Date: Mar.17 / 14

Section 5.3 – RELATED RATES

Ex. 1. ***Differentiate each of the following formulas with respect to time, t.***

CIRCLE

$C = 2\pi r$

$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$

$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

TRIANGLE

$A = \frac{1}{2}bh$

$\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt} \cdot h + \frac{dh}{dt} \cdot \frac{1}{2}b$

SPHERE

$S.A. = 4\pi r^2$

$\frac{dS.A.}{dt} = 8\pi r \cdot \frac{dr}{dt}$

CYLINDER

$V = \pi r^2 h$

$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} \cdot h + \frac{dh}{dt} \cdot \pi r^2$

Ex. 2. A raindrop falls in a puddle and the ripples spread in circles. If the radius is growing at a rate of 2 *cm/s*, find the rate of increase of the area of such a circle when its area is $36\pi \ cm^2$.

Find: $\frac{dA}{dt}$ when $A = 36\pi \ cm^2$

Given: $\frac{dr}{dt} = 2 cm/s$

✱ Find r if
$A = 36\pi \ cm^2$
$\pi r^2 = 36\pi$
$r^2 = 36$
$r = 6, \ r \geq 0$

$A = \pi r^2$

diff. w.r.t. t

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ ✱

$\frac{dA}{dt} = 2\pi(6)(2)$

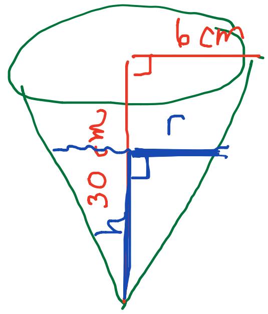
$\frac{dA}{dt} = 24\pi$

∴ the area of the circle is increasing at an exact rate of $24\pi \ cm^2/s$ or an approximate rate of $75.4 \ cm^2/s$

Ex. 3. A conical flower vase is 30 *cm* high with a radius of 6 *cm* at the top. If it is being filled with water at a rate of $10\ cm^3/s$, find the rate at which the water level is rising when the depth is 20 *cm*.

Given: $\frac{dV}{dt} = 10\ cm^3/s$

Find: $\frac{dh}{dt}$ when $h = 20\ cm$



Find r in terms of the water level "h". Using similar $\triangle$s,

$\frac{r}{h} = \frac{6}{30}$

$r = \frac{1}{5}h$

$V = \frac{1}{3}\pi r^2 h$

$V_{water} = \frac{1}{3}\pi\left(\frac{1}{5}h\right)^2 \cdot h$

$V = \frac{1}{3}\pi \cdot \frac{1}{25}h^2 \cdot h$

$V = \frac{\pi}{75}h^3$

diff. w.r.t. "t"

$\frac{dV}{dt} = \frac{\pi}{25}h^2 \cdot \frac{dh}{dt}$

$10 = \frac{\pi}{25}(20)^2 \frac{dh}{dt}$

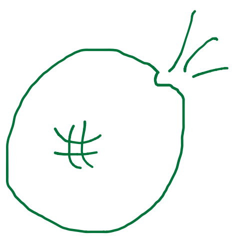
$10 = 16\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{10}{16\pi}$

$\frac{dh}{dt} = \frac{5}{8\pi}$

$\therefore$ the water level is rising at an exact rate of $\frac{5}{8\pi}$ cm/s or approximate rate of 0.2 cm/s when the depth is 20 cm.

Ex. 4. A spherical weather balloon with radius 9 *m* springs a leak losing air at the rate of $171\pi\ m^3/\text{min}$. Find the rate of decrease of the radius after 4 minutes.



$r_0 = 9m$

Given: $\frac{dV}{dt} = -171\pi\ m^3/min$

Find: $\frac{dr}{dt}$ after 4 min.

$V = \frac{4}{3}\pi r^3$

diff. w.r.t. "t"

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-171\cancel{\pi} = 4\cancel{\pi}(6)^2 \frac{dr}{dt}$

$-\frac{171}{144} = \frac{dr}{dt}$

$-\frac{19}{16} = \frac{dr}{dt}$

Find r after 4 min.

$V = V_0 - V_{lost}$

$= \frac{4}{3}\pi(9)^3 - 171\pi \times 4$

$= 972\pi - 684\pi$

$= 288\pi$

Find r

$\frac{4}{3}\cancel{\pi}r^3 = 288\cancel{\pi}$

$r^3 = 216$

$r = \sqrt[3]{216}$

$r = 6$

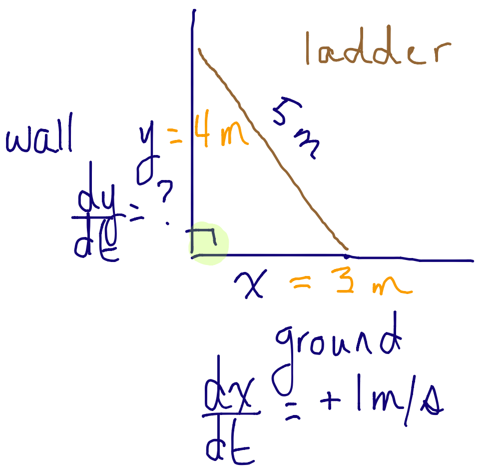
$\therefore$ the rate of decrease of the radius is $1\frac{3}{16}$ m/min.

HW: p. 193 #1ab, 3 to 7, 12, 13, 15, 16, 21

Date: March 18/14

RELATED RATES – I

Ex. 1. A ladder 5 m long rests against a vertical wall. The base of the ladder begins to slide outwards at a rate of 1 m/s. How fast is the top of the ladder descending when the base is 3 m away from the wall?



Find $\frac{dy}{dt}$ when $x = +3$ m

$$x^2 + y^2 = 25$$

diff. w.r.t. "t"

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$\div$ by 2

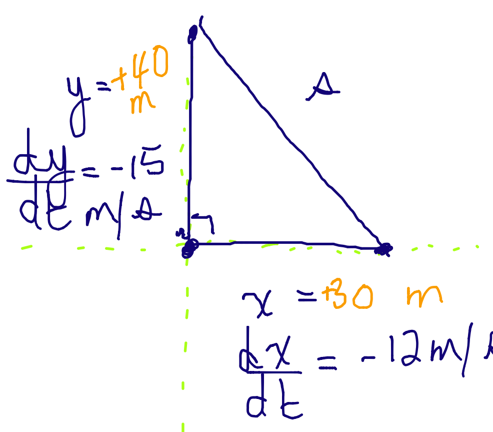
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$(3)(1) + (4)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{3}{4}$$

$\therefore$ the top of the ladder is descending at a rate of $\frac{3}{4}$ m/s.

Ex. 2. Car A approaches an intersection from the east at a rate of 12 m/s and Car B approaches from the north at a rate of 15 m/s. How fast is the distance between the cars decreasing at the instant Car A is 30 m east of the intersection and Car B is 40 m north of the intersection?



Find $\frac{ds}{dt}$ when $x = +30$ m & $y = +40$ m

$$s^2 = x^2 + y^2$$

diff. w.r.t. "t"

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$(50)\frac{ds}{dt} = (+30)(-12) + (+40)(-15)$$

$$\frac{ds}{dt} = -19.2$$

* Find s if $x = +30$ & $y = +40$

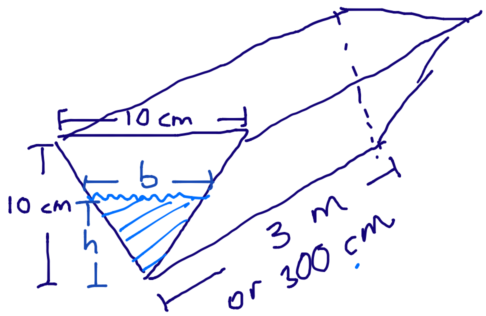
$$s^2 = (30)^2 + (40)^2$$

$$s^2 = 2500$$

$$s = 50$$

$\therefore$ the distance between the cars is decreasing at a rate of 19.2 m/s.

Ex. 3. A horizontal eavestrough 3 m long has a triangular cross-section 10 cm across the top and 10 cm deep. During a rainstorm, the water in the trough is rising at a rate of 1 cm/min. How fast is the volume of water in the trough increasing when the depth of water is 5 cm?



Given: $\frac{dh}{dt} = +1$ cm/min

Find: $\frac{dV}{dt}$ when $h = 5$ cm

$V_{\text{water in } cm^3} = A_{\text{triangle}} \times 300$

$= \frac{bh}{2} \times 300$

$V = 150\,bh$

$V = 150(h)(h)$

$V = 150h^2$

diff. w.r.t. "t"

$\frac{dV}{dt} = 300\,h\,\frac{dh}{dt}$

$= 300(5)(1)$

$= 1500$

∴ the volume of water in the trough is increasing at a rate of 1500 cm^3/min. or 0.0015 m^3/min.

✱Find b in terms of h using similar △s.

$\frac{b}{h} = \frac{10}{10}$

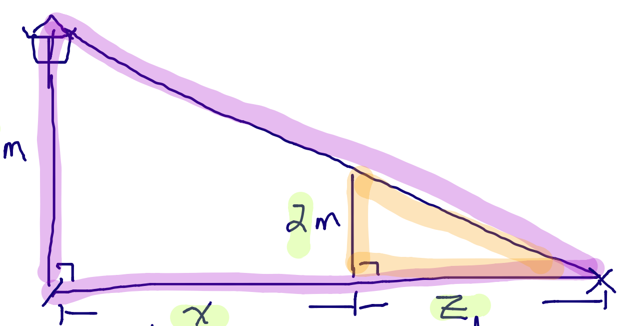
$b = h$

Ex. 4. A woman 2 m tall walks away from a streetlight that is 6 m high at a rate of 1.5 m/s.

a) At what rate is her shadow lengthening when she is 3 m from the base of the light?

b) At what rate is the tip of her shadow moving when she is 3 m from the base of the light?

a)



$\frac{dx}{dt} = +1.5$ m/s a) $\frac{dz}{dt} = 0.75$ m/s

Find $\frac{dz}{dt}$ when $x = 3$ m

Using similar △'s

$\frac{6}{2} = \frac{x+z}{z}$

$\frac{3}{1} = \frac{x+z}{z}$

$3z = x + z$

$2z = x$

$z = \frac{1}{2}x$

diff. w.r.t t

$\frac{dz}{dt} = \frac{1}{2}\frac{dx}{dt}$

$= \frac{1}{2}\left(\frac{3}{2}\right)$

$= \frac{3}{4}$

∴ her shadow is lengthening at a rate of $\frac{3}{4}$ m/s when she is 3 m from the base of the light.

b) $\frac{dx}{dt} + \frac{dz}{dt}$

$= 1.5 + 0.75$

$= 2.25$

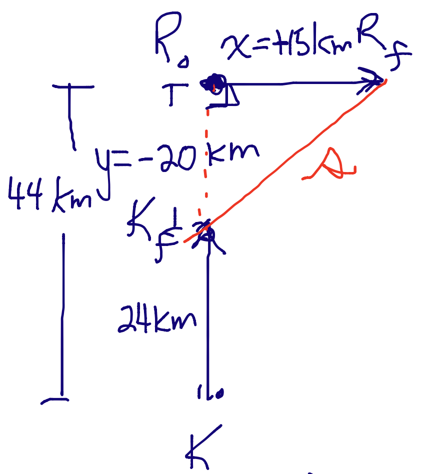
∴ the tip of her shadow is moving at a rate of 2.25 m/s.

HW: p. 193 #1cd, 2a, 8 to 10, 18, 19; Related Rates – Sheet I

Date: Mar.19/14

RELATED RATES – II

Ex. 1. A ship *K* is sailing due north at 16 km/h, and a second ship *R*, which is 44 km north of *K*, is sailing due east at 10 km/h. At what rate is the distance between ships *K* and *R* changing 90 minutes later? Are they approaching one another or separating at this time?



Given $\frac{dy}{dt}=+16$ km/h & $\frac{dx}{dt}=+10$ km/h

Find $\frac{ds}{dt}$ after 90 min.

After 90 min, ship K sails 24 km N. & ship R sails 15 km E.

$s^2=x^2+y^2$

diff. w.r.t. t

$\cancel{2}s\frac{ds}{dt}=\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}$

Find s if $x=15$ & $y=-20$

$s^2=(15)^2+(-20)^2$

$s^2=625$

$s=\sqrt{625}$

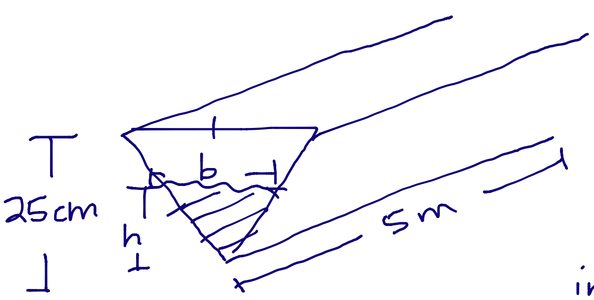
$s=25$

$(25)\frac{ds}{dt}=(+15)(+10)+(-20)(+16)$

$\frac{ds}{dt}=-6.8$

$\therefore$ the distance between the ships is changing at a rate of -6.8 km/h. They are approaching one another at this time.

Ex. 2. The cross-section of a water trough is an equilateral triangle with a horizontal top edge. If the trough is 5 m long and 25 cm deep, and water is flowing in at a rate of 0.25 m^3/min, how fast is the water level rising when the water is 10 cm deep at the deepest point?



Given: $\frac{dV}{dt}=0.25\ m^3/min$

Find: $\frac{dh}{dt}$ when $h=10$ cm or $h=0.1$ m

V_{water} in m^3 $=\frac{1}{2}bh\times 5$

$V=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}h\cdot h\cdot 5$

$V=\frac{5}{\sqrt{3}}h^2$

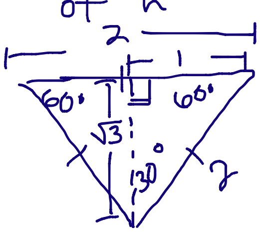
diff. w.r.t. t

$\frac{dV}{dt}=\frac{10}{\sqrt{3}}h\frac{dh}{dt}$

$\frac{1}{4}=\frac{\cancel{10}}{\sqrt{3}}\left(\frac{1}{\cancel{10}}\right)\frac{dh}{dt}$

$\frac{\sqrt{3}}{4}=\frac{dh}{dt}$

Find b in terms of h



$\frac{b}{h}=\frac{2}{\sqrt{3}}$

$b=\frac{2}{\sqrt{3}}h$

$\therefore$ the water level is rising at an exact rate of $\frac{\sqrt{3}}{4}$ m/min or an approximate rate of 43 cm/min.

Ex. 3. A conveyor belt system at a gravel pit pours washed sand onto the ground at the rate of 180 m^3/h. The sand forms a conical pile with height one-third the diameter of the base. Determine how fast the height of the pile is increasing at the instant the radius of the base is 6 m.

Given: $\frac{dV}{dt} = 180 m^3/h$ & $h = \frac{1}{3}d$

Find: $\frac{dh}{dt}$ when $r = 6$ m

$h = \frac{1}{3}d$

* Find r in terms of h

$h = \frac{2r}{3}$

$\frac{3h}{2} = r$

** Find h if r = 6 m

$h = \frac{2(6)}{3}$

$h = 4$

$V = \frac{1}{3}\pi r^2 h$ *

$V = \frac{1}{3}\pi\left(\frac{3h}{2}\right)^2 \cdot h$

$V = \frac{1}{\cancel{3}_1}\pi \cdot \frac{\cancel{9}^3 h^2}{4} \cdot h$

$V = \frac{3\pi h^3}{4}$

diff. w.r.t. "t"

$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$

**

$180 = \frac{9}{4}\pi(4)^2 \frac{dh}{dt}$

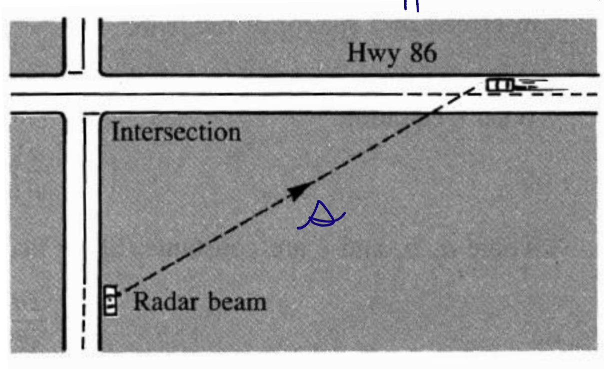
$180 = 36\pi \frac{dh}{dt}$

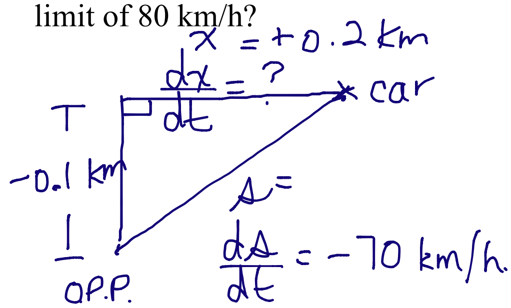
$\frac{180}{36\pi} = \frac{dh}{dt}$

$\frac{5}{\pi} = \frac{dh}{dt}$

∴ the height is increasing at an exact rate of $\frac{5}{\pi}$ m/h or approx. 1.59 m/h

Ex. 4. An OPP officer is operating a radar speed trap on a sideroad 100 m from Highway 86, near Listowel. When a car is 200 m from the intersection, its velocity of approach is measured as 70 km/h. Is the car exceeding the speed limit of 80 km/h?





Find $\frac{dx}{dt}$

* Find s

$s^2 = (-0.1)^2 + (0.2)^2$

$s^2 = 0.01 + 0.04$

$s^2 = 0.05$

$s = \sqrt{0.05}$

$x^2 + (-0.1)^2 = s^2$

$x^2 + 0.01 = s^2$

diff. w.r.t. "t"

$\cancel{2}x\frac{dx}{dt} = \cancel{2}s\frac{ds}{dt}$

$(+0.2)\frac{dx}{dt} = (\sqrt{0.05})(-70)$

$\frac{dx}{dt} = -78.3$

∴ the car is not exceeding the speed limit.

HW: Related Rates – Sheet II

REVIEW FOR TEST: p. 219 #1-9, 11, 12, 17-22, 23a; p. 223 #1-7, 9; p. 291 #1-6, 11, 12