2x-3)^{-4}(2)$

$y'' = \frac{24}{(2x-3)^4}$

"y double prime"

Velocity and Acceleration

Distance vs Time Graph



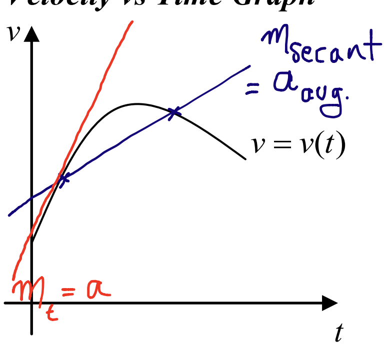
Average Velocity

$$v_{avg} = m_{secant}$$
$$= \frac{s_2 - s_1}{t_2 - t_1}$$
$$\boxed{\therefore v_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}}$$

Velocity

$$v = m_t$$
$$= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$
$$= \frac{ds}{dt} \ \ or \ \ s'$$
$$\boxed{\therefore v(t) = s'(t)}$$

Velocity vs Time Graph



Average Acceleration

$$a_{avg} = m_{secant}$$
$$= \frac{v_2 - v_1}{t_2 - t_1}$$
$$\boxed{\therefore a_{avg} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}}$$

Acceleration

$$a = m_t$$
$$= \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$
$$= \frac{dv}{dt} \ \ or \ \ v'$$
$$= \frac{d^2s}{dt^2} \ \ or \ \ s''$$
$$\boxed{\therefore a(t) = v'(t) \ \ or \ \ s''(t)}$$

Note: Concerning the velocity:

If $v(t) > 0$, the object is moving to the right or up (in the +ve direction).

If $v(t) < 0$, the object is moving to the left or down (in the –ve direction).

If $v(t) = 0$, the object is stationary, or at rest.

Concerning acceleration: If $a(t) = 0$, braking begins.

The object is *accelerating* if the product of $a(t)$ and $v(t)$ is positive.

The object is *decelerating* if the product of $a(t)$ and $v(t)$ is negative.

Ex. 2. Starting at $t = 0$, a dragster accelerates down a strip and then brakes and comes to a stop. The distance in metres at t seconds is given by $s(t) = 6t^2 - \frac{1}{5}t^3$.

a) Find the dragster's velocity and acceleration after 2 seconds.
b) After how many seconds does the dragster stop?
c) What distance does the dragster travel?
d) At what time does braking commence?

$s(t) = 6t^2 - \frac{1}{5}t^3$

$v(t) = s'(t)$

$v(t) = 12t - \frac{3}{5}t^2$

$a(t) = v'(t)$

$a(t) = 12 - \frac{6}{5}t$

a) $v(2) = 12(2) - \frac{3}{5}(2)^2$

$= 21.6$

$a(2) = 12 - \frac{6}{5}(2)$

$= 9.6$

$\therefore$ after 2 seconds the velocity is 21.6 m/s & the acceleration is 9.6 m/s^2.

b) Find t if $v(t) = 0$

$12t - \frac{3}{5}t^2 = 0$

$60t - 3t^2 = 0$

$3t(20 - t) = 0$

$\therefore t = 0$ or $t = 20$ ✓

$\therefore$ the dragster stops after 20 seconds.

c) $s(20)$

$= 6(20)^2 - \frac{1}{5}(20)^3$

$= 800$

$\therefore$ the dragster travels 800 m.

d) Find t if $a(t) = 0$

$12 - \frac{6}{5}t = 0$

$60 - 6t = 0$

$-6t = -60$

$t = 10$

$\therefore$ braking commences at 10 s.

Ex. 3. The position function of an object moving in a straight line is $s(t) = 3t^2 - \frac{1}{4}t^4$.

Is the object moving towards or away from its starting position at:

a) $t = 2$? **b)** $t = 4$? **c)** $t = 3$?

$s(t) = 3t^2 - \frac{1}{4}t^4$

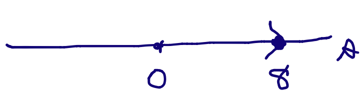
$v(t) = 6t - t^3$

For the starting position, $t = 0$

$s(0) = 0$

a) $s(2) = +8$

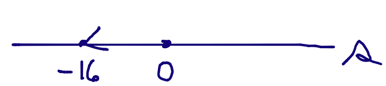
$v(2) = +4$



$\therefore$ at $t = 2$ the object is moving away from its starting position

b) $s(4) = -16$

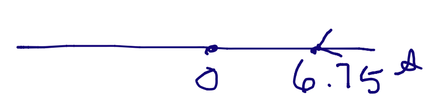
$v(4) = -40$



$\therefore$ at $t = 4$ the object is moving away from its starting position.

c) $s(3) = 6.75$

$v(3) = -9$

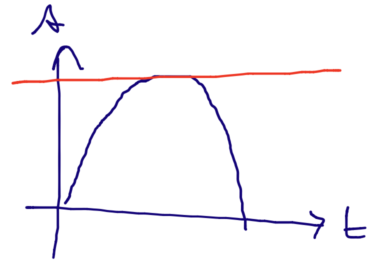


$\therefore$ at $t = 3$ the object is moving towards its starting position

Ex. 4. A fly ball is hit vertically upward. The position function of the ball is $s(t) = -5t^2 + 30t$ where s is its height, in metres, above the ground after t seconds.

a) Find the maximum height reached by the ball.
b) Find the velocity of the ball as it strikes the ground.

$m = v$
$t = 0$

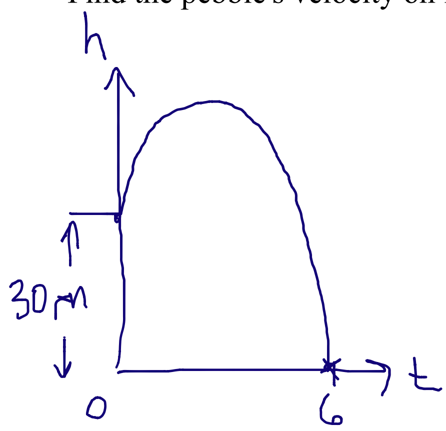


$s(t) = -5t^2 + 30t$
$v(t) = -10t + 30$
$a(t) = -10$

a) Find t if
$v(t) = 0$
$-10t + 30 = 0$
$-10t = -30$
$t = 3$
$s(3) = -5(3)^2 + 30(3)$
$= 45$
$\therefore$ the ball reaches a maximum height of 45 m

b) Find t if
$s(t) = 0$
$-5t^2 + 30t = 0$
$-5t(t-6) = 0$
$t = 0$ or $t = 6$ ✓
$v(6) = -10(6) + 30$
$= -30$
$\therefore$ the ball strikes the ground with a velocity of -30 m/s.

Ex. 5. A pebble is thrown vertically upwards from the top of a cliff of height 30 m. The pebble's height, h, in metres, in relation to the bottom of the cliff t seconds later is $h(t) = -5t^2 + 25t + 30$. Find the pebble's velocity on impact at the base of the cliff.



$h(t) = -5t^2 + 25t + 30$
$v(t) = -10t + 25$

Find t if
$h(t) = 0$
$-5t^2 + 25t + 30 = 0$
$-5(t^2 - 5t - 6) = 0$
$-5(t-6)(t+1) = 0$
$\therefore t = 6$ or $t = -1$
$\because t \geq 0$, $t = 6$

$v(6) = -10(6) + 25$
$= -35$
$\therefore$ the pebble's velocity on impact at the base of the cliff is -35 m/s.

HW: p. 185 #2 to 9, 12, 14, 15; p. 219 #4 to 9, 11ab, 17